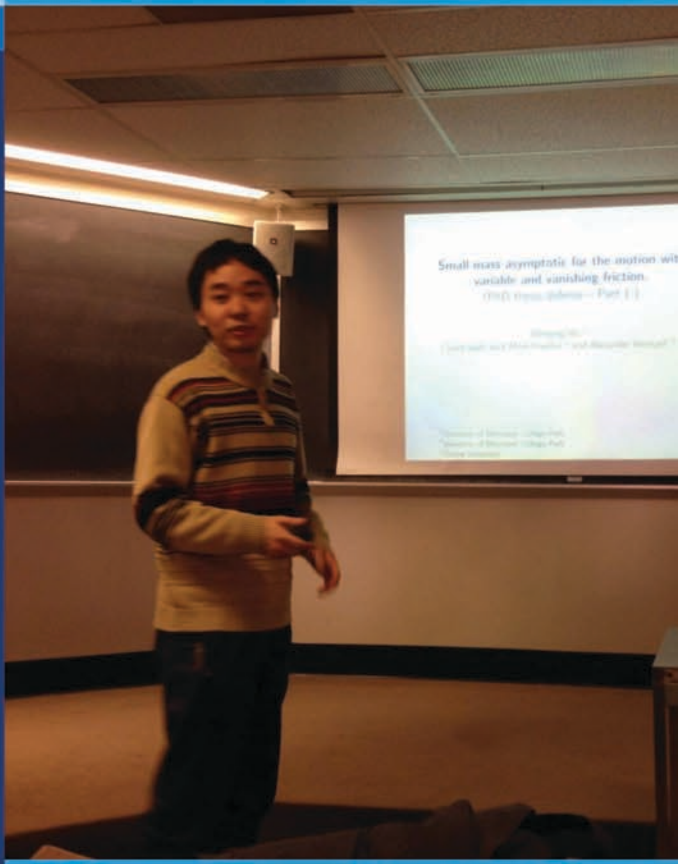
北京大學

PEKING UNIVERSITY

School of Mathematical Sciences

Center for Statistical Science

Lectures on Stochastic Fluid Mechanics



Dr. Wenqing Hu will give a short course about stochastic fluid mechanics and the following topics will be covered: basic existence and uniqueness results for deterministic and stochastic 2–d and 3–d Navier–Stokes equations; existence and uniqueness of invariant measures for 2–d hydrodynamical systems subject to degenerate random forcing; inviscid limit and related problems in turbulence; the 2–d deterministic and stochastic Euler equations and related problems; motion of incompressible ideal fluids from group theoretic and Hamiltonian dynamical point of view.

Dr. Hu currently is a postdoc at the School of Mathematics, University of Minnesota, Twin Cities, working under the supervision of Professor Vladimir Sverak. Dr. Hu completed a Ph.D. in Mathematics at the Department of mathematics, University of Maryland, College Park, under the supervision of Professor Mark Freidlin.

Schedule:

Lecture 1:	**7.9**	**(Thu)**	**8:30-10:30 a.m.**
Lecture 2:	**7.10**	**(Fri)**	**8:30-10:30 a.m.**
Lecture 3:	**7.13**	**(Mon)**	**8:30-10:30 a.m.**
Lecture 4:	**7.15**	**(Wed)**	**8:30-10:30 a.m.**
Lecture 5:	**7.17**	**(Fri)**	**8:30-11:30 a.m.**

Venue:

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Lecture Abstracts.
Stochastic Fluid Mechanics.

Wenqing Hu*

Peking University, July 2015.

Lecture 1.
Basic existence and uniqueness results for deterministic and stochastic 2–d and 3–d Navier–Stokes equations.

We first discuss some background knowledge as well as recent progresses in 2–d and 3–d Navier–Stokes equations. We then show basic existence and uniqueness results for deterministic and stochastic 2–d Navier–Stokes equations.

References

[1] Kuksin, S., Shirikyan, A., *Mathematics of 2–dimensional turbulence*. Cambridge Tracts in Mathematics, **194**, November 2012.

[2] Šverák, V., Lecture notes of *Selected Topics in Fluid Mechanics*, 2011–2012.

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[6] Mikulevicius, R., Rozovskii, B.L., Stochastic Navier–Stokes equations for turbulent flows. *SIAM Journal of Mathematical Analysis*, **35**(5), pp.1250–1310, 2004.

[7] Cafferelli, L., Kohn, R., Nirenberg, J., Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Communications in Pure and Applied Mathematics*, **35** (1982), no. 6, pp. 771–831.

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[9] Jia, H., Šverák, V., Are the incompressible 3d Navier–Stokes equations locally ill–posed in the natural energy space? arXiv:1306.2136

Lecture 2.
Existence and uniqueness of invariant measures for 2–d hydrodynamical systems subject to degenerate random forcing.

We present a model problem in finite dimensions showing the existence and uniqueness of invariant measure for general white–noise driven stochastic systems with hydrodynamical background. We will also mention the result as well as difficulties for infinite dimensional systems corresponding to general semilinear evolutionary SPDEs. Application of the theoretical method includes hydrodynamical systems such as the 2–d Navier–Stokes system, the 2–d Boussinesq system, and a 2–d Euler system subject to fractional dissipation.

References

[1] Hairer, M., Mattingly, J.C., Ergodicity of the 2-d Navier–Stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, (2), **164**(3), pp. 567–600, 2009.

[2] Hairer, M., Mattingly, J.C., A theory of hypoellipticity and unique ergodicity for semilinear parabolic SPDEs. *Electronic Journal of Probability*, **16** (2011), pp. 658–738.

[3] Kuksin, S., Shirikyan, A., *Mathematics of 2–dimensional turbulence*. Cambridge Tracts in Mathematics, **194**, November 2012.

[4] Constantin, P., Glatt–Holtz, N., Vicol, V., Unique ergodicity for fractionally dissipated, stochasticaly forced 2–d Euler equations. arXiv:1304.2022[MATH.AP]

[5] Földes, J., Glatt–Holtz, N., Richards, G., Thomann, E., Ergodic and mixing properties of the Boussinesq equations with a degenerate random forcing. arXiv:1311.3620v1

[6] Glatt–Holtz, N., Notes on statistically invariant states in stochastically driven fluid flows. available at

http://www.math.vt.edu/people/negh/research/index.html

Lecture 3.

Inviscid limit and related problems in turbulence.

Asymptotic problems of sending the viscosity $\nu \to 0$ will be discussed in this lecture. In particular, we discuss the application of de Giorgi–Nash–Moser iteration method to stochastic Navier–Stokes equations. We show the corresponding invariant mesure is concentrated in L_∞ in vorticity space. The relationship between inviscid limit and turbulence will be explained.

References

[1] Glatt–Holtz, N., Šverák, V., Vicol, V., On inviscid limits for the stochastic Navier-Stokes equation. `arXiv:1302.0542[MATH.AP]`

[2] Kuksin, S., Shirikyan, A., *Mathematics of 2–dimensional turbulence.* Cambridge Tracts in Mathematics, **194**, November 2012.

