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Risk Aversion and Risk Vulnerability in the Continuous and Discrete Case

A Unified Treatment with Extensions

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Abstract This paper discusses utility functions for money, where allowable money values are from an arbitrary nonempty closed subset of the real numbers. Thus the classical case, where this subset is a closed interval (bounded or not) of the real line, is included in the study. The discrete case, where this subset is the set of all integer numbers, is also included. In a sense the discrete case (which has not been addressed in the literature thus far) is more suitable for real-world applications than the continuous case. In this general setting, the concepts of risk aversion and risk premium are defined, an analogue of Pratt's fundamental theorem is proved, and temperance, prudence, and risk vulnerability are examined.

Keywords Utility function · time scale · delta derivative · risk aversion · risk vulnerability

Mathematics Subject Classification (2000) AMS Subject Classification 91B16, 34A10; JEL Classification C02, D81

1 Introduction

Expected utility analysis has been the work horse for decision-making under risk. [Arrow(1965)] and [Pratt(1964)] (see also [Machina(1987)]) presented a path-breaking work in which they define a utility function property that guarantees intuitive behaviorial results in a class of economic decision problems, under the assumption that economic agents exhibit risk-averse behavior, i.e., the utility function is strictly increasing and concave on the set of all wealth levels. In particular, [Arrow(1976)] showed that in portfolios of one risky and

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one riskless asset such that the expected value of the return of the risky asset is larger than the return of the riskless asset, the optimal quantity of the risky asset held is positive and increasing in the wealth level if and only if the utility function u exhibits the property that $-u''(w)/u'(w)$ is decreasing in w. This property is commonly referred to as Arrow–Pratt DARA (decreasing absolute risk aversion).

[Pratt(1964)] analyzed the problem of optimal insurance payments for individuals facing unwanted risks. In particular, he showed that, if we define the risk premium π^*, for a mean-zero risk $\tilde{\varepsilon}$ added to initial wealth w_0, by $\mathbb{E}(u(w_0+\tilde{\varepsilon})) = u(w_0-\pi^*)$, then π^* is a decreasing function of w_0 if and only if u exhibits Arrow–Pratt DARA.

Since this seminal work, a number of researchers have extended the Arrow–Pratt results by replacing the assumption of fixed initial wealth with the more general case of random initial wealth. For instance, [Gollier and Pratt(1996)] define a utility function property called "risk vulnerability". Defining π^* and $\mathit{\Pi}^*$ by $\mathbb{E}(u(w+X)) = u(w-\pi^*)$ and $\mathbb{E}(u(w+X+Y)) = \mathbb{E}(u(w+Y-\mathit{\Pi}^*))$ and assuming $\mathbb{E}(Y) \leq 0$, $\mathit{\Pi}^* \geq \pi^*$ for all w is equivalent to u being risk vulnerable. [Gollier and Pratt(1996)] show that if u exhibits Arrow–Pratt DARA, then $t \geq r(2-r/p)$ is sufficient for u being risk vulnerable, where p, the coefficient of absolute prudence, equals $-u'''(w)/u''(w)$ and t, the coefficient of absolute temperance, equals $-u''''(w)/u'''(w)$.

The work of Arrow and Pratt, and the many researchers who have labored to extend their results to a broader class of risk problems, has created a rich tapestry within expected utility theory. However, one weakness of the traditional approach is the assumption that money, the domain of our utility functions, can be represented as a continuous variable. While this assumption allows us to use traditional calculus in our analysis, it is in fact unrealistic as it is impractical to talk of $\sqrt{2}$ dollars. Money is best modelled as a discrete set of integers (cents, for instance) rather than as a continuous set of real numbers. What is needed is an approach that would allow for a calculus-based analysis on a discrete domain.

A solution to the problem of unifying continuous and discrete analysis can be found in the theory of time scales. Time scale analysis was introduced by Stefan Hilger in 1988, see [Hilger(1990), Hilger(1988)]. The theory allows for the performance of calculus on functions, where the domain can be an arbitrary nonempty closed subset of the real numbers. The time scale calculus has been applied to problems in a number of areas, including biology (see, e.g., [Bohner and Warth(2007), Bohner et al(2007)Bohner, Fan, and Zhang]), engineering (see, e.g., [Bohner and Martynyuk(2007), Kloeden and Zmorzynska(2006), Sheng(2005)]), and physics (see, e.g., [Bohner and Ünal(2005), Bekker et al(2010)Bekker, Bohner, Herega, and Voulov]).

Time scale analysis also has been applied to problems in business and economics. For instance, [Atıcı et al(2006)Atıcı, Biles, and Lebedinsky] apply a calculus of variations model based on time scales to a dynamic optimization problem with a consumer seeking to maximize lifetime utility under consumption constraints. [Atıcı and Uysal(2008)] use the same calculus of variations on time scales applied to the problem of optimal production and inventory paths. [Tisdell and Zaidi(2008), Section 6] present another short application to a simple model from economics, known as the Keynesian–Cross model with lagged income. However, there has not been to date an application of time scale calculus to the economic problems of decision making under risk and expected utility analysis. The current paper represents a first contribution to this area.

The set up of this paper is as follows. In the next section we recall some basic properties from the calculus on time scales (see [Bohner and Peterson(2001), Bohner and Peterson(2003)]), but only those that are essential for the studies in this paper and only in a very dense form. Examples for the three most important cases of time scales are given in Section 2 as well. In Section 3 we introduce the concepts of risk aversion and risk premium of a utility function and give the main definitions used in this paper, state the main theorem, and illustrate it with some examples. Section 4 contains the proof of Jensen's inequality, which is interesting in its own right and is needed in the proof of the main theorem. This proof is presented in Section 4 using a series of auxiliary results. Section 5 discusses applications and examples, and we also prove that the risk aversion is dominated by the reciprocal of the graininess in the case of isolated time scales. In Section 6 we introduce the time scales analogue of risk vulnerability and exhibit various conditions that are sufficient for the utility function to be risk vulnerable. Some of these conditions involve the coefficients of absolute temperance and prudence, which are introduced in Section 6 as well (see also [Maggi et al(2006)Maggi, Magnani, and Menegatti]). Finally, an appendix supplies some results instrumental in proving the required time scales version of Jensen's inequality.

Time scales analogues of many results concerning risk aversion and risk vulnerability from the two papers [Pratt(1964), Gollier and Pratt(1996)] are included in our study. This combines two active areas of research, namely the study of dynamic equations on time scales in mathematics (see [Bohner and Peterson(2001), Bohner and Peterson(2003)]) and the study of utility functions and risk aversion in economics (see [Gollier(2001)]). Just like the papers [Pratt(1964), Gollier and Pratt(1996)] initiated much research (see [Diamond and Gelles(1995), Diamond and Gelles(1999), Gelles and Mitchell(1999), Kihlstrom et al(1981)Kihlstrom, Romer, and Williams, Kimball(1990), Kimball(1993), Machina and Neilson(1987), Pratt and Zeckhauser(1987)] for a small selection), we anticipate that this paper will be the foundation for further studies in this area.

2 Time Scales Calculus

In this section we will give the rules necessary for the time scales related calculations in this paper. For their proofs and further details we refer the reader to [Bohner and Peterson(2001), Bohner and Peterson(2003)].

2.1 Time Scales

A time scale is an arbitrary nonempty closed subset of the real numbers. It is usually denoted by $\mathbb{T}$. Elements in $\mathbb{T}$ are usually denoted by t, as they represent time. In this paper, however, the elements should represent money rather than time and are therefore denoted by x, while the "time scale" itself is denoted by $\mathbb{X}$. Some examples of such sets $\mathbb{X}$ are the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, $h\mathbb{Z} = \{hk : \ k \in \mathbb{Z}\}$ with $h > 0$, and $q^{\mathbb{N}_0} = \{q^k : \ k \in \mathbb{N}_0\}$ with $q > 1$.

2.2 Jump Operators

The forward and backward jump operators $\sigma : \mathbb{X} \to \mathbb{X}$ and $\rho : \mathbb{X} \to \mathbb{X}$ are defined by

$$\sigma(x) = \inf\{y \in \mathbb{X} : \ y > x\} \qquad \text{and} \qquad \rho(x) = \sup\{y \in \mathbb{X} : \ y < x\}$$

for all $x \in \mathbb{X}$, where we put $\inf \emptyset := \sup \mathbb{X}$ and $\sup \emptyset := \inf \mathbb{X}$. For a function $f : \mathbb{X} \to \mathbb{R}$, we write $f^\sigma = f \circ \sigma$.

2.3 Graininess

The graininess $\mu : \mathbb{X} \to [0, \infty)$ is defined by

$$\mu(x) = \sigma(x) - x \quad \text{ for all } \quad x \in \mathbb{X}.$$

2.4 (Delta) Derivative

Put $\mathbb{X}^\kappa := \mathbb{X}$ if $\sup \mathbb{X} = \infty$; otherwise put $\mathbb{X}^\kappa := \mathbb{X} \setminus (\rho(\sup \mathbb{X}), \sup \mathbb{X}]$. For $f : \mathbb{X} \to \mathbb{R}$, the derivative $f^\Delta(x)$ at $x \in \mathbb{X}^\kappa$ is defined to be the number (if it exists) with the property that for any $\varepsilon > 0$ there exists a neighborhood U of x such that

$$|f(\sigma(x)) - f(y) - f^\Delta(x)(\sigma(x) - y)| \leq \varepsilon |\sigma(x) - y| \quad \text{ for all } \quad y \in U \cap \mathbb{X}.$$

The function f is called differentiable if $f^\Delta(x)$ exists for each $x \in \mathbb{X}^\kappa$. In this case, the derivative function of f is denoted by $f^\Delta : \mathbb{X}^\kappa \to \mathbb{R}$.

2.5 Integral

If $F : \mathbb{X} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{X} \to \mathbb{R}$, i.e., $F^\Delta = f$ on $\mathbb{X}^\kappa$, then the integral of f is defined by

$$\int_x^y f(\xi) \Delta\xi = F(y) - F(x) \quad \text{ for all } \quad x, y \in \mathbb{X} \tag{1}$$

(note that by [Bohner and Peterson(2001), Corollary 1.68 (iii)] the integral does not depend on the particular choice of F).

2.6 Some Formulas

Here we collect some of the time scales results that will be frequently used in this paper. In Theorem 1 below, propositions (a)–(f) are, respectively, [Bohner and Peterson(2001), Theorems 1.16 (iv), 1.20 (i), (iii), (v), 1.93, and 1.97] and proposition (g) is [Bohner and Peterson(2003), Theorem 1.17].

Theorem 1 (a) *Simple Useful Formula: If $f : \mathbb{X} \to \mathbb{R}$ is differentiable, then*

$$f^{\sigma} = f + \mu f^{\Delta}. \tag{2}$$

(b) *Sum Rule: If $f, g : \mathbb{X} \to \mathbb{R}$ are differentiable, then so is $f + g$, and*

$$(f + g)^{\Delta} = f^{\Delta} + g^{\Delta}. \tag{3}$$

(c) *Product Rule: If $f, g : \mathbb{X} \to \mathbb{R}$ are differentiable, then so is fg, and*

$$(fg)^{\Delta} = f^{\Delta} g^{\sigma} + f g^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}. \tag{4}$$

(d) *Quotient Rule: If $f, g : \mathbb{X} \to \mathbb{R}$ are differentiable and $gg^{\sigma} \neq 0$, then so is f/g, and*

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \tag{5}$$

(e) *Chain Rule: Assume that $g : \mathbb{X} \to \mathbb{R}$ is strictly increasing and let $\mathbb{Y} = g(\mathbb{X})$ be a time scale. Let $f : \mathbb{Y} \to \mathbb{R}$. If g^{Δ_x} and $f^{\Delta_y} \circ g$ exist on $\mathbb{X}^{\kappa}$, then $f \circ g$ is differentiable, and*

$$(f \circ g)^{\Delta_x} = (f^{\Delta_y} \circ g) g^{\Delta_x}. \tag{6}$$

(f) *Derivative of the Inverse: Assume that $g : \mathbb{X} \to \mathbb{R}$ is strictly increasing and let $\mathbb{Y} = g(\mathbb{X})$ be a time scale. If $g : \mathbb{X} \to \mathbb{Y}$ is differentiable, then so is $g^{-1} : \mathbb{Y} \to \mathbb{X}$, and*

$$(g^{-1})^{\Delta_y} = \frac{1}{g^{\Delta_x}} \circ g^{-1} \tag{7}$$

at points where g^{Δ_x} is different from zero.

(g) *Mean Value Inequality: Let $a, b \in \mathbb{X}$. If $f, g : \mathbb{X} \to \mathbb{R}$ are continuous functions on $[a, b] \cap \mathbb{X}$ which are differentiable on $[a, b) \cap \mathbb{X}$ such that $g^{\Delta} > 0$ on $[a, b) \cap \mathbb{X}$, then there exist $\xi, \tau \in [a, b) \cap \mathbb{X}$ such that*

$$\frac{f^{\Delta}}{g^{\Delta}}(\tau) \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq \frac{f^{\Delta}}{g^{\Delta}}(\xi). \tag{8}$$

2.7 Exponential Function

The exponential function $e_p(\cdot, x_0)$ is defined to be the unique solution of the initial value problem

$$f^{\Delta}(x) = p(x)f(x), \quad f(x_0) = 1, \qquad x \in \mathbb{X}^{\kappa},$$

where $x_0 \in \mathbb{X}$ and $p : \mathbb{X} \to \mathbb{R}$ is continuous and regressive, i.e., $1+\mu(x)p(x) \neq 0$ for all $x \in \mathbb{X}$. This is denoted by $p \in \mathcal{R}$. We say p is positively regressive and write $p \in \mathcal{R}^+$ if $1 + \mu(x)p(x) > 0$ for all $x \in \mathbb{X}$. If $p \in \mathcal{R}^+$, then $e_p(x, x_0) > 0$ for all $x \in \mathbb{X}$. For more details, see [Bohner and Peterson(2001), Section 2.2].

2.8 Examples

The most important time scales are $\mathbb{R}$ and $\mathbb{Z}$. Let $a < b$. For $\mathbb{X} = \mathbb{R}$, we have

$$\sigma(x) = x, \quad \mu(x) = 0, \quad f^{\Delta}(x) = f'(x), \quad \int_a^b f(x)\Delta x = \int_a^b f(x)\mathrm{d}x$$

and

$$e_p(x, y) = \exp\left\{\int_y^x p(\xi)\Delta\xi\right\}, \quad e_\alpha(x, 0) = e^{\alpha x}.$$

For $\mathbb{X} = \mathbb{Z}$, we have

$$\sigma(x) = x+1, \quad \mu(x) = 1, \quad f^{\Delta}(x) = f(x+1)-f(x), \quad \int_a^b f(x)\Delta x = \sum_a^{b-1} f(x),$$

and

$$e_p(x, y) = \prod_{\xi=y}^{x-1} \{1 + p(\xi)\} \text{ for } y < x, \quad e_\alpha(x, 0) = (1 + \alpha)^x.$$

For $\mathbb{X} = q^{\mathbb{N}_0}$ with $q > 1$, we have

$$\sigma(x) = qx, \quad \mu(x) = (q-1)x, \quad f^{\Delta}(x) = \frac{f(qx) - f(x)}{(q-1)x},$$
$$\int_a^b f(x)\Delta x = (q-1) \sum_{k=\log_q a}^{\log_q b-1} q^k f(q^k),$$

and

$$e_p(x, y) = \prod_{k=\log_q y}^{\log_q x-1} \left\{1 + (q-1)q^k p(q^k)\right\} \text{ for } y < x.$$

3 Comparative Risk Aversion Theorem

In this section we introduce the main notions and present a version of Pratt's comparative risk aversion theorem. The proof of this theorem will be given in Section 4. Throughout this paper, we will use the notation and assumptions as introduced in the following definition.

Definition 1 Let $\mathbb{X}$ and $\mathbb{Y}$ be time scales.

(a) A function $u : \mathbb{X} \to \mathbb{Y}$ with $u(\mathbb{X}) = \mathbb{Y}$ is called a *risk-averse utility function* (henceforth *utility function*) if it is twice differentiable, strictly increasing (i.e., $u^{\Delta} > 0$ on $\mathbb{X}^{\kappa}$), and concave (i.e., $u^{\Delta\Delta} \leq 0$ on $\mathbb{X}^{\kappa\kappa} := (\mathbb{X}^{\kappa})^{\kappa}$). Moreover, when higher derivatives of utility functions occurring in this paper are needed, we assume that they exist.

(b) The *risk aversion coefficient* r of a utility function u is defined by

$$r(x) = -\frac{u^{\Delta\Delta}(x)}{u^{\Delta}(x)} \quad \text{for} \quad x \in \mathbb{X}^{\kappa\kappa}.$$

(c) The *risk premium* π for a utility function u is defined by

$$\pi(X) = \mathbb{E}(X) - u^{-1}(\lfloor \mathbb{E}(u(X)) \rfloor_{\mathbb{Y}})$$

for any random variable X with values in $\mathbb{X}$ (throughout, any random variable occurring in this paper assumes an underlying probability space $(\Omega, \mathcal{A}, P)$, which is always understood since it is enough to know the probability law of the random variable — it is also assumed that all expectations of random variables occurring in this paper are finite), where we define

$$\lfloor x \rfloor_{\mathbb{Y}} = \sup \{y \in \mathbb{Y} : \ y \leq x\} \quad \text{for} \quad x \in \mathbb{R}$$

(note that $\lfloor x \rfloor_{\mathbb{Y}} \in \mathbb{Y} = u(\mathbb{X})$ since $\mathbb{Y}$ is closed and since $\sup \emptyset = \inf \mathbb{Y}$).

(d) The *probability premium* for a utility function u is defined by

$$p(x_1, x_2, x_3) = \frac{2u(x_2) - u(x_1) - u(x_3)}{u(x_3) - u(x_1)}$$

for $x_1, x_2, x_3 \in \mathbb{X}$ with $x_1 \leq x_2 \leq x_3$ and $x_1 < x_3$.

(e) The *Pratt function* for a utility function u is defined by

$$\varphi(x_1, x_2, x_3, x_4) = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}$$

for $x_1, x_2, x_3, x_4 \in \mathbb{X}$ with $x_1 < x_2 \leq x_3 < x_4$.

The main result in this paper is the following analogue of [Pratt(1964), Theorem 1].

Theorem 2 *Let $\mathbb{X}$, $\mathbb{Y}_1$, and $\mathbb{Y}_2$ be time scales. Let r_i, π_i, p_i, and φ_i, $i \in \{1, 2\}$, be the functions introduced in Definition 1 for two given twice continuosly differentiable utility functions $u_1 : \mathbb{X} \to \mathbb{Y}_1$ and $u_2 : \mathbb{X} \to \mathbb{Y}_2$ with $u_1(\mathbb{X}) = \mathbb{Y}_1$ and $u_2(\mathbb{X}) = \mathbb{Y}_2$. Then the following conditions are equivalent.*

(a) $r_1 \geq r_2$.
(b) $\pi_1 \geq \pi_2$.
(c) $p_1 \geq p_2$.
(d) $\varphi_1 \leq \varphi_2$.
(e) u_2^Δ / u_1^Δ *is increasing.*
(f) $u_1 \circ u_2^{-1}$ *is concave.*

The following example explains Theorem 2 in the finite case.

Example 1(i) Let $\mathbb{X} = \{1, 2, 4, 8\}$ and define

$$u_1(1) = 4, \quad u_1(2) = 5, \quad u_1(4) = 6, \quad u_1(8) = 7$$

and

$$u_2(1) = 1, \quad u_2(2) = 3, \quad u_2(4) = 6, \quad u_2(8) = 10.$$

Hence $\mathbb{Y}_1 = \{4, 5, 6, 7\}$ and $\mathbb{Y}_2 = \{1, 3, 6, 10\}$. Thus we have

$$u_1^\Delta(1) = \frac{5-4}{2-1} = 1, \quad u_1^\Delta(2) = \frac{6-5}{4-2} = \frac{1}{2}, \quad u_1^\Delta(4) = \frac{7-6}{8-4} = \frac{1}{4},$$
$$u_2^\Delta(1) = \frac{3-1}{2-1} = 2, \quad u_2^\Delta(2) = \frac{6-3}{4-2} = \frac{3}{2}, \quad u_2^\Delta(4) = \frac{10-6}{8-4} = 1,$$
$$u_1^{\Delta\Delta}(1) = \frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}, \quad u_1^{\Delta\Delta}(2) = \frac{\frac{1}{4}-\frac{1}{2}}{4-2} = -\frac{1}{8},$$
$$u_2^{\Delta\Delta}(1) = \frac{\frac{3}{2}-2}{2-1} = -\frac{1}{2}, \quad u_2^{\Delta\Delta}(2) = \frac{1-\frac{3}{2}}{4-2} = -\frac{1}{4}.$$