[3] Arnold, V.I., Meshchalkin, L.D., Kolmogorov seminar on selected questions of analysis 1958–1959, *Russian Mathematical Surverys*, **15**, 1, pp. 247–250. (1960)

Lecture 4.

The 2–d deterministic and stochastic Euler equations and related problems.

The L_∞–well posedness (in vorticity formulation) of 2–d deterministic Euler equations will be discussed. Recent results in singularity formation in the sense of double exponential growth of the gradient of the vorticity at the boundary will be mentioned. The area–preserving scheme of adding the stochastic noise to the equation will be considered. Well–posedness results in the case of stochastic equation will also be discussed.

References

[1] Bertozzi, A., Majda, A., Vorticity and incompressible flow. *Cambridge Texts in Applied Mathematics*, **27**, Cambridge University, 2002.

[2] Kiselev, A., Šverák, V., Small scale creation for solutions of the incompressible 2–dimensional Euler equation. *Annals of Mathematics* **180** (2014), pp. 1205–1220.

[3] Brzeźniak, Z., Flandoli, F., Maurelli, M., Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. `arXiv:1401.5938`

Lecture 5.

Motion of incompressible ideal fluids from group theoretic and Hamiltonian dynamical point of view.

Classical mechanics from group theoretic point of view will be discussed, including the motion of incompressible ideal fluids. The Hamiltonian formulation of Euler equations will be presented. If time permits, we will also discuss finite dimensional model problems and some open questions related to 2–d turbulence.

References

[1] Arnold, V.I., *Mathematical methods of classical mechanics*, Springer, 1978.

[2] Arnold, V.I., Khesin, B., *Topological methods in hydrodynamics*, Springer, 1998.

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[4] Šverák, V., Lecture notes of *Selected Topics in Fluid Mechanics*, 2011–2012.

Lecture 1 On existence and uniqueness of 2-d deterministic & Stochastic Navier–Stokes equations and related topics

General Remarks

Navier - Stokes equations

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p &= \nu \Delta u \\ \operatorname{div} u &= 0 \\ u(x,0) &= u_0(x) \end{aligned}\right\} \text{ on } D \times [0,\infty), \quad D \subset \mathbb{R}^d \qquad (1)$$

Existence of weak solution: Leray - Hopf weak solution

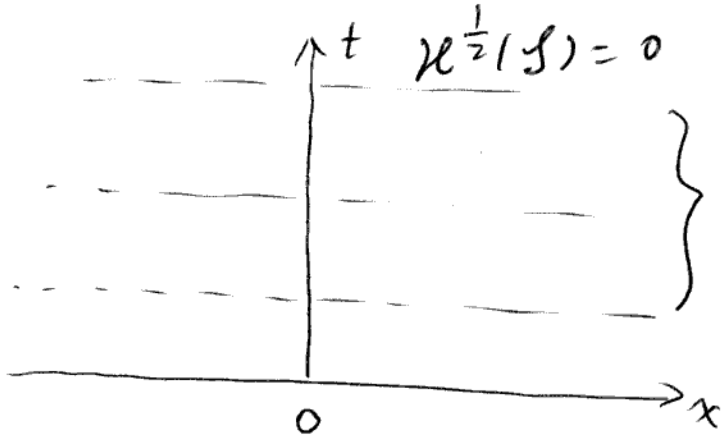
$d=2$ Uniqueness of Leray-Hopf weak solution (Ladyzhenskaya)

$d=3$ Uniqueness is open

Many blow-up criterions: Ladyzheskaya, Serrin, Seregin, Prodi, Kato ...

Regularity of Leray - Hopf weak solutions

Leray's argument already predicts



$\mathcal{H}^{\frac{1}{2}}(\mathcal{S}) = 0$

singularity set $\mathcal{S}$

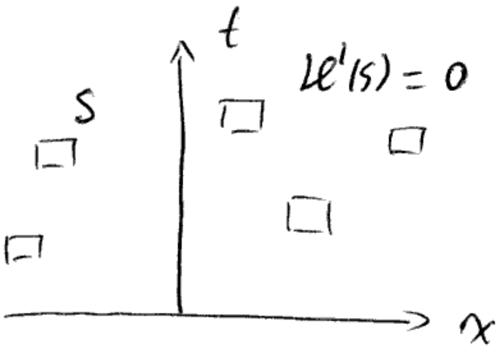
$\mathcal{H}^{d}$ — Hausdorff measure in dimension d

Leray's argument $\rightsquigarrow$ $\mathcal{H}^{\frac{1}{2}}(\mathcal{S}) = 0$

"localized Leray's argument" $\longrightarrow$ partial regularity

"only available estimate is energy estimate"

Suitable weak solutions = weak solutions that satisfy local energy inequality



$\mathcal{H}^{1}(S) = 0$

"partial regularity"

Cafferelli - Kohn - Nirenberg

Ladyzhenskaya - Seregin

Fanghua Lin's paper

weak - strong uniquess theorem : If weak solution exist and strong solution exist then they agree.

search for regularity

Mild solution : (Kato) = local strong solution

perturbation theory $(u \cdot \nabla) u$ is only a perturbation

"critical" "subcritical" "supercritical equations"

depends on the choice of function space

Koch-Tataru : Small BMO^{-1} norm initial data

$\rightsquigarrow$ unique small global solution

We will focus on 2-d hydrodynamics in these lectures

existence + uniqueness is available

more concentration is the dynamical and stochastic issue

"turbulence".

Our ambient space in which the fluids is moving is $\mathbb{T}^2$ most of the discussion

No boundary issue. $\nu = 1$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p - \Delta u = 0 & \\ \operatorname{div} u = 0 & x \in \mathbb{T}^2 \\ & t \geq 0. \\ u(x,0) = u_0(x) & \end{cases} \qquad (2)$$

Definition 1 (Leray-Hopf weak solution)

Let $V = \left\{ \varphi : \mathbb{T}^2 \to \mathbb{R}^2,\ \operatorname{div}\varphi = 0,\ \int_{\mathbb{T}^2} |\nabla\varphi|^2 dx < \infty \right\}$

We say that the function $u(x,t) = \sum_{k \in \mathbb{Z}^2} u_k(x)\, \psi_k(x)$

$\in L_2(\mathbb{T}^2; \mathbb{R}^2)$

is a (Leray-Hopf) weak solution to (2) on $[0,T]$ if for any $\varphi \in V$ we have

$$\int_{\mathbb{T}^2} (u_t \cdot \varphi + (u\cdot\nabla)u\cdot\varphi)\, dx + \int_{\mathbb{T}^2} \nabla u \cdot \nabla\varphi\, dx = 0 \qquad (4)$$

$$t \in [0,T]$$

"very weak solution" = moving all derivatives to φ

Therem 1 There exists a unique weak solution $u(x,t)$ in the space $L_\infty([0,T]; L_2(\mathbb{T}^2; \mathbb{R}^2))$ $\cap L_2([0,T]; H^1(\mathbb{T}^2; \mathbb{R}^2)) \cap L_4(\mathbb{T}^2 \times [0,T]; \mathbb{R}^2)$

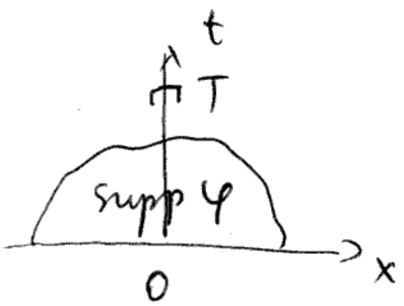
which is continuous as a function of t into the space $(L_2(\mathbb{T}^2; \mathbb{R}^2)$; strong-topology) s.t. for any $\varphi \in C^{(1)}_{div,0}$ and any $0 \le t_1 \le t_2 \le T$ we have

$$\int_{\mathbb{T}^2} u \cdot \varphi \, dx \Big|_{t=t_2}^{t=t_1} + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} [-u\varphi_t - u_k u_l \varphi_{l,k} + \nabla u \cdot \nabla \varphi] dx = 0 \qquad (5)$$

Moreover the function u satisfies an energy identity

$$\frac{1}{2} \int_{\mathbb{T}^2} |u|^2(x, t_1)\, dx = \frac{1}{2} \int_{\mathbb{T}^2} |u|^2(x, t_2)\, dx + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u|^2 dx \qquad (6)$$

Space $C^1_{div,0} = \{ \varphi(x,t)$; once continuously differentiable in t, compactly supported in $\mathbb{T}^2 \times [0,T)$, $\operatorname{div} \varphi = 0$, $\int_{\mathbb{T}^2} |\nabla \varphi|^2 dx < \infty$ for each $t \}$



Proof Main idea

① Galerkin approximation
② Compactness argument
③ Uniqueness

} Let us perform a very careful proof!

First we write the weak formulation (4) out

$$\int_{\mathbb{T}^2} \left(u_t \cdot \varphi + (u\cdot\nabla)u\cdot\varphi + \nabla u \cdot \nabla \varphi \right) dx = 0 \qquad (4)$$

Let ξ_k, $k = 1, 2, \ldots, m$ be an orthonormal basis for a finite m-dimensional subspace $V^m \subset V$.

Suppose $\xi_k(x)$ is orthogonal normal system. Then

$$u^{(m)}(x,t) = \sum_{k=1}^{m} \gamma^{(k)}(t)\, \xi_k(x), \quad \text{such that}$$

$$\dot{\gamma}^{(l)}(t) + \sum_{k=1}^{m} \gamma^{(k)}(t) \left(\sum_{j=1}^{m} c_{jkl}\, \gamma^{(j)}(t) + b_{kl} \right) = 0$$

$$l = 1, 2, \ldots m \qquad (7)$$

where $b_{kl} = \int_{\mathbb{T}^2} \nabla\xi_k \cdot \nabla \xi_l(x)\, dx$

$$c_{jkl} = \int_{\mathbb{T}^2} (\xi_k(x)\cdot\nabla)\,\xi_j(x)\cdot\xi_l(x)\,dx$$

(In particular $b_{kl} = b_{lk}$ and $c_{jkl} = -c_{lkj}$)

Initial condition

$$\int_{\mathbb{T}^2} u^{(m)}(x,0)\varphi(x)\,dx = \int_{\mathbb{T}^2} u_0(x)\varphi(x)\,dx, \quad \forall\varphi\in V^m \qquad (8)$$

This gives existence and uniqueness of a solution $u^{(m)}(x,t)$ to $\langle 7\rangle$.

How to pass to the limit? Contribution of Leray.

Let $\varphi = u^{(m)}(x,t)$ in (4)

$$\frac{1}{2}\int_{\mathbb{T}^2} |u^{(m)}|^2(x,t_2)\,dx = \frac{1}{2}\int_{\mathbb{T}^2} |u^{(m)}|^2(x,t_1)\,dx + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u^{(m)}|^2(x,t)\,dx \qquad (9)$$

Now we write (4) in the following form:

for $0 \le t_1 \le t_2 \le T$ we have

$$\int_{\mathbb{T}^2} u \cdot \varphi \, dx \Big|_{t=t_2}^{t=t_1} + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} \left(-u \cdot \varphi_t + (u \cdot \nabla) u \cdot \varphi + \nabla u \cdot \nabla \varphi \right) dx = 0 \tag{10}$$

in which ~~for each t fixed~~ we assume the test function $\varphi(x,t)$ to be compactly supported in $\mathbb{T}^2 \times [0,T)$, smooth in t, and as a function of x belongs to the space V.

Integration by parts in (10)

$$\int_{\mathbb{T}^2} u \cdot \varphi \, dx \Big|_{t=t_2}^{t=t_1} + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} \left(-u \cdot \varphi_t - u_j u_i \varphi_{i,j} + \nabla u \cdot \nabla \varphi \right) dx = 0 \tag{11}$$

By (9) we know that $u^{(m)}$ is uniformly bounded in $L_\infty([0,T]; L_2(\mathbb{T}^2; \mathbb{R}^2)) \cap L_2([0,T]; H^1(\mathbb{T}^2; \mathbb{R}^2))$

we see that there exist a subsequence $u^{(m')}$ (which we assume it is just $u^{(m)}$), such that $u^{(m)}$ ($u^{(m')}$) converges weakly - * in $L_\infty([0,T]; L_2(\mathbb{T}^2;\mathbb{R}^2))$ and weakly in $L_2([0,T]; H^1(\mathbb{T}^2;\mathbb{R}^2))$ to some $u \in L_\infty([0,T]; L_2(\mathbb{T}^2;\mathbb{R}^2)) \cap L_2([0,T]; H^1(\mathbb{T}^2;\mathbb{R}^2))$ and the function u is defined in $L_2(\mathbb{T}^2;\mathbb{R}^2) \cap H^1(\mathbb{T}^2;\mathbb{R}^2)$ for <u>almost every</u> $t \in [0,T]$.