Hence

$$r_1(1) = \frac{1}{2}, \quad r_2(1) = \frac{1}{4} \quad \text{so that} \quad r_1(1) \geq r_2(1)$$

and

$$r_1(2) = \frac{1}{4}, \quad r_2(2) = \frac{1}{6} \quad \text{so that} \quad r_1(2) \geq r_2(2).$$

Thus $r_1 \geq r_2$. Moreover, u_2^Δ / u_1^Δ is increasing. Indeed,

$$\frac{u_2^\Delta(1)}{u_1^\Delta(1)} = 2, \quad \frac{u_2^\Delta(2)}{u_1^\Delta(2)} = 3, \quad \text{and} \quad \frac{u_2^\Delta(4)}{u_1^\Delta(4)} = 4.$$

Now we verify that $u_1 \circ u_2^{-1}$ is concave. Indeed,

$$(u_1 \circ u_2^{-1})(1) = 4, \quad (u_1 \circ u_2^{-1})(3) = 5, \quad (u_1 \circ u_2^{-1})(6) = 6, \quad (u_1 \circ u_2^{-1})(10) = 7$$

so that

$$(u_1 \circ u_2^{-1})^{\Delta_2}(1) = \frac{5-4}{3-1} = \frac{1}{2}, \quad (u_1 \circ u_2^{-1})^{\Delta_2}(3) = \frac{6-5}{6-3} = \frac{1}{3},$$
$$(u_1 \circ u_2^{-1})^{\Delta_2}(6) = \frac{7-6}{10-6} = \frac{1}{4}, \quad (u_1 \circ u_2^{-1})^{\Delta_2\Delta_2}(1) = \frac{\frac{1}{3}-\frac{1}{2}}{3-1} = -\frac{1}{12} \leq 0,$$

and

$$(u_1 \circ u_2^{-1})^{\Delta_2 \Delta_2}(2) = \frac{\frac{1}{4} - \frac{1}{3}}{6 - 3} = -\frac{1}{36} \leq 0.$$

Next, we will check Theorem 2 (c) and (d), but only for one set of arguments each:

$$\begin{aligned}
p_1(1,2,4) &= \frac{2u_1(2) - u_1(1) - u_1(4)}{u_1(4) - u_1(1)} = 0, \\
p_2(1,2,4) &= \frac{2u_2(2) - u_2(1) - u_2(4)}{u_2(4) - u_2(1)} = -\frac{1}{5}, \\
\varphi_1(1,2,4,8) &= \frac{u_1(8) - u_1(4)}{u_1(2) - u_1(1)} = \frac{7-6}{5-4} = 1, \\
\varphi_2(1,2,4,8) &= \frac{u_2(8) - u_2(4)}{u_2(2) - u_2(1)} = \frac{10-6}{3-1} = 2.
\end{aligned}$$

Finally, we illustrate Theorem 2 (b) by choosing a random variable X which assumes the value 1 with probability $1/7$, the value 2 with probability $1/2$, and the value 8 with probability $5/14$. Then $\mathbb{E}(X) = 4$,

$$\mathbb{E}(u_1(X)) = \frac{39}{7}, \quad \lfloor \mathbb{E}(u_1(X)) \rfloor_{\mathbb{Y}_1} = 5 \quad \text{so that} \quad \pi_1 = 2$$

and

$$\mathbb{E}(u_2(X)) = \frac{73}{14}, \quad \lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_2} = 3 \quad \text{so that} \quad \pi_2 = 2.$$

(ii) In (i), $\pi_1 = \pi_2$, but it is possible that $\pi_1 > \pi_2$. Indeed, let $\mathbb{X} = \{0,1,2,3\}$ and define

$$u_1(0) = 0, \quad u_1(1) = 2, \quad u_1(2) = 4, \quad u_1(3) = 5$$

and

$$u_2(0) = 2, \quad u_2(1) = 3, \quad u_2(2) = 4, \quad u_2(3) = 5.$$

Hence $\mathbb{Y}_1 = \{0,2,4,5\}$ and $\mathbb{Y}_2 = \{2,3,4,5\}$. We can verify $r_1 \geq r_2$:

$$r_1(0) = 0, \quad r_2(0) = 0, \quad r_1(1) = \frac{1}{2}, \quad r_2(1) = 0.$$

Moreover, by choosing a random variable X which assumes the values 1 and 3 with probability $1/2$ each, we have $\mathbb{E}(X) = 2$,

$$\mathbb{E}(u_1(X)) = 3.5, \quad \lfloor \mathbb{E}(u_1(X)) \rfloor_{\mathbb{Y}_1} = 2 \quad \text{so that} \quad \pi_1 = 1$$

and

$$\mathbb{E}(u_2(X)) = 4, \quad \lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_2} = 4 \quad \text{so that} \quad \pi_2 = 0.$$

4 Jensen's Inequality and Proof of the Fundamental Theorem

We start this section by proving a version of Jensen's inequality.

Theorem 3 *Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be time scales and Z a random variable assuming values in $\mathbb{Y}_1$. If $f : \mathbb{Y}_1 \to \mathbb{Y}_2$ satisfies $f^\Delta \geq 0$ and $f^{\Delta\Delta} \geq 0$, then*

$$\lfloor \mathbb{E}(f(Z)) \rfloor_{\mathbb{Y}_2} \geq f(\lfloor \mathbb{E}(Z) \rfloor_{\mathbb{Y}_1}).$$

Proof We first prove that the auxiliary function $\bar{f}$ defined in Appendix A satisfies the inequality

$$\left\lfloor \bar{f}(\mathbb{E}(Z)) \right\rfloor_{\mathbb{Y}_2} \geq \bar{f}(\lfloor \mathbb{E}(Z) \rfloor_{\mathbb{Y}_1}). \tag{9}$$

Denote

$$\bar{x} = \left\lfloor \bar{f}(\mathbb{E}(Z)) \right\rfloor_{\mathbb{Y}_2} \quad \text{and} \quad \bar{y} = \lfloor \mathbb{E}(Z) \rfloor_{\mathbb{Y}_1}.$$

Now, by definition,

$$\bar{y} = \sup \{ y \in \mathbb{Y}_1 : \; y \leq \mathbb{E}(Z) \}$$

so that $\bar{y} \leq \mathbb{E}(Z)$ and (since $\mathbb{Y}_1$ is closed) $\bar{y} \in \mathbb{Y}_1$. Hence, by Lemma 9, we have

$$\bar{f}(\bar{y}) \leq \bar{f}(\mathbb{E}(Z)).$$

Moreover, $\bar{f}(\bar{y}) = f(\bar{y}) \in \mathbb{Y}_2$ since $\bar{y} \in \mathbb{Y}_1$. Altogether,

$$\bar{x} = \sup \left\{ x \in \mathbb{Y}_2 : \; x \leq \bar{f}(\mathbb{E}(Z)) \right\} \geq \bar{f}(\bar{y}),$$

which confirms (9). Now, since $f^{\Delta\Delta} \geq 0$, $\bar{f}$ is convex due to Lemma 10, and we conclude by applying the usual Jensen inequality to $\bar{f}$ that

$$\mathbb{E}(\bar{f}(Z)) \geq \bar{f}(\mathbb{E}(Z)) \tag{10}$$

holds. Using (9) and (10), we find

$$\lfloor \mathbb{E}(f(Z)) \rfloor_{\mathbb{Y}_2} = \left\lfloor \mathbb{E}(\bar{f}(Z)) \right\rfloor_{\mathbb{Y}_2} \geq \left\lfloor \bar{f}(\mathbb{E}(Z)) \right\rfloor_{\mathbb{Y}_2} \geq \bar{f}(\lfloor \mathbb{E}(Z) \rfloor_{\mathbb{Y}_1}) = f(\lfloor \mathbb{E}(Z) \rfloor_{\mathbb{Y}_1}),$$

and this completes the proof. □

Now we present a series of lemmas that establish Theorem 2. We use the notation as introduced in Section 3.

Lemma 1 *If $r_1 \geq r_2$, then u_2^Δ / u_1^Δ is increasing.*

Proof Using the quotient rule on time scales (5), we find

$$\begin{aligned} \left(\frac{u_2^\Delta}{u_1^\Delta} \right)^\Delta &= \frac{u_2^{\Delta\Delta} u_1^\Delta - u_2^\Delta u_1^{\Delta\Delta}}{u_1^\Delta u_1^{\Delta\sigma}} \;=\; \frac{u_2^{\Delta\Delta}}{u_1^{\Delta\sigma}} - \frac{u_2^\Delta u_1^{\Delta\Delta}}{u_1^\Delta u_1^{\Delta\sigma}} \\ &= \left(\frac{u_2^{\Delta\Delta}}{u_2^\Delta} - \frac{u_1^{\Delta\Delta}}{u_1^\Delta} \right) \frac{u_2^\Delta}{u_1^{\Delta\sigma}} \;=\; (r_1 - r_2) \frac{u_2^\Delta}{u_1^{\Delta\sigma}} \;\geq\; 0. \end{aligned}$$

Hence u_2^Δ / u_1^Δ is increasing. □

Lemma 2 *If u_2^Δ/u_1^Δ is increasing, then $u_1 \circ u_2^{-1}$ is concave.*

Proof Using the chain rule (6) and the derivative of the inverse (7) (note that all required assumptions are satisfied), we find

$$\left(u_1 \circ u_2^{-1}\right)^{\Delta_2} \circ u_2 = \left[\left(u_1^\Delta \circ u_2^{-1}\right)\left(u_2^{-1}\right)^{\Delta_2}\right] \circ u_2 = u_1^\Delta \left[\left(u_2^{-1}\right)^{\Delta_2} \circ u_2\right] = \frac{u_1^\Delta}{u_2^\Delta},$$

and thus

$$\left(u_1 \circ u_2^{-1}\right)^{\Delta_2} = \frac{u_1^\Delta}{u_2^\Delta} \circ u_2^{-1} \quad \text{is decreasing}$$

since u_2^{-1} is strictly increasing. Hence $u_1 \circ u_2^{-1} : \mathbb{Y}_2 \to \mathbb{Y}_1$ is concave. □