[redacted]

First, for any $\varphi \in V^m$, the function

$$t \longrightarrow \int_{\mathbb{T}^2} u(x,t)\,\varphi(x)\,dx$$

is uniformly continuous in t. To see this, One can set

$$I(t_1,t_2) = I(t_1,t_2;u,\varphi) = \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} [-u\varphi_t - u_j u_i \varphi_{i,j} + \nabla u \cdot \nabla \varphi]\,dx$$

Suppose $0 \le t_1 \le t_2 \le T$. Then

$$I(t_1, t_2) + I(t_2, t_3) = I(t_1, t_3)$$

and hence it is enough to look at $I(t_1, t_2)$ as $(t_2 - t_1) \downarrow 0$. Suppose

$$|\varphi_t| \le A, \quad |\nabla \varphi| \le B, \quad |\varphi| \le C$$

Then

$$\int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |u \varphi_t| \, dx \le A (t_2 - t_1) [\mathrm{Vol}(\mathbb{T}^2)]^{\frac{1}{2}} \sup_{t_1 \le t \le t_2} \left(\int_{\mathbb{T}^2} |u|^2 dx \right)^{\frac{1}{2}}$$

$$\int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |u_j u_i \varphi_{i,j}| \, dx \le B (t_2 - t_1) \sup_{t_1 \le t \le t_2} \int_{\mathbb{T}^2} |u|^2 dx$$

$$\int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} \nabla u \cdot \nabla \varphi \, dx \le B \sqrt{t_2 - t_1} \, (\mathrm{Vol}(\mathbb{T}^2)) \left(\int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}}$$

From here we check that the function $t_1 \to I(t_1, t_2)$ is uniformly continuous in t_1 independent of t_2.

Next let $h^{(m)}(t) = \int_{\mathbb{T}^2} u^{(m)}(x,t)\,\varphi(x)\,dx$

and $h(t) = \int_{\mathbb{T}^2} u(x,t)\,\varphi(x)\,dx$

We note that while the functions $h^{(m)}(t)$ are defined for any t; the function h is at the moment defined only for almost every t. Also the functions $h^{(m)}$ and h are uniformly bounded as $u^{(m)}$ are uniformly bounded in $L_\infty([0,T]; L_2(\mathbb{T}^2; \mathbb{R}^2))$.

For each smooth test function η we know

$$\int_0^T h^{(m)}(t)\,\eta(t)\,dt \to \int_0^T h(t)\,\eta(t)\,dt$$

as $m \to \infty$. At the same time we have already established that the functions $h^{(m)}$ are equicontinuous (i.e. they share a common modulus of continuity). Together with their uniform boundedness this implies, by Ascoli-Arzela, that the sequence $h^{(m)}$ is pre-compact in the topology of uniform convergence. Since we have

$$\int_0^T h^{(m)}(t)\,\eta(t)\,dt \to \int_0^T h(t)\,\eta(t)\,dt$$

and the uniform boundedness of $h^{(m)}$ we see that $h^{(m)}$ converges to h weakly-$*$ in $L_\infty([0,T];\mathbb{R})$ (or weakly in $L_2([0,T];\mathbb{R})$). Thus

$$\sup_{0\le t\le T} |h^{(m)}(t) - h(t)| \to 0$$

as $m\to\infty$. This conclusion is reached for any fixed $\varphi\in V^{(m_0)}$. Since m_0 can be chosen arbitrary and $\bigcup_m V^{(m)}$ is dense in V, we see that for each t and each $\varphi\in V$ we have

$$\int_{\mathbb{T}^2} u^{(m)}(x,t)\,\varphi(x)\,dx \to \int_{\mathbb{T}^2} u(x,t)\,\varphi(x)\,dx$$

as $m \to \infty$. It is not hard to show that V is L_2-dense in the space $L_{2,div} = \{a \in L_2(\mathbb{T}^2) : div\, a = 0\}$. (Since otherwise, there exist some $a \in L_{2,div}$, $a \neq 0$, $\int_{\mathbb{T}^2} a v \, dx = 0$, $v \in V$. By Helmholtz decomposition this implies that $a = \nabla \alpha$ for some $\alpha \in W_2^1(\mathbb{T}^2)$. But $div\, a = 0$ so $\Delta \alpha = 0 \Rightarrow \alpha = const$ $a = 0$ contradiction) Thus we see that in fact

$$\int_{\mathbb{T}^2} u^{(m)}(x,t)\, b(x)\, dx \to \int_{\mathbb{T}^2} u(x,t)\, b(x)\, dx$$

as $m \to \infty$ for all $b = (b_1, b_2) \in L_2(\mathbb{T}^2; \mathbb{R}^2)$.

We conclude this step that the function $x \to u(x,t)$ is well-defined as an $L_2(\mathbb{T}^2; \mathbb{R}^2)$ function for each $t \in [0,T]$ and $u^{(m)}(\cdot, t) \to u(\cdot, t)$ weakly in $L_2(\mathbb{T}^2; \mathbb{R}^2)$ for each $t \in [0,T]$. Also $t \to u(\cdot, t)$ is continuous as a function from $[0,T]$ to $(L_2(\mathbb{T}^2; \mathbb{R}^2);$ weak topology$)$

Now comes Leray's argument

let $e^{(m)}(t) = \int_{\Pi^2} |u^{(m)}(x,t)|^2 dx$

$$e(t) = \int_{\Pi^2} |u(x,t)|^2 dx$$

It is easy to see, from the fact that $e^{(m)}(t)$ is a decreasing function of t that the total variation $\int_0^T \left| \frac{de^{(m)}}{dt}(t) \right| dt$ of the functions $e^{(m)}$ is uniformly bounded. So we can assume

$e^{(m)}(t) \to e^*(t)$ as $m \to \infty$ for each $t \in [0,T]$

Clearly $\int_0^T e^{(m)}(t)\,dt \to \int_0^T e^*(t)\,dt$ as $m \to \infty$

Let $D^{(m)}(t) = \int_{\Pi^2} |\nabla u^{(m)}(x,t)|^2 dx$

$$D^*(t) = \liminf_{m\to\infty} D^{(m)}(t)$$

Then by (9)

$$\int_0^T D^{(m)}(t)\,dt \le C < \infty, \quad m = 1, 2, \ldots$$

and hence by Fatou's lemma we have

$$\int_0^T D^*(t)\,dt = \int_0^T \liminf_{m\to\infty} D^{(m)}(t)\,dt \le \liminf_{m\to\infty} \int_0^T D^{(m)}(t)\,dt \le C < \infty$$

So $D^*(t)$ is finite for almost every $t \in [0,T]$. If $D^*(t)<\infty$ then since $u^{(m)}(\cdot,t) \to u(\cdot,t)$ in $(L_2(\Pi^2;\mathbb{R}^2)$; weak convergence), by Rellich's compactness theorem we know that $u^{(m')}(\cdot,t) \to u(\cdot,t)$ in $L_2(\Pi^2;\mathbb{R}^2)$ for some subsequence $u^{(m')}(\cdot,t)$ of $u^{(m)}(\cdot,t)$. So $e(t) = e^*(t)$ for each t for which $D^*(t)<\infty$. This gives strong convergence $u^{(m)} \to u$ in $L_2([0,T];L_2(\Pi^2;\mathbb{R}^2))$

Using the above discussion we can pass to the limit to get Leray-Hopf weak solution. For $0\le t_1\le t_2\le T$ we have

$$\int_{\Pi^2} u^{(m)}\cdot\varphi\,dx\Big|_{t=t_2}^{t=t_1} + \int_{t_1}^{t_2} dt \int_{\Pi^2} \left(-u^{(m)}\cdot\varphi_t - u_j^{(m)} u_i^{(m)} \varphi_{i,j} + Du^{(m)}\cdot D\varphi\right) dx = 0$$

Pass to the limit

$$\lim_{m\to\infty} \int_{\mathbb{T}^2} u^{(m)}\cdot\varphi\, dx = \int_{\mathbb{T}^2} u\cdot\varphi\, dx$$

$$\lim_{m\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} -u^{(m)}\cdot\varphi_t\, dx = \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} -u\cdot\varphi_t\, dx$$

$$\lim_{m\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} -u_j^{(m)} u_i^{(m)} \varphi_{i,j}\, dx = \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} -u_j u_i \varphi_{i,j}\, dx$$

$$\lim_{m\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} \nabla u^{(m)}\cdot\nabla\varphi\, dx = \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} \nabla u\cdot\nabla\varphi\, dx$$

Recall in dimension 2 the multiplicative inequality

$$\int_{\mathbb{T}^2} |u|^4 dx \le c \left(\int_{\mathbb{T}^2} |u|^2 dx \right)\left(\int_{\mathbb{T}^2} |\nabla u|^2 dx \right)$$

$$\int_0^T dt \int_{\mathbb{T}^2} |u|^4 dx \le c \left(\operatorname*{ess\,sup}_{0\le t\le T} \int_{\mathbb{T}^2} |u|^2 dx \right)\left(\int_0^T dt \int_{\mathbb{T}^2} |\nabla u|^2 dx \right)$$

(Exercise!) so that $u \in L_4(\mathbb{T}^2 \times [0,T]; \mathbb{R}^2)$.

Taking limit in (9) we get, due to weak lower semicontinuity of $\overset{\circ}{H}{}^1(\mathbb{T}^2; \mathbb{R}^2)$ - norm

$$\liminf_{m\to\infty} \int_{\mathbb{T}^2} |\nabla u^{(m)}|^2 dx \geq \int_{\mathbb{T}^2} |\nabla u|^2 dx$$

Thus

$\boxed{\text{whenever } e(t_2) = e^*(t_2)}$

(general phenomenon)

$$\frac{1}{2}\int_{\mathbb{T}^2} |u|^2(x, t_2)\,dx \geqslant \frac{1}{2}\int_{\mathbb{T}^2} |u|^2(x, t_1)\,dx + \int_{\mathbb{T}^2} |\nabla u|^2 dx$$

Next goal is to claim energy inequality is actually <u>energy identity</u>, this is done by showing further regularity property of weak solution.

What we already have: existence of a weak solution in the space $u \in L_\infty([0,T]; L_2(\mathbb{T}^2; \mathbb{R}^2)) \cap L_2([0,T]; H^1(\mathbb{T}^2; \mathbb{R}^2)) \cap L_4(\mathbb{T}^2 \times [0,T]; \mathbb{R}^2)$.

<u>Exercise</u> The definition of weak solution can be extended to test functions $\varphi \in C^{(1)}_{\mathrm{div},0}$.

Further regularity of $u(x,t)$

(Reference: [Ladyzhenskaya, O.A.
Solonnikov, V.A.
Ural'tseva, N.N.

Linear and quasilinear parabolic equations
AMS, 1988
Section 3.4]

Let $\zeta : (-\infty, \infty) \to [0, \infty)$ be a smooth function symmetric w.r.t. origin. $\operatorname{supp} \zeta \subset (-1,1)$ $\int_{-\infty}^{+\infty} \zeta(s)\, ds = 1$. Let $\zeta_\varepsilon(s) = \frac{1}{\varepsilon} \zeta\left(\frac{s}{\varepsilon}\right)$. For any function f defined on $\mathbb{T}^2 \times [0,T]$ we have defined

$$f_\varepsilon(x,t) = \int_{\mathbb{T}^2} f(x, t+s)\, \zeta_\varepsilon(s)\, ds$$

Note: if $f \in L_2([0,T]; L_2(\mathbb{T}^2))$ then

$f_\varepsilon \in C([-\varepsilon, T+\varepsilon]; L_2(\mathbb{T}^2))$. Moreover if $g(x,t) : \mathbb{T}^2 \times [0,T] \to \mathbb{R}$ is compactly supported in $(0,T)$ as function of t then $\int_0^T dt \int_{\mathbb{T}^2} f_\varepsilon(x,t)\, g(x,t)\, dx = \int_0^T dt \int_{\mathbb{T}^2} f(x,t)\, g_\varepsilon(x,t)\, dx$

From weak formulation we know for any test function $\varphi(x,t)$ compactly supported as a function of $t \in (0, T-\varepsilon)$ we have

$$\int_0^T dt \int_{\Pi^2} (-u \cdot \varphi_t)\, dx = \int_0^T dt \int_{\Pi^2} [(u_j u_i)\varphi_{i,j} - \nabla u \cdot \nabla\varphi]\, dx$$

Replace φ by φ_ε

$$\int_0^T dt \int_{\Pi^2} (-u_\varepsilon \cdot \varphi_t)\, dx = \int_0^T dt \int_{\Pi^2} [(u_j u_i)_\varepsilon \varphi_{i,j} - \nabla u_\varepsilon \cdot \nabla\varphi]\, dx$$

Suppose $0 < \varepsilon, \varepsilon' < \varepsilon_1$ and $v = u_\varepsilon - u_{\varepsilon'}$. Put $b_{i,j} = (u_j u_i)_\varepsilon - (u_j u_i)_{\varepsilon'}$. Thus v satisfies the equation

$$\int_0^T dt \int_{\Pi^2} (-v \cdot \varphi_t)\, dx = \int_0^T dt \int_{\Pi^2} [b_{i,j}\varphi_{i,j} - \nabla v \cdot \nabla\varphi]\, dx$$

Choose $\eta : [0, T] \to \mathbb{R}$ smooth and compactly supported as a function of t in $(0, T-\varepsilon)$. Pick $\varphi(x,t) = \eta(t)v(x,t)$

Thus

$$\int_0^T dt \int_{\Pi^2} (-v \cdot v_t \eta - |v|^2 \eta_t)\, dx = \int_0^T dt \int_{\Pi^2} (b_{i,j} v_{i,j}\eta - |\nabla v|^2 \eta)\, dx$$

We see now it is legal to say $v \cdot v_t = \frac{1}{2}(|v|^2)_t$

we pick a sequence of smooth functions $\eta = \eta^\delta = \bar{\eta} * \zeta_\delta$

to approximate

$$\bar{\eta}(t) = 0 \quad \text{for } t \notin [t_1, t_2], \quad \bar{\eta}(t) = \frac{t_2 - t}{t_2 - t_1} \quad \text{for } t \in [t_1, t_2]$$