Lemma 3 *If $u_1 \circ u_2^{-1}$ is concave, then $\pi_1 \geq \pi_2$.*

Proof We let $f = u_2 \circ u_1^{-1}$. Then $f^{\Delta_1} \geq 0$ since f is strictly increasing. Moreover, recalling that

$$0 \geq \left(u_1 \circ u_2^{-1}\right)^{\Delta_2\Delta_2} = \left(\frac{u_1^\Delta}{u_2^\Delta} \circ u_2^{-1}\right)^{\Delta_2},$$

we have that u_1^Δ/u_2^Δ is decreasing. Consequently, u_2^Δ/u_1^Δ is increasing, so that $f^{\Delta_1} = (u_2^\Delta/u_1^\Delta) \circ u_1^{-1}$ is also increasing. Therefore, $f^{\Delta_1\Delta_1} \geq 0$. Now let X be a random variable with values in $\mathbb{X}$ and define $Y = u_1(X)$. By applying Theorem 3, we have

$$\begin{aligned}
\pi_1(X) - \pi_2(X) &= \mathbb{E}(X) - u_1^{-1}\left(\lfloor \mathbb{E}(u_1(X))\rfloor_{\mathbb{Y}_1}\right) \\
&\quad - \left[\mathbb{E}(X) - u_2^{-1}\left(\lfloor \mathbb{E}(u_2(X))\rfloor_{\mathbb{Y}_2}\right)\right] \\
&= u_2^{-1}\left(\lfloor \mathbb{E}(u_2(X))\rfloor_{\mathbb{Y}_2}\right) - u_1^{-1}\left(\lfloor \mathbb{E}(u_1(X))\rfloor_{\mathbb{Y}_1}\right) \\
&= u_2^{-1}\left(\lfloor \mathbb{E}(f(Y))\rfloor_{\mathbb{Y}_2}\right) - u_1^{-1}\left(\lfloor \mathbb{E}(Y)\rfloor_{\mathbb{Y}_1}\right) \\
&\geq u_2^{-1}\left(f\left(\lfloor \mathbb{E}(Y)\rfloor_{\mathbb{Y}_1}\right)\right) - u_1^{-1}\left(\lfloor \mathbb{E}(Y)\rfloor_{\mathbb{Y}_1}\right) \\
&= 0,
\end{aligned}$$

on noting that u_2^{-1} is an increasing function. □

Lemma 4 *If u_2^Δ/u_1^Δ is increasing, then $\varphi_1 \leq \varphi_2$.*

Proof Suppose $x_1, x_2, x_3, x_4 \in \mathbb{X}$ are such that $x_1 < x_2 \leq x_3 < x_4$. Let $\xi \leq x_3$ and define

$$f(x) = \frac{u_2(x)}{u_2^\Delta(\xi)} - \frac{u_1(x)}{u_1^\Delta(\xi)}.$$

Then

$$\frac{u_2(x_4) - u_2(x_3)}{u_2^\Delta(\xi)} - \frac{u_1(x_4) - u_1(x_3)}{u_1^\Delta(\xi)} = f(x_4) - f(x_3) \geq f^\Delta(\xi^*)(x_4 - x_3)$$

for some $\xi^* \in \mathbb{X}$ with $x_3 \le \xi^* < x_4$, by the mean value theorem (8). Note

$$f^\Delta(\xi^*) = \frac{u_2^\Delta(\xi^*)}{u_2^\Delta(\xi)} - \frac{u_1^\Delta(\xi^*)}{u_1^\Delta(\xi)} = \left[\left(\frac{u_2^\Delta}{u_1^\Delta}\right)(\xi^*) - \left(\frac{u_2^\Delta}{u_1^\Delta}\right)(\xi)\right]\frac{u_1^\Delta(\xi^*)}{u_2^\Delta(\xi)} \ge 0$$

due to the increasing nature of u_2^Δ/u_1^Δ and since $\xi^* \ge x_3 \ge \xi$. Hence we have shown that

$$0 < \frac{u_1(x_4) - u_1(x_3)}{u_1^\Delta(\xi)} \le \frac{u_2(x_4) - u_2(x_3)}{u_2^\Delta(\xi)} \quad \text{whenever} \quad \xi \le x_3 \tag{11}$$

holds. Now we define

$$g(x) = \frac{u_1(x)}{u_1(x_4) - u_1(x_3)} - \frac{u_2(x)}{u_2(x_4) - u_2(x_3)}.$$

Then

$$\frac{u_1(x_2) - u_1(x_1)}{u_1(x_4) - u_1(x_3)} - \frac{u_2(x_2) - u_2(x_1)}{u_2(x_4) - u_2(x_3)} = g(x_2) - g(x_1) \ge g^\Delta(\tilde{\xi})(x_2 - x_1)$$

for some $\tilde{\xi} \in \mathbb{X}$ with $x_1 \le \tilde{\xi} < x_2$, by the mean value theorem (8). Note

$$g^\Delta(\tilde{\xi}) = \frac{u_1^\Delta(\tilde{\xi})}{u_1(x_4) - u_1(x_3)} - \frac{u_2^\Delta(\tilde{\xi})}{u_2(x_4) - u_2(x_3)} \ge 0$$

due to (11) and since $\tilde{\xi} < x_2 \le x_3$. Hence we have shown

$$\frac{u_1(x_2) - u_1(x_1)}{u_1(x_4) - u_1(x_3)} \ge \frac{u_2(x_2) - u_2(x_1)}{u_2(x_4) - u_2(x_3)},$$

i.e., $\varphi_1(x_1, x_2, x_3, x_4) \le \varphi_2(x_1, x_2, x_3, x_4)$. □

Before we study the equivalence concerning the probability premium p, let us state some properties that can easily be verified by calculation:

$$1 - p(x_1, x_2, x_3) = 2\frac{u(x_3) - u(x_2)}{u(x_3) - u(x_1)} \ge 0, \tag{12}$$

$$1 + p(x_1, x_2, x_3) = 2\frac{u(x_2) - u(x_1)}{u(x_3) - u(x_1)} \ge 0, \tag{13}$$

$$|p| \le 1, \tag{14}$$

$$\frac{1 - p(x_1, x_2, x_3)}{1 + p(x_1, x_2, x_3)} = \frac{u(x_3) - u(x_2)}{u(x_2) - u(x_1)} \ge 0 \quad \text{for} \quad x_1 < x_2. \tag{15}$$

Lemma 5 *If $\varphi_1 \le \varphi_2$, then $p_1 \ge p_2$.*

Proof Since $x_1 = x_2$ or $x_2 = x_3$ implies $p_1 = p_2$, let $x_1 < x_2 < x_3$. By (15), we have

$$\frac{1 - p_1(x_1, x_2, x_3)}{1 + p_1(x_1, x_2, x_3)} = \varphi_1(x_1, x_2, x_2, x_3)$$

and

$$\frac{1 - p_2(x_1, x_2, x_3)}{1 + p_2(x_1, x_2, x_3)} = \varphi_2(x_1, x_2, x_2, x_3).$$

Since $\varphi_1(x_1, x_2, x_2, x_3) \leq \varphi_2(x_1, x_2, x_2, x_3)$, we have

$$\frac{1 - p_1(x_1, x_2, x_3)}{1 + p_1(x_1, x_2, x_3)} \leq \frac{1 - p_2(x_1, x_2, x_3)}{1 + p_2(x_1, x_2, x_3)}$$

and hence

$$\begin{aligned} 0 &\leq \frac{1 - p_2(x_1, x_2, x_3)}{1 + p_2(x_1, x_2, x_3)} - \frac{1 - p_1(x_1, x_2, x_3)}{1 + p_1(x_1, x_2, x_3)} \\ &= 2\frac{p_1(x_1, x_2, x_3) - p_2(x_1, x_2, x_3)}{(1 + p_1(x_1, x_2, x_3))(1 + p_2(x_1, x_2, x_3))}. \end{aligned}$$

Hence the claim follows from (13). □

Lemma 6 *Let $\tilde{x}, \bar{x}, \hat{x} \in \mathbb{X}$ with $\tilde{x} < \bar{x} < \hat{x}$. Define*

$$p = \frac{u_1(\hat{x}) - u_1(\bar{x})}{u_1(\hat{x}) - u_1(\tilde{x})}.$$

Then $p \in (0, 1)$. Now let X be a random variable which assumes the value $\tilde{x}$ with probability p and the value $\hat{x}$ with probability $1 - p$. Then

$$\pi_1(X) = \mathbb{E}(X) - \bar{x}$$

and there exist $\tau \in [\tilde{x}, \bar{x}) \cap \mathbb{X}$ and $\xi \in [\bar{x}, \hat{x}) \cap \mathbb{X}$ such that

$$u_2(\bar{x}) - \mathbb{E}(u_2(X)) \geq p\left(u_1(\bar{x}) - u_1(\tilde{x})\right)\left(\frac{u_2^{\Delta}}{u_1^{\Delta}}(\tau) - \frac{u_2^{\Delta}}{u_1^{\Delta}}(\xi)\right).$$

Proof Since u_1 is strictly increasing, we have $0 < p < 1$. Moreover,

$$\mathbb{E}(u_1(X)) = pu_1(\tilde{x}) + (1 - p)u_1(\hat{x}) = u_1(\hat{x}) - p\left(u_1(\hat{x}) - u_1(\tilde{x})\right) = u_1(\bar{x})$$

so that $\pi_1(X) = \mathbb{E}(X) - \bar{x}$. Finally,

$$\begin{aligned} &u_2(\bar{x}) - \mathbb{E}(u_2(X)) = u_2(\bar{x}) - pu_2(\tilde{x}) - (1 - p)u_2(\hat{x}) \\ &= p\left(u_2(\hat{x}) - u_2(\tilde{x})\right) - \left(u_2(\hat{x}) - u_2(\bar{x})\right) \\ &= \frac{\left(u_1(\hat{x}) - u_1(\bar{x})\right)\left(u_2(\hat{x}) - u_2(\tilde{x})\right) - \left(u_1(\hat{x}) - u_1(\tilde{x})\right)\left(u_2(\hat{x}) - u_2(\bar{x})\right)}{u_1(\hat{x}) - u_1(\tilde{x})} \\ &= \frac{\left(u_1(\hat{x}) - u_1(\bar{x})\right)\left(u_2(\bar{x}) - u_2(\tilde{x})\right) - \left(u_1(\bar{x}) - u_1(\tilde{x})\right)\left(u_2(\hat{x}) - u_2(\bar{x})\right)}{u_1(\hat{x}) - u_1(\tilde{x})} \\ &= p\left(u_1(\bar{x}) - u_1(\tilde{x})\right)\left(\frac{u_2(\bar{x}) - u_2(\tilde{x})}{u_1(\bar{x}) - u_1(\tilde{x})} - \frac{u_2(\hat{x}) - u_2(\bar{x})}{u_1(\hat{x}) - u_1(\bar{x})}\right). \end{aligned}$$

Using the mean value theorem (8) completes the proof. □

Proof (Proof of Theorem 2) In this section we have now proved that (a) implies (e) implies (f) implies (b) and that (e) implies (d) implies (c). The proof is completed if we can show that (b) implies (a) and that (c) implies (a). We first show that (b) implies (a). To this end, suppose (b) holds and assume (a) is wrong, i.e., there exists $\tilde{x} \in \mathbb{X}^{\kappa\kappa}$ such that $r_1(\tilde{x}) < r_2(\tilde{x})$. As in the proof of Lemma 1, we have

$$\left(\frac{u_2^\Delta}{u_1^\Delta}\right)^\Delta (\tilde{x}) = (r_1(\tilde{x}) - r_2(\tilde{x}))\,\frac{u_2^\Delta(\tilde{x})}{u_1^\Delta(\sigma(\tilde{x}))} < 0. \tag{16}$$

Without loss of generality, we may assume $\tilde{x} \neq \sup \mathbb{X}$ (if $\sup \mathbb{X}$ is left-scattered, then $\tilde{x} \neq \sup \mathbb{X}$ as $\tilde{x} \in \mathbb{X}^{\kappa\kappa}$; if $\sup \mathbb{X}$ is left-dense and $\tilde{x} = \sup \mathbb{X}$, then, by the continuity of r_1 and r_2, (16) holds in a left neighborhood of $\tilde{x}$, so that $\tilde{x}$ can be replaced with one of those other points in this neighborhood). Only the following three cases are possible:

1. $\tilde{x}$ is right-scattered and $\sigma(\tilde{x})$ is right-scattered.
2. $\tilde{x}$ is right-scattered and $\sigma(\tilde{x})$ is right-dense.
3. $\tilde{x}$ is right-dense.