Thus

$$\frac{1}{2}\int_{\mathbb{T}^2} |v|^2(x, t_1)\,dx = \int_{t_1}^{T} dt \int_{\mathbb{T}^2} \left(b_{ij} v_{i,j} \bar{\eta} - |\nabla v|^2 \bar{\eta} + \frac{1}{2}|v|^2 \bar{\eta}_t \right) dx$$

Since $u \in L_2([0,T]; H^1(\mathbb{T}^2; \mathbb{R}^2)) \cap L_4(\mathbb{T}^2 \times [0,T]; \mathbb{R}^2)$

we see from above that as $\varepsilon_1 \downarrow 0$ we have

$$\frac{1}{2}\int_{\mathbb{T}^2} |v|^2(x, t_1)\,dx \longrightarrow 0$$

uniformly in $t_1 \in [0, T-\kappa]$ for any $\kappa > 0$. Thus

$u_\varepsilon \to u$ uniformly in $C([0, T-\kappa]; L_2(\mathbb{T}^2; \mathbb{R}^2))$

and $u \in C([0, T-\kappa]; L_2(\mathbb{T}^2; \mathbb{R}^2))$. We can repeat

the argument $T \to 0$ so $u \in C([0,T]; L_2(\mathbb{T}^2; \mathbb{R}^2))$

Now let $v = u_\varepsilon$ $\quad \eta = \eta^\delta = \bar{\eta} * \zeta_\delta$, $\bar{\eta} = \mathbb{1}(t_1, t_2)$

let $\varepsilon \downarrow 0$ get energy identity.

Our final step is to show uniqueness (of the weak solution). Let us fix some $T>0$. Suppose there are two different weak solutions u_1, u_2 so that $u_1 - u_2 \stackrel{\text{def}}{=} \delta$. Then $\delta(x,0)=0$ and the function δ is a weak solution of the equation

$$\frac{\partial \delta}{\partial t} - \Delta \delta + \nabla p = -(\delta \cdot \nabla) u_1 + (u_2 \cdot \nabla)\delta .$$

As we know that $u_1, u_2, \delta \in L_4(\mathbb{T}^2 \times [0,T]; \mathbb{R}^2)$, similar considerations borrowed from linear theory (as in the previous step when deriving the energy identity) give us an energy identity for δ:

$$\frac{1}{2}\int_{\mathbb{T}^2} |\delta|^2 dx + \int_0^t ds \int_{\mathbb{T}^2} |\nabla \delta|^2 dx = \int_0^t ds \int_{\mathbb{T}^2} \sum_{i,j=1}^{2} (u_1)_j \frac{\partial \delta_j}{\partial x_i} \delta_i dx$$

Integration by parts then gives

$$\frac{1}{2}\int_{\mathbb{T}^2} |\delta|^2 dx + \int_0^t ds \int_{\mathbb{T}^2} |\nabla \delta|^2 dx = -\int_0^t ds \int_{\mathbb{T}^2} \sum_{i,j=1}^{2} \frac{\partial (u_1)_j}{\partial x_i} \delta_j \delta_i dx$$

We apply Hölder inequality, Ladyzhenskaya's multiplicative inequality and Young's inequality so that

$$\operatorname*{ess\,sup}_{0\le s\le t} \frac{1}{2}\int_{\mathbb{T}^2} |\delta|^2(x,t)\,dx + \int_0^t ds \int_{\mathbb{T}^2} |\nabla\delta|^2\,dx$$

$$\le C\int_0^t ds \left(\int_{\mathbb{T}^2} |\nabla u_1|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} |\delta|^4\,dx\right)^{\frac{1}{2}}$$

$$\le C\int_0^t ds \left(\int_{\mathbb{T}^2} |\nabla u_1|^2\,dx\right)^{1/2} \left(\int_{\mathbb{T}^2} |\delta|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} |\nabla\delta|^2\,dx\right)^{\frac{1}{2}}$$

$$\le \frac{1}{2}\int_0^t ds \int_{\mathbb{T}^2} |\nabla\delta|^2\,dx + C\int_0^t ds \left(\int_{\mathbb{T}^2} |\delta|^2\,dx\right)\left(\int_{\mathbb{T}^2} |\nabla u_1|^2\,dx\right)$$

so if we let $m(t) = \operatorname*{ess\,sup}_{0\le s\le t} \frac{1}{2}\int_{\mathbb{T}^2} |\delta|^2(x,t)\,dx$ we get

$$m(t) \le C\int_0^t \left(\int_{\mathbb{T}^2} |\nabla u_1|^2\,dx\right) m(s)\,ds$$

$$\Rightarrow\quad m(t) \le C\exp\left(\int_0^t ds \int_{\mathbb{T}^2} |\nabla u_1|^2\,dx\right) m(0)$$

$$\Rightarrow\quad m(t) = 0 \quad \text{i.e.}\quad \delta = 0 \quad \text{for } 0\le t\le T$$

$\Rightarrow$ uniqueness $\square$

Let us briefly discuss now randomly forced 2-D Navier-Stokes equation.

$\mathcal{D} = \mathcal{D}(\mathbb{T}^2)$ — space of all C^∞ class vector fields on $\mathbb{T}^2$

$\mathbb{L}_p = L_p(\mathbb{T}^2) \times L_p(\mathbb{T}^2)$, $p > 0$

$H^\alpha(\mathbb{T}^2)$, $L_2 = H^0$, $\mathbb{H}^\alpha(\mathbb{T}^2) = \mathbb{H}^\alpha = H^\alpha \times H^\alpha$

$\overset{\parallel}{H^\alpha}$

$\mathbb{W}_p^k = W_p^k(\mathbb{T}^2) \times W_p^k(\mathbb{T}^2)$

$\mathcal{U} = \{u \in \mathcal{D},\ \operatorname{div} u = 0\}$

$V^\alpha = \overline{\mathcal{U}}^{[\mathbb{H}^\alpha]^2}$ $\quad H = V^0 \quad V = V^1$

Helmholtz decomposition $\quad L_2(\mathbb{T}^2; \mathbb{R}^2) = H \oplus H^\perp$

$H = \{u \in E,\ \operatorname{div} u = 0\}$

$H^\perp = \{u \in E,\ u = \nabla p,\ p \in H^1\}$

$E = E(\mathbb{T}^2) = \{u \in \mathbb{L}_2,\ \operatorname{div} u \in L_2(\mathbb{T}^2)\}$

$$\langle u, v\rangle_E = (u, v)_{\mathbb{L}_2} + (\operatorname{div} u, \operatorname{div} v)_{L_2(\mathbb{T}^2)}$$