In each case, we will construct a random variable as in Lemma 6 and show that $u_2(\bar{x}) - \mathbb{E}(u_2(X)) > 0$, which will yield $\lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_2} < u_2(\bar{x})$ and hence

$$\pi_2(X) = \mathbb{E}(X) - u_2^{-1}\left(\lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_2}\right) > \mathbb{E}(X) - \bar{x} = \pi_1(X),$$

contradicting (b) and concluding the proof. In Case 1, use $\bar{x} = \sigma(\tilde{x})$ and $\hat{x} = \sigma(\bar{x})$ so that $\tau = \tilde{x}$ and $\xi = \bar{x}$. In this case, (16) implies $u_2^\Delta(\tau)/u_1^\Delta(\tau) > u_2^\Delta(\xi)/u_1^\Delta(\xi)$. In Case 2, use $\bar{x} = \sigma(\tilde{x})$ so that $\tau = \tilde{x}$. In this case, (16) implies $u_2^\Delta(\tau)/u_1^\Delta(\tau) > u_2^\Delta(\bar{x})/u_1^\Delta(\bar{x})$, and hence there exists $\hat{x} > \bar{x}$ with $\hat{x} \in \mathbb{X}$ such that $u_2^\Delta(\tau)/u_1^\Delta(\tau) > u_2^\Delta(\xi)/u_1^\Delta(\xi)$ for all $\xi \in [\bar{x}, \hat{x}) \cap \mathbb{X}$. In Case 3, (16) implies that there exists $\hat{x} > \tilde{x}$ with $\hat{x} \in \mathbb{X}$ such that $u_2^\Delta(\tau)/u_1^\Delta(\tau) > u_2^\Delta(\xi)/u_1^\Delta(\xi)$ for all $\tau, \xi \in \mathbb{X}$ such that $\tilde{x} \leq \tau < \xi < \hat{x}$, and we can choose $\bar{x}$ to be any point in $(\tilde{x}, \hat{x}) \cap \mathbb{X}$.

Now we show that (c) implies (a). To this end, suppose (c) holds and assume (a) is wrong, i.e., there exists $\tilde{x} \in \mathbb{X}^{\kappa\kappa}$, $\tilde{x} \neq \sup \mathbb{X}$, such that $r_1(\tilde{x}) < r_2(\tilde{x})$, and therefore (16) holds. Now, consider Case 1. Since the function f defined by $f(x) = (1-x)/(1+x)$ is strictly decreasing on $(0,1)$ and since (16) implies

$$\frac{u_1(\hat{x}) - u_1(\bar{x})}{u_1(\bar{x}) - u_1(\tilde{x})} = \frac{\mu(\bar{x}) u_1^\Delta(\bar{x})}{\mu(\tilde{x}) u_1^\Delta(\tilde{x})} > \frac{\mu(\bar{x}) u_2^\Delta(\bar{x})}{\mu(\tilde{x}) u_2^\Delta(\tilde{x})} = \frac{u_2(\hat{x}) - u_2(\bar{x})}{u_2(\bar{x}) - u_2(\tilde{x})},$$

we obtain by (15)

$$p_1(\tilde{x}, \bar{x}, \hat{x}) = \frac{1 - \frac{u_1(\hat{x}) - u_1(\bar{x})}{u_1(\bar{x}) - u_1(\tilde{x})}}{1 + \frac{u_1(\hat{x}) - u_1(\bar{x})}{u_1(\bar{x}) - u_1(\tilde{x})}} < \frac{1 - \frac{u_2(\hat{x}) - u_2(\bar{x})}{u_2(\bar{x}) - u_2(\tilde{x})}}{1 + \frac{u_2(\hat{x}) - u_2(\bar{x})}{u_2(\bar{x}) - u_2(\tilde{x})}} = p_2(\tilde{x}, \bar{x}, \hat{x}),$$

contradicting (c). Case 2 and Case 3 can be proved analogously by choosing $\tilde{x}, \bar{x}, \hat{x}$ as in the proof of (b) implies (a). This concludes the proof. $\square$

5 Applications and Examples

Now we give an application of the main result, Theorem 2. It utilizes the equivalence in Theorem 2 (b) and shows that an agent B with a higher risk aversion than agent A rejects all lotteries / gambles that A rejects (and maybe more). This intuitively justifies the comparison of risk aversion of two agents by the risk aversion coefficients of their utility functions.

Definition 2 A utility function $\tilde{u}$ is called *more risk averse* than u if $\tilde{r} \geq r$.

Below, in Theorem 4, Corollary 1, and Definition 5, "always implies" means that the statement is true for any random variable X with values in $\mathbb{X}$ and for all $x^* \in \mathbb{X}$.

Theorem 4 *u_1 is more risk averse than u_2 if and only if*

$$\lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_1} \leq u_2(x^*) \quad \textit{always implies} \quad \lfloor \mathbb{E}(u_1(X)) \rfloor_{\mathbb{Y}_2} \leq u_1(x^*). \tag{17}$$

Proof First we assume $r_1 \geq r_2$. Then $\pi_1 \geq \pi_2$ by Theorem 2 (b). Assume $\lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_1} \leq u_2(x^*)$. Then

$$u_2(x^*) \geq \lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_1} = u_2(\mathbb{E}(X) - \pi_2).$$

Since u_2 is increasing, this implies

$$x^* \geq \mathbb{E}(X) - \pi_2 \geq \mathbb{E}(X) - \pi_1.$$

Since u_1 is increasing, this implies

$$u_1(x^*) \geq u_1(\mathbb{E}(X) - \pi_1) = \lfloor \mathbb{E}(u_1(X)) \rfloor_{\mathbb{Y}_2}.$$

Hence condition (17) is satisfied. Now, in turn, we assume that (17) is satisfied. Let X be arbitrary and define $x^* = \mathbb{E}(X) - \pi_2$. Then

$$u_2(x^*) = u_2(\mathbb{E}(X) - \pi_2) = \lfloor \mathbb{E}(u_2(X)) \rfloor_{\mathbb{Y}_1}.$$

By (17), we conclude

$$u_1(x^*) \geq \lfloor \mathbb{E}(u_1(X)) \rfloor_{\mathbb{Y}_2} = u_1(\mathbb{E}(X) - \pi_1) = u_1(x^* + \pi_2 - \pi_1).$$

Since u_1 is increasing, this implies $x^* \geq x^* + \pi_2 - \pi_1$ and thus $\pi_1 \geq \pi_2$. By Theorem 2 (b), $r_1 \geq r_2$ follows. $\square$

In order to get a corollary of Theorem 4, we introduce the notion of DARA function.

Definition 3 A utility function u is called *decreasingly risk averse* (henceforth DARA) if its risk aversion r is decreasing.

Corollary 1 *Assume $x - y \in \mathbb{X}$ whenever $x, y \in \mathbb{X}$. Then u is DARA if and only if for all $K \in \mathbb{X} \cap (0, \infty)$,*

$$\lfloor \mathbb{E}(u(X)) \rfloor_{\mathbb{Y}} \leq u(x^*) \quad \textit{always implies} \quad \lfloor \mathbb{E}(u(X - K)) \rfloor_{\mathbb{Y}} \leq u(x^* - K).$$

Examples of time scales satisfying the condition in Corollary 1 are $\mathbb{X} = \mathbb{R}$, $\mathbb{X} = \mathbb{Z}$, $\mathbb{X} = h\mathbb{Z}$ with $h > 0$.

The next result will be needed in Section 6.

Theorem 5 *If u is DARA, then $-u^{\Delta}$ is more risk averse than u.*

Proof For $r = -u^{\Delta\Delta}/u^{\Delta}$ we use the quotient rule (5) to find

$$r^{\Delta} = \left(-\frac{u^{\Delta\Delta}}{u^{\Delta}}\right)^{\Delta} = -\frac{u^{\Delta\Delta\Delta}u^{\Delta} - u^{\Delta\Delta}u^{\Delta\Delta}}{u^{\Delta}u^{\Delta\sigma}} = \frac{(u^{\Delta\Delta})^2 - u^{\Delta}u^{\Delta\Delta\Delta}}{u^{\Delta}u^{\Delta\sigma}}. \quad (18)$$

If u is DARA, then $r^{\Delta} \leq 0$ and hence (recall $r \geq 0$ and $u^{\Delta} > 0$)

$$(u^{\Delta\Delta})^2 \leq u^{\Delta\Delta\Delta}u^{\Delta} \quad \text{so that} \quad -\frac{(-u^{\Delta})^{\Delta\Delta}}{(-u^{\Delta})^{\Delta}} \geq -\frac{u^{\Delta\Delta}}{u^{\Delta}},$$

and hence the risk aversion of $-u^{\Delta}$ is greater than or equal to the risk aversion of u. □

Now we give some examples of utility functions. We first derive a relation between a utility function and its risk aversion in terms of the exponential function on time scales.

Theorem 6 *Given a utility function u, the function $-r$ is positively regressive, i.e.,*

$$1 - \mu(x)r(x) > 0 \quad \textit{for all} \quad x \in \mathbb{X}^{\kappa\kappa}.$$

Proof Using the simple useful formula (2), the calculation

$$1 - \mu r = 1 + \mu\frac{u^{\Delta\Delta}}{u^{\Delta}} = \frac{u^{\Delta} + \mu u^{\Delta\Delta}}{u^{\Delta}} = \frac{u^{\Delta\sigma}}{u^{\Delta}} > 0$$

shows that $-r$ is positively regressive. □

By Theorem 6, if $\mathbb{X}$ is an isolated time scale (i.e., the graininess is always positive), we get

$$r(x) \in \left[0, \frac{1}{\mu(x)}\right) \quad \text{for all} \quad x \in \mathbb{X}^{\kappa\kappa}.$$

Note that the continuous case $\mathbb{X} = \mathbb{R}$ (where $\mu \equiv 0$) may be seen as a limit case by putting $1/0^+ = +\infty$.

Theorem 7 *We have*

$$e_{-r}(x, x_0) = \frac{u^{\Delta}(x)}{u^{\Delta}(x_0)} \quad \textit{for all} \quad x \in \mathbb{X}^{\kappa} \quad (19)$$

and

$$u(x) = u^{\Delta}(x_0)\int_{x_0}^{x} e_{-r}(\xi, x_0)\Delta\xi + u(x_0) \quad \textit{for all} \quad x \in \mathbb{X}. \quad (20)$$

Proof Define $f = u^\Delta / u^\Delta(x_0)$. Then $f(x_0) = u^\Delta(x_0)/u^\Delta(x_0) = 1$ and

$$f^\Delta = \frac{u^{\Delta\Delta}}{u^\Delta(x_0)} = \frac{u^{\Delta\Delta}}{u^\Delta}\frac{u^\Delta}{u^\Delta(x_0)} = -rf.$$

Since $-r$ is positively regressive by Theorem 6, it is also regressive and hence the exponential function $e_{-r}(\cdot, x_0)$ is well defined and (19) follows from the properties of the exponential function on time scales. By integrating (19) between x_0 and x, we obtain (20). □

Definition 4 For two utility functions u_1 and u_2 defined on the same set $\mathbb{X}$ we write $u_1 \sim u_2$ and say that u_1 is equivalent to u_2 if there exist $a, b \in \mathbb{R}$ with $a > 0$ such that

$$u_1(x) = au_2(x) + b \quad \text{for all} \quad x \in \mathbb{X}.$$

The next two results follow immediately from Theorem 7.

Theorem 8 *Two utility functions are equivalent iff they have the same risk aversion.*

Theorem 9 *Exactly the following utility functions have constant risk aversion:*

(a) $u(x) \sim x$,
(b) $u(x) \sim -e_{-c}(x, x_0)$ *with* $c > 0$ *such that* $-c \in \mathcal{R}^+$.

We conclude this section with two examples, where we assume that $\mathbb{X}$ is a time scale with $\sup \mathbb{X} = \infty$.

Example 2 Unlike the logarithmic utility which is DARA for any time scale (as is easily verified), the property of being DARA of the following utility function depends on the involved time scale. Let $x_0 \in \mathbb{X}$ with $x_0 > 0$ and define

$$u(x) = \int_{x_0}^{x} \frac{\Delta\xi}{\xi^2} \quad \text{for} \quad x > 0.$$

Then

$$u^\Delta(x) = \frac{1}{x^2} > 0 \quad \text{and} \quad u^{\Delta\Delta}(x) = -\frac{x + \sigma(x)}{x^2(\sigma(x))^2} < 0 \quad \text{for} \quad x > 0$$

so that

$$r(x) = \frac{x + \sigma(x)}{(\sigma(x))^2} \quad \text{for} \quad x > 0.$$

Note that r is decreasing for some time scales (e.g., $r(x) = 2/x$ for $\mathbb{X} = [1, \infty)$, $r(x) = (2x+1)/(x+1)^2$ for $\mathbb{X} = \mathbb{N}_0$, $r(x) = (q+1)/(q^2 x)$ for $\mathbb{X} = q^{\mathbb{N}_0}$), but for some time scales, r is not decreasing. As an example, let $x_0 > 0$ and $a, b > 0$ and consider $\mathbb{X} = \{x_n : n \in \mathbb{N}_0\}$ with

$$x_{n+1} = \begin{cases} x_n + a & \text{if } n \text{ is even} \\ x_n + b & \text{if } n \text{ is odd}, \end{cases} \qquad n \in \mathbb{N}_0.$$

Hence

$$r(\sigma(x_n)) - r(x_n) = r(x_{n+1}) - r(x_n) = \frac{x_{n+1} + x_{n+2}}{x_{n+2}^2} - \frac{x_n + x_{n+1}}{x_{n+1}^2}.$$

Put $x = x_n$. If n is even, then $x_{n+1} = x + a$ and $x_{n+2} = x_{n+1} + b = x + a + b$ so that

$$\begin{aligned} r(\sigma(x)) - r(x) &= \frac{2x + 2a + b}{(x + a + b)^2} - \frac{2x + a}{(x + a)^2} \\ &= \frac{(a - 3b)x^2 + (2a^2 - 4ab - 2b^2)x + a^3 - ab^2 - a^2 b}{(x + a)^2 (x + a + b)^2}, \end{aligned}$$

which is eventually ($x \to \infty$) positive if e.g., $a = 4$ and $b = 1$. If n is odd, then $x_{n+1} = x + b$ and $x_{n+2} = x_{n+1} + a = x + a + b$ so that

$$\begin{aligned} r(\sigma(x)) - r(x) &= \frac{2x + 2b + a}{(x + a + b)^2} - \frac{2x + b}{(x + b)^2} \\ &= \frac{(b - 3a)x^2 + (2b^2 - 4ab - 2a^2)x + b^3 - ab^2 - a^2 b}{(x + b)^2 (x + a + b)^2}, \end{aligned}$$

which is eventually ($x \to \infty$) negative if $a = 4$ and $b = 1$.

Example 3 This example shows that the risk aversion of the cubic utility, which in the classical case is increasing, may be not increasing and/or not decreasing (according to the choice of the time scale). Let $x_0 \in \mathbb{X}$ and define

$$u(x) = \int_{x_0}^{x} \xi^2 \Delta\xi \quad \text{for} \quad \sigma(x) < 0.$$

Then

$$u^{\Delta}(x) = x^2 > 0 \quad \text{and} \quad u^{\Delta\Delta}(x) = x + \sigma(x) < 0 \quad \text{for} \quad \sigma(x) < 0$$

so that

$$r(x) = -\frac{x + \sigma(x)}{x^2} \quad \text{for} \quad \sigma(x) < 0.$$

Note that r is increasing for some time scales (e.g., $r(x) = -2/x$ for $\mathbb{X} = (-\infty, -1]$, $r(x) = -2/x - 1/x^2$ for $\mathbb{X} = -\mathbb{N} \setminus \{-1\}$, $r(x) = -q(q+1)/x$ for $\mathbb{X} = -q^{\mathbb{N}_0}$), but for some time scales, r is not increasing. As an example, let $x_0 < 0$ and $a, b > 0$ and consider $\mathbb{X} = \{x_n : n \in \mathbb{N}_0\}$ with

$$x_{n+1} = \begin{cases} x_n - a & \text{if } n \text{ is even} \\ x_n - b & \text{if } n \text{ is odd}, \end{cases} \qquad n \in \mathbb{N}_0.$$

As in Example 2 we can show that r alternately increases and decreases eventually ($x \to -\infty$) provided e.g., $a = 1$ and $b = 4$.

6 Risk Vulnerability

In this section we introduce risk vulnerability and characterize it. The following definition can be interpreted in that any unfair background risk makes risk-averse agents behave in a more risk-averse way. In this section we assume that X is a random variable with values in $\mathbb{X}^{\kappa\kappa}$ and that $x^* \in \mathbb{X}^{\kappa\kappa\kappa} := (X^{\kappa\kappa})^{\kappa}$.

Definition 5 The utility function u is called *risk vulnerable* if

$$\mathbb{E}(X) \le x^* \quad \text{always implies} \quad -\frac{\mathbb{E}(u^{\Delta\Delta}(X))}{\mathbb{E}(u^{\Delta}(X))} \ge -\frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}.$$

We illustrate Definition 5 with the following example.

Example 4 Let $\mathbb{X} = \{0, 1, 2, 3, 4, 5\}$ and define $u : \mathbb{X} \to \mathbb{R}$ by

$$u(0) = 0, \quad u(1) = 2, \quad u(2) = 4, \quad u(3) = 5, \quad u(4) = 6, \quad u(5) = 7.$$

Then

$$r(0) = 0, \quad r(1) = \frac{1}{2}, \quad r(2) = 0, \quad r(3) = 0.$$

Now let $x^* = 1$ and X be a random variable taking the values 0 and 2 with probability 1/2 each. Then

$$\mathbb{E}(X) = 1 = x^* \quad \text{but} \quad -\frac{\mathbb{E}(u^{\Delta\Delta}(X))}{\mathbb{E}(u^{\Delta}(X))} = 0 < \frac{1}{2} = r(1) = -\frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}.$$

Thus u is not risk vulnerable.

It is then interesting to supply some conditions assuring risk vulnerability. To this end, the main result is the following extension of [Gollier and Pratt(1996), Proposition 3].

Theorem 10 *If the utility function u satisfies*

$$r^{\Delta} \le 0 \quad \text{and} \quad \left[(1 - \mu r) r^{\Delta}\right]^{\Delta} \ge r r^{\Delta}, \tag{21}$$

then it is risk vulnerable.

In order to prove this theorem, we first present the following technical lemma.