Leray projection: $\Pi : \mathbb{L}_2 \longrightarrow H$

(Exercise) $\quad Lu = -\Pi \Delta u = -\Delta u$

$B : V \times V \longrightarrow V^{-1} \qquad z \in V$

$$(B(u,v), z)_H = \int_{\mathbb{T}^2} z(x) \cdot [(u(x)\cdot\nabla) v(x)]\, dx$$

(Exercise)

$$(B(u,v), v)_H = 0, \quad (B(u,v), z)_H = -(B(u,z),v)_H \tag{10}$$

(Exercise) (cf. Flandoli paper '94)

$$|(B(v+z, v+z), v)_H| \le \varepsilon \|\nabla v\|^2_{\mathbb{L}_2} + C(\varepsilon)\|z\|^4_{\mathbb{L}_4}\|v\|^2_{\mathbb{L}_2} + C(\varepsilon)\|z\|^4_{\mathbb{L}_4}. \tag{11}$$

$\nabla \times u = \omega \vec{e_3}$

Vorticity equation $\quad \nabla \times u = \omega \vec{\mathbf{e}_3}$

Stream function $\quad u = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right) = \nabla^\perp \psi$

$\Delta \psi = \omega \qquad u = \nabla^\perp \Delta^{-1} \omega = K\omega$

$\downarrow$

Biot-Savart operator

$$\|K\omega\|_{\mathbb{H}^k} \le C\|\omega\|_{\mathbb{H}^{k-1}}, \quad k \in \mathbb{N}. \tag{12}$$

Idea Reduce stochastic eq to deterministic one

$$\frac{\partial u}{\partial t} + B(u,u) - \Delta u = \frac{d\eta}{dt} \quad (13)$$

$$\eta = \eta(x,t) = \sum_{k\in\mathbb{Z}^2} b_k \phi_k(x) W^k(t)$$

$$\phi_k(x) = \left(-\frac{ik_2}{|k|}, \frac{ik_1}{|k|}\right)\psi_k(x), \quad \psi_k(x) = \frac{1}{2\pi}e^{ik\cdot x}$$

Vorticity eq. $\frac{\partial w}{\partial t} + (u\cdot\nabla)w - \Delta w = \frac{d\zeta}{dt}$ (14)

$$\zeta = \zeta(x,t) = \sum_{k\in\mathbb{Z}^2} |k| b_k \psi_k(x) W^k(t), \quad u = Kw = \nabla^{\perp}\Delta^{-1}w$$

Definition 2 We say an H-valued process $u(t)$ $t\geq 0$ is a weak solution to (13) if

(a) The process $u(t)$ is adapted to a filtration $\mathcal{F}_t$ and its almost every trajectory belongs to $C([0,T];H) \cap L_2([0,T];V)$ for any $T>0$

(b). With probability 1 for any $\varphi\in H$ we have

$$(u(t),\varphi)_H + \int_0^t (-\Delta u(s),\varphi)_H\, ds - \int_0^t (B(u(s),\varphi),u(s))_H\, ds$$

$$= (u(0),\varphi)_H + (\eta(t),\varphi)_H \quad , \quad t \geq 0 \qquad (15)$$

<u>Theorem 2</u> Existence + Uniquess of weak solution to (13)

<u>Proof</u> (sketch)

<u>Stochastic Convolution</u> $z^\eta(t) = \int_0^t e^{(t-s)\Delta}\, d\eta(s)$

(Exercise) $z^\eta(t) \in C([0,T];H) \cap L_2([0,T];V)$ solves

$$\frac{dz^\eta}{dt} - \Delta z^\eta = \frac{d\eta}{dt}$$

Make necessary assumptions on $\{b_k\}_{k\in\mathbb{Z}^2}$

show $\mathbb{E}\, \|z^\eta(t)\|^4_{\mathbb{H}^{\frac{1}{2}}} \leq C$

Decompose $u = z^\eta + \alpha$ so that α satisfies

$$\frac{\partial\alpha}{\partial t} + B(z^\eta+\alpha,\ z^\eta+\alpha) - \Delta\alpha = 0 \qquad (16)$$

The eq (16) is Navier-Stokes type

Show existence and uniqueness using (11) and (12).

$\square$

Lecture 2

On existence and uniqueness of invariant measures for white-noise driven stochastic systems of hydrodynamical nature

2-D Navier-Stokes system

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p &= \nu\Delta u \\ \operatorname{div} u &= 0 \\ u = u(x,t), \quad x \in \Pi^2 &= [-\pi,\pi]^2 \end{aligned}\right\} \qquad (1)$$

In vorticity formulation $\omega\vec{e_3} = \nabla \times u$

i.e. $\omega = \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1}$

$$d\omega = \nu\Delta\omega\, dt + B(K\omega, \omega)\, dt \qquad (2)$$

$B(u,\omega) = -(u\cdot\nabla)\omega$

K — Biot-Savart operator, $K\omega = u$

$K = \nabla^{\perp}\Delta^{-1}$

In Fourier coordinates

$$(K\omega)_k = -i\, w_k \frac{k^{\perp}}{\|k\|^2}$$

$$\|k\|^2 = k_1^2 + k_2^2 , \qquad (k_1, k_2)^{\perp} = (k_2, -k_1)$$

$$\omega = \sum_{k \in \mathbb{Z}^2} w_k \cdot \frac{1}{2\pi} \exp(i k \cdot x)$$

(2) in terms of infinite dimensional dynamical systems has the following form

$$\dot{w}_k = -\nu |k|^2 w_k - \frac{1}{4\pi} \sum_{j+l=k} \langle j^{\perp}, l \rangle \left(\frac{1}{|l|^2} - \frac{1}{|j|^2} \right) w_j w_l \tag{2}$$

(Exercise : Check this ! It is a nice clear form)

Here $\langle j^{\perp}, l \rangle = \langle (j_2, -j_1), (l_1, l_2) \rangle = j_2 l_1 - j_1 l_2$

Analyze the dynamics of (2)

"$-\nu |k|^2 w_k$" — dissipation

when ν is large or $|k|$ is large, dissipation is the dominant part

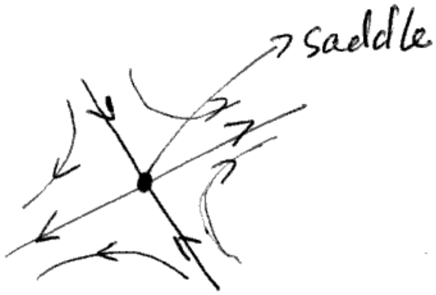
$$\text{"} -\frac{1}{4\pi} \sum_{j+l=k} \langle j^\perp, l \rangle \left(\frac{1}{|l|^2} - \frac{1}{|j|^2} \right) w_j w_l \text{"}$$

— nonlinearity

When $|k|$ is small, nonlinearity is the dominant part

$\rightsquigarrow$ system on these modes can be unstable

Complete dynamics : Hamiltonian nature with dissipation



contracting directions: high modes

expanding directions: low modes

<u>Remark</u> First mode $k = (1,0)$ or $(0,1)$ is stable for Euler (lowest mode stable)

If we add dissipation it is even more stable

Reference : [Marchioro, C., An example of absense of turbulence for any Reynolds number, CMP. <u>105</u> pp. 99–106, 1986]

(The implications of above stability can be fruitful which I currently not fully understood.)