Lemma 7 *We have*

$$1 - \mu r = \frac{u^{\Delta\sigma}}{u^{\Delta}}, \tag{22}$$

$$r^{\Delta} = -\frac{1}{u^{\Delta\sigma}} \left[u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}}\right], \tag{23}$$

$$\left[u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}}\right]^{\Delta} = -u^{\Delta\sigma} \left\{\left[(1 - \mu r) r^{\Delta}\right]^{\Delta} - r r^{\Delta}\right\}, \tag{24}$$

$$u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}} \quad \text{is decreasing iff} \quad \left[(1 - \mu r) r^{\Delta}\right]^{\Delta} \ge r r^{\Delta}. \tag{25}$$

Proof Using the simple useful formula (2), the calculation

$$1-\mu r = 1+\mu\frac{u^{\Delta\Delta}}{u^{\Delta}} = \frac{u^{\Delta}+\mu u^{\Delta\Delta}}{u^{\Delta}} = \frac{u^{\Delta\sigma}}{u^{\Delta}}$$

shows that (22) holds. Next, rewriting (18) establishes (23). To prove (24), we first use (23) and then (22) to write

$$u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}} = -u^{\Delta\sigma}r^{\Delta} = -u^{\Delta}(1-\mu r)r^{\Delta}.$$

Differentiating the above equation and using the product rule (4), we find

$$\begin{aligned}\left[u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}}\right]^{\Delta} &= -\left[u^{\Delta}(1-\mu r)r^{\Delta}\right]^{\Delta} \\ &= -\left\{u^{\Delta\sigma}\left[(1-\mu r)r^{\Delta}\right]^{\Delta} + u^{\Delta\Delta}(1-\mu r)r^{\Delta}\right\} \\ &= -\left\{u^{\Delta\sigma}\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - ru^{\Delta}(1-\mu r)r^{\Delta}\right\} \\ &= -\left\{u^{\Delta\sigma}\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - ru^{\Delta\sigma}r^{\Delta}\right\} \\ &= -u^{\Delta\sigma}\left\{\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - rr^{\Delta}\right\},\end{aligned}$$

where we have used again (22). Finally, (25) follows directly from (24). □

Proof (Proof of Theorem 10) By the assumption and (25),

$$u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}} \quad \text{is decreasing.}$$

Note also that we may assume $u^{\Delta\Delta}(x^*) \neq 0$. Then

$$\begin{aligned}&\frac{1}{u^{\Delta\Delta}(x^*)}\left[\left\{u^{\Delta\Delta\Delta}(x) - \frac{(u^{\Delta\Delta}(x))^2}{u^{\Delta}(x)}\right\} - \left\{u^{\Delta\Delta\Delta}(x^*) - \frac{(u^{\Delta\Delta}(x^*))^2}{u^{\Delta}(x^*)}\right\}\right] \\ &\qquad + \frac{u^{\Delta\Delta}(x)}{u^{\Delta\Delta}(x^*)}\left[\frac{u^{\Delta\Delta}(x)}{u^{\Delta}(x)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}\right] \\ &\qquad = \frac{u^{\Delta\Delta\Delta}(x)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x)}{u^{\Delta}(x^*)} - \left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}\right],\end{aligned}$$

and hence, recalling that $r^{\Delta} \leq 0$, the last expression is nonnegative if $x \geq x^*$ and nonpositive if $x \leq x^*$, so that we can conclude

$$\begin{aligned}0 &\leq \int_{x^*}^{x}\left\{\frac{u^{\Delta\Delta\Delta}(\xi)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(\xi)}{u^{\Delta}(x^*)} - \left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}\right]\right\}\Delta\xi \\ &= \frac{u^{\Delta\Delta}(\xi)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta}(\xi)}{u^{\Delta}(x^*)} - \xi\left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}\right]\Bigg|_{\xi=x^*}^{\xi=x} \\ &= \frac{u^{\Delta\Delta}(x)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta}(x)}{u^{\Delta}(x^*)} - (x-x^*)\left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}\right].\end{aligned}$$

Thus
$$\frac{u^{\Delta\Delta}(x)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta}(x)}{u^{\Delta}(x^*)} \geq (x - x^*) \left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)} \right]$$
holds for arbitrary $x \in \mathbb{X}^{\kappa\kappa}$. Therefore, if X is a random variable with values in $\mathbb{X}^{\kappa\kappa}$, we deduce
$$\frac{\mathbb{E}(u^{\Delta\Delta}(X))}{u^{\Delta\Delta}(x^*)} - \frac{\mathbb{E}(u^{\Delta}(X))}{u^{\Delta}(x^*)} \geq (\mathbb{E}(X) - x^*) \left[\frac{u^{\Delta\Delta\Delta}(x^*)}{u^{\Delta\Delta}(x^*)} - \frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)} \right]. \quad (26)$$
By Theorem 5, the second factor on the right-hand side of (26) is nonpositive. The assumption $\mathbb{E}(X) \leq x^*$ implies that the expression on the left-hand side of (26) is nonnegative. Hence
$$-\frac{\mathbb{E}(u^{\Delta\Delta}(X))}{\mathbb{E}(u^{\Delta}(X))} \geq -\frac{u^{\Delta\Delta}(x^*)}{u^{\Delta}(x^*)}.$$
This means that u is risk vulnerable. □

Next, we will analyze when condition (21) holds. To do so, we first introduce the following two coefficients.

Definition 6 We define the coefficients p of absolute *prudence* and t of absolute *temperance* for a utility function u by
$$p = -\frac{u^{\Delta\Delta\Delta}}{u^{\Delta\Delta}} \quad \text{and} \quad t = -\frac{u^{\Delta\Delta\Delta\Delta}}{u^{\Delta\Delta\Delta}}.$$

Some relations of p, t, and r are given next.

Lemma 8 *We have*
$$p = r^{\sigma} - \frac{r^{\Delta}}{r} = r - \frac{(1-\mu r)r^{\Delta}}{r}, \quad (27)$$
$$t = \frac{1-\mu r}{pr} \left\{ \left[(1-\mu r)r^{\Delta}\right]^{\Delta} - r r^{\Delta} \right\} + r + r^{\sigma} - \frac{r r^{\sigma}}{p}, \quad (28)$$
$$\frac{1-\mu r}{pr} \left[(1-\mu r)r^{\Delta}\right]^{\Delta} = t - \left[2r + r^{\sigma} - \frac{r(r+r^{\sigma})}{p} \right], \quad (29)$$
$$\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - (r + r^{\sigma})r^{\Delta} = \frac{pr}{1-\mu r}(t - r). \quad (30)$$

Proof We use the quotient rule (5) and (22) from Lemma 7:
$$\begin{aligned} r - p = -\frac{u^{\Delta\Delta}}{u^{\Delta}} + \frac{u^{\Delta\Delta\Delta}}{u^{\Delta\Delta}} &= \frac{u^{\Delta\Delta\Delta}u^{\Delta} - (u^{\Delta\Delta})^2}{u^{\Delta}u^{\Delta\sigma}} \frac{u^{\Delta\sigma}}{u^{\Delta\Delta}} \\ = \left(\frac{u^{\Delta\Delta}}{u^{\Delta}}\right)^{\Delta} \frac{u^{\Delta}(1-\mu r)}{u^{\Delta\Delta}} &= \frac{r^{\Delta}(1-\mu r)}{r}. \end{aligned}$$
Moreover, using (2),
$$r^{\sigma} - \frac{r^{\Delta}}{r} = r + \mu r^{\Delta} - \frac{r^{\Delta}}{r} = r - \frac{(1-\mu r)r^{\Delta}}{r}.$$

This proves (27). To show (28), we use the quotient rule (5) and (24) from Lemma 7:

$$
\begin{aligned}
&\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - rr^{\Delta} = -\frac{1}{u^{\Delta\sigma}}\left[u^{\Delta\Delta\Delta} - \frac{(u^{\Delta\Delta})^2}{u^{\Delta}}\right]^{\Delta} \\
&= -\frac{1}{u^{\Delta\sigma}}\left[u^{\Delta\Delta\Delta\Delta} - \frac{u^{\Delta\Delta\Delta}(u^{\Delta\Delta}+u^{\Delta\Delta\sigma})u^{\Delta} - (u^{\Delta\Delta})^3}{u^{\Delta}u^{\Delta\sigma}}\right] \\
&= \frac{u^{\Delta\Delta\Delta}}{u^{\Delta\sigma}}\left[-\frac{u^{\Delta\Delta\Delta\Delta}}{u^{\Delta\Delta\Delta}} + \frac{u^{\Delta\Delta}+u^{\Delta\Delta\sigma}}{u^{\Delta\sigma}} - \frac{(u^{\Delta\Delta})^3}{u^{\Delta}u^{\Delta\sigma}u^{\Delta\Delta\Delta}}\right] \\
&= \frac{u^{\Delta\Delta\Delta}}{u^{\Delta\Delta}}\frac{u^{\Delta\Delta}}{u^{\Delta}}\frac{u^{\Delta}}{u^{\Delta\sigma}}\left[t + \frac{u^{\Delta\Delta}}{u^{\Delta}}\frac{u^{\Delta}}{u^{\Delta\sigma}} - r^{\sigma} - \frac{\left(\frac{u^{\Delta\Delta}}{u^{\Delta}}\right)^2 \frac{u^{\Delta}}{u^{\Delta\sigma}}}{\frac{u^{\Delta\Delta\Delta}}{u^{\Delta\Delta}}}\right] \\
&= \frac{pr}{1-\mu r}\left[t - \frac{r}{1-\mu r} - r^{\sigma} + \frac{r^2}{(1-\mu r)p}\right],
\end{aligned}
$$

where we have used in the last step three times (22). This together with

$$
\frac{1}{1-\mu r} - \frac{r}{(1-\mu r)p} = 1 - \frac{r^{\sigma}}{p} \tag{31}
$$

proves (28). It remains to show (31). By the second and first part of (27), we have

$$
\begin{aligned}
\frac{1}{1-\mu r} - \frac{r}{(1-\mu r)p} &= \frac{p-r}{(1-\mu r)p} = \frac{-\frac{(1-\mu r)r^{\Delta}}{r}}{(1-\mu r)p} \\
&= -\frac{r^{\Delta}}{rp} = \frac{p-r^{\sigma}}{p} = 1 - \frac{r^{\sigma}}{p}.
\end{aligned}
$$

This establishes (31), and hence (28). Next, by (27) and (28), we have

$$
\begin{aligned}
\frac{1-\mu r}{pr}\left[(1-\mu r)r^{\Delta}\right]^{\Delta} &= t - \left[r + r^{\sigma} - \frac{rr^{\sigma}}{p}\right] + \frac{(1-\mu r)r^{\Delta}}{p} \\
&= t - \left[r + r^{\sigma} - \frac{rr^{\sigma}}{p}\right] - \frac{(p-r)r}{p} \\
&= t - \left[2r + r^{\sigma} - \frac{r(r+r^{\sigma})}{p}\right].
\end{aligned}
$$

This establishes (29). Finally, by (28) and the second part of (27), we have

$$
\begin{aligned}
&\left[(1-\mu r)r^{\Delta}\right]^{\Delta} - rr^{\Delta} - r^{\sigma}r^{\Delta} = \frac{pr}{1-\mu r}\left[t - r - r^{\sigma} + \frac{rr^{\sigma}}{p}\right] - r^{\sigma}r^{\Delta} \\
&= \frac{pr}{1-\mu r}(t-r) - \frac{prr^{\sigma}}{1-\mu r} + \frac{r^2r^{\sigma}}{1-\mu r} - r^{\sigma}\left[\frac{r(r-p)}{1-\mu r}\right] \\
&= \frac{pr}{1-\mu r}(t-r).
\end{aligned}
$$

This establishes (30). □

The following three theorems are direct consequences of Lemma 8 and Theorem 6.

Theorem 11 *A utility function is DARA iff its coefficient of absolute prudence exceeds its risk aversion.*

Proof Since
$$r^{\Delta} \leq 0 \quad \text{iff} \quad p \geq r$$
holds by (27), the statement follows. □

Theorem 12 *A utility function is risk vulnerable if it is DARA and*
$$t \geq r + r^{\sigma} - \frac{rr^{\sigma}}{p}$$
or
$$t \geq 2r + r^{\sigma} - \frac{r(r + r^{\sigma})}{p}$$
holds.

Proof Since $r^{\Delta} \leq 0$, by (28), we have
$$\left[(1-\mu r)r^{\Delta}\right]^{\Delta} \geq rr^{\Delta} \quad \text{iff} \quad t \geq r + r^{\sigma} - \frac{rr^{\sigma}}{p},$$
so that the first statement follows from Theorem 10. Next, since $r^{\Delta} \leq 0$, by Theorem 11 and (29), we have
$$\left[(1-\mu r)r^{\Delta}\right]^{\Delta} \geq 0 \quad \text{iff} \quad t \geq 2r + r^{\sigma} - \frac{r(r + r^{\sigma})}{p},$$
so that, on noting that $rr^{\Delta} \leq 0$, the second statement follows from Theorem 10, too. □

Theorem 13 *A utility function is DARA and satisfies*
$$\left[(1-\mu r)r^{\Delta}\right]^{\Delta} \geq (r + r^{\sigma})r^{\Delta}$$
iff its coefficients of absolute prudence and temperance both exceed its risk aversion.

Proof Since
$$r^{\Delta} \leq 0, \ \left[(1-\mu r)r^{\Delta}\right]^{\Delta} \geq (r + r^{\sigma})r^{\Delta} \quad \text{iff} \quad p \geq r, \ t \geq r$$
holds by Theorem 11 and (30), the proof is complete. □

Remark 1 In order to obtain the results in [Gollier and Pratt(1996)] from the statements given in this section, one has to take $\mathbb{X} = \mathbb{R}$. Indeed, the conditions considered in Theorems 10, 12, and 13 become the same as in [Gollier and Pratt(1996), Proposition 3, Corollary 1, and Formula (13)].

A An Auxiliary Function

For a given function $f : \mathbb{X} \to \mathbb{R}$, we define an auxiliary function $\bar{f} : [\inf \mathbb{X}, \sup \mathbb{X}] \to \mathbb{R}$ (with the corresponding endpoints excluded if $\inf \mathbb{X} = -\infty$ or $\sup \mathbb{X} = \infty$) by

$$\bar{f}(t) = f(\lfloor x \rfloor) + (t - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) \quad \text{for} \quad \lfloor x \rfloor \le t \le \sigma(\lfloor x \rfloor) \quad \text{if} \quad \lfloor x \rfloor := \lfloor x \rfloor_{\mathbb{X}} \in \mathbb{X}^{\kappa}$$

and $\bar{f}(x) = f(x)$ if $x \in \mathbb{X} \setminus \mathbb{X}^{\kappa}$.

Lemma 9 *Suppose $f : \mathbb{X} \to \mathbb{R}$ satisfies $f^{\Delta} \ge 0$ on $\mathbb{X}^{\kappa}$. Then $\bar{f}$ is increasing.*

Proof Let $\inf \mathbb{X} \le x < y \le \sup \mathbb{X}$ (with the corresponding endpoints excluded if $\inf \mathbb{X} = -\infty$ or $\sup \mathbb{X} = \infty$). Then by (1) and (2), if $\lfloor y \rfloor \in \mathbb{X}^{\kappa}$,

$$\begin{aligned} \bar{f}(y) - \bar{f}(x) &= f(\lfloor y \rfloor) + (y - \lfloor y \rfloor) f^{\Delta}(\lfloor y \rfloor) - f(\lfloor x \rfloor) - (x - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) \\ &= \int_{\lfloor x \rfloor}^{\lfloor y \rfloor} f^{\Delta}(t) \Delta t + (y - \lfloor y \rfloor) f^{\Delta}(\lfloor y \rfloor) - (x - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) \\ &= \int_{\sigma(\lfloor x \rfloor)}^{\lfloor y \rfloor} f^{\Delta}(t) \Delta t + \mu(\lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) + (y - \lfloor y \rfloor) f^{\Delta}(\lfloor y \rfloor) - (x - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) \\ &= \int_{\sigma(\lfloor x \rfloor)}^{\lfloor y \rfloor} f^{\Delta}(t) \Delta t + (\sigma(\lfloor x \rfloor) - x) f^{\Delta}(\lfloor x \rfloor) + (y - \lfloor y \rfloor) f^{\Delta}(\lfloor y \rfloor) \\ &\ge 0 \end{aligned}$$

since $f^{\Delta} \ge 0$, and, similarly, if $y \in \mathbb{X} \setminus \mathbb{X}^{\kappa}$,

$$\bar{f}(y) - \bar{f}(x) = f(y) - f(\lfloor x \rfloor) - (x - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) = \int_{\sigma(\lfloor x \rfloor)}^{y} f^{\Delta}(t) \Delta t + (\sigma(\lfloor x \rfloor) - x) f^{\Delta}(\lfloor x \rfloor) \ge 0.$$

Thus $\bar{f}$ is increasing. □

Lemma 10 *Suppose $f : \mathbb{X} \to \mathbb{R}$ satisfies $f^{\Delta\Delta} \ge 0$ on $\mathbb{X}^{\kappa\kappa}$. Then $\bar{f}$ is convex.*

Proof Let $x, y \in \mathbb{X}^{\kappa}$ with $x < y$. Then

$$f^{\Delta}(y) - f^{\Delta}(x) = \int_{x}^{y} f^{\Delta\Delta}(t) \Delta t \ge 0$$

so that $f^{\Delta} : \mathbb{X}^{\kappa} \to \mathbb{R}$ is increasing. Now note that for any x such that $\lfloor x \rfloor \in \mathbb{X}^{\kappa}$ we have

$$(x - \lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) = \bar{f}(x) - f(\lfloor x \rfloor),$$

and, using the simple useful formula (2),

$$\begin{aligned} (\sigma(\lfloor x \rfloor) - x) f^{\Delta}(\lfloor x \rfloor) &= (\sigma(\lfloor x \rfloor) - \lfloor x \rfloor + \lfloor x \rfloor - x) f^{\Delta}(\lfloor x \rfloor) \\ &= \mu(\lfloor x \rfloor) f^{\Delta}(\lfloor x \rfloor) + (\lfloor x \rfloor - x) f^{\Delta}(\lfloor x \rfloor) \\ &= f(\sigma(\lfloor x \rfloor)) - f(\lfloor x \rfloor) + (\lfloor x \rfloor - x) f^{\Delta}(\lfloor x \rfloor) \\ &= f(\sigma(\lfloor x \rfloor)) - \bar{f}(x). \end{aligned}$$

We let $\inf \mathbb{X} \le x < y \le \sup \mathbb{X}$ (with the corresponding endpoints excluded if $\inf \mathbb{X} = -\infty$ or $\sup \mathbb{X} = \infty$), $0 < \lambda < 1$, and define $z = \lambda x + (1 - \lambda) y$. Then we use the properties of the integral (1) to find, if $\lfloor y\rfloor \in \mathbb{X}^\kappa$ (and, as before in the proof of Lemma 9, similarly, if $y \in \mathbb{X}\setminus\mathbb{X}^\kappa$)

$$\begin{aligned}
&\lambda\bar f(x) + (1-\lambda)\bar f(y) - \bar f(z) = -\lambda(\bar f(z) - \bar f(x)) + (1-\lambda)(\bar f(y) - \bar f(z))\\
&= -\lambda\left(\bar f(z) - f(\lfloor z\rfloor) + f(\sigma(\lfloor x\rfloor)) - \bar f(x) + \int_{\sigma(\lfloor x\rfloor)}^{\lfloor z\rfloor} f^{\Delta}(t)\Delta t\right)\\
&\quad +(1-\lambda)\left(\bar f(y) - f(\lfloor y\rfloor) + f(\lfloor z\rfloor) - \bar f(z) + \int_{\lfloor z\rfloor}^{\lfloor y\rfloor} f^{\Delta}(t)\Delta t\right)\\
&= -\lambda\left((z-\lfloor z\rfloor)f^{\Delta}(\lfloor z\rfloor) + (\sigma(\lfloor x\rfloor) - x)f^{\Delta}(\lfloor x\rfloor) + \int_{\sigma(\lfloor x\rfloor)}^{\lfloor z\rfloor} f^{\Delta}(t)\Delta t\right)\\
&\quad +(1-\lambda)\left((y-\lfloor y\rfloor)f^{\Delta}(\lfloor y\rfloor) - (z-\lfloor z\rfloor)f^{\Delta}(\lfloor z\rfloor) + \int_{\lfloor z\rfloor}^{\lfloor y\rfloor} f^{\Delta}(t)\Delta t\right)\\
&= -(z-\lfloor z\rfloor)f^{\Delta}(\lfloor z\rfloor) - \lambda(\sigma(\lfloor x\rfloor) - x)f^{\Delta}(\lfloor x\rfloor) + (1-\lambda)(y-\lfloor y\rfloor)f^{\Delta}(\lfloor y\rfloor)\\
&\quad +(1-\lambda)\int_{\lfloor z\rfloor}^{\lfloor y\rfloor} f^{\Delta}(t)\Delta t - \lambda\int_{\sigma(\lfloor x\rfloor)}^{\lfloor z\rfloor} f^{\Delta}(t)\Delta t\\
&\geq -(z-\lfloor z\rfloor)f^{\Delta}(\lfloor z\rfloor) - \lambda(\sigma(\lfloor x\rfloor) - x)f^{\Delta}(\lfloor x\rfloor) + (1-\lambda)(y-\lfloor y\rfloor)f^{\Delta}(\lfloor y\rfloor)\\
&\quad +(1-\lambda)(\lfloor y\rfloor - \lfloor z\rfloor)f^{\Delta}(\lfloor z\rfloor) - \lambda(\lfloor z\rfloor - \sigma(\lfloor x\rfloor))f^{\Delta}(\lfloor z\rfloor)\\
&= (1-\lambda)(y-\lfloor y\rfloor)\left[f^{\Delta}(\lfloor y\rfloor) - f^{\Delta}(\lfloor z\rfloor)\right] + \lambda(\sigma(\lfloor x\rfloor) - x)\left[f^{\Delta}(\lfloor z\rfloor) - f^{\Delta}(\lfloor x\rfloor)\right] \geq 0
\end{aligned}$$

since $x \leq z \leq y$ and f^{Δ} is increasing. Thus $\bar f$ is convex. □

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