Stochastic forcing at lowest modes

$$dw = \nu \Delta w \, dt + B(Kw, w) \, dt + Q \, dW \qquad (3)$$

$Q e_n = q_n f_{k_n} \; : \; \mathbb{R}^m \longrightarrow \mathcal{H} = L^2_0$ (vanishing mean)

$m < \infty$ finitely many modes are forced

W —— m-dimensional BM

$$f_k(x) = \begin{cases} \sin(k \cdot x) & \text{if } k \in \mathbb{Z}^2_+ \\ \cos(k \cdot x) & \text{if } k \in \mathbb{Z}^2_- \end{cases}$$

$q_n > 0$

$\mathcal{Z}_0 = \{ k_n \mid n = 1 \dots m \} \subset \mathbb{Z}^2 \setminus \{(0,0)\}$

Theorem 1 [Hairer - Mattingly 2006]

If $\mathcal{Z}_0$ satisfy the following two assumptions

(A1) There exist at least two elements in $\mathcal{Z}_0$ with different Euclidean norms

(A2) Integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}^2$

Then (3) has a unique invariant measure in $\mathcal{H}$.

Statistical Relevancy

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\omega_t)\,dt = \int_{\mathcal{H}} \varphi(\omega)\,d\mu(\omega)$$

μ — unique invariant measure

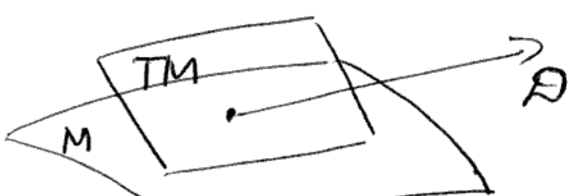
What is unique ergodicity for Markovian stochastic systems?

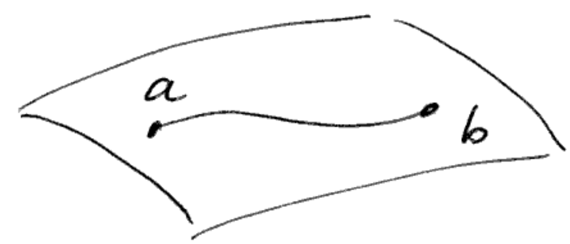
Sub-Riemannian geometry

M — Riemannian manifold

TM — tangent bundle

$\mathcal{D}$ — a distribution of vector fields on M





Chow's theorem [Chow 1939, Rashevskii 1938]

If $\mathcal{D}$ is bracket generating then from $\forall a \in M$ along $\mathcal{D}$ we reach any other $b \in M$

(controllability)

see [Montegomery, R. A tour of subriemannian geometries, their geodesics and applications]

The stochastic version of Chow's theorem is Hörmander's hypo-ellipticity theorem.

What is the idea?

$$f(q) = f(\exp(t_1 X_1)\exp(t_2 X_2) \cdots \exp(t_d X_d) q)$$

Differentiating $\Rightarrow$ $X_{i_1} \cdots X_{i_k} f = 0$

Then use bracket generating condition

"brackets generate the additional degrees of freedom"

Hörmander's theorem: Differentiability is replaced by Itô's formula

more analytical techinques such as Norris' lemma

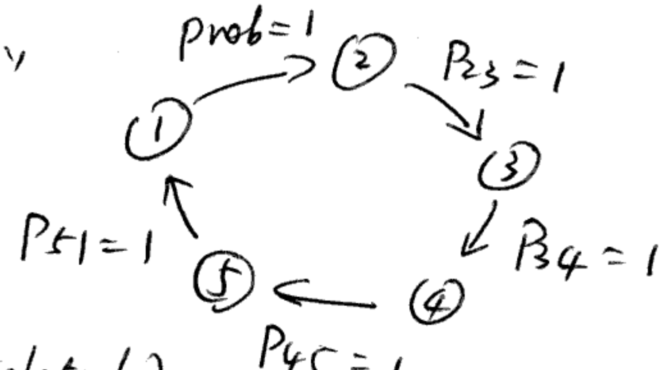
see Appendix to this lecture.

Chow's theorem is deterministic $\longrightarrow$ irreducibility

irreducibility = "there exists an "accessible point" which must belong to the topological support of every invariant probability measure"

Think of a finite Markov chain

"irreducible" but "periodic"



This example may not be exact

but it reveals that ("mixing" is violated)

"aperiodicity" = "loss of memory" ("irreversibility")

"irreducibility" + "aperiodicity" = "$p_{ij}^{(n)} \to \pi_i$" $\hookrightarrow$ "mixing"

Process "forgets" where it come from.

In continuous case it means "transition semigroup $(P_t)_{t \geq 0}$ has smoothing property.

Roughly speaking

Smoothing $\Longrightarrow$ invariant measures are <u>more</u> than mutually singular and moreover $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \phi$

irreducibility $\Longrightarrow$ invariant measure covers all points in phase space

Invariant measure is unique.

Dynamical Smoothing : contracting dynamics

Probabilistic Smoothing : injection of noise

Asymptotic Strong Feller (Hairer-Mattingly 2006)

$$\|\nabla P_t \varphi_{(u)}\| \leq C(\|u\|)\,(\|\varphi\|_\infty + \delta_n \|D\varphi\|_\infty)$$

$\delta_n \to 0$ $\qquad$ $\|\varphi\|_\infty$: Probabilistic Smoothing $\qquad$ $\delta_n \|D\varphi\|_\infty$: Dynamical Smoothing (Lasota-Yorke inequality)

Let us come to formulation of classical Feller category.

Strong Feller $\quad P_t: M_b(H) \to C_b(H)$

Implied by the quantitative bound

$$\|\nabla P_t \varphi(u)\| \leq C(\|u\|)\,\|\varphi\|_\infty$$

Theorem 2 (Doob-Khasminskii)

Suppose that $\{P_t\}_{t\geq 0}$ is a Markov semigroup on a metric space (H,ρ) and assume that the set of invariant measures $\mathcal{I}$ is compact in the topology of weak convergence. If

(1). P_t is weakly irreducible, i.e. $\exists\ x_0^*$ which is in the support of every invariant measure

(2). P_t is strong Feller

Then $\mathcal{I}$ has at most one element (unique ergodicity)

$(P_t)_{t\geq 0}$ corresponds to Kolmogorov equation

$$\frac{\partial \psi}{\partial t} = L\psi \quad , \quad \psi(0, x) = \phi(x)$$

Then $\psi(t, x) = P_t\,\phi(x)$

$(P_t)_{t\geq 0}$ is smoothing $\Rightarrow$ L is (parabolic) hypoelliptic

<u>Model equation</u>

$$dU + (\nu A U + B(U, U))\,dt = \sigma dW = \sum_{k=1}^{d} \sigma_k\, dW^k \quad (5)$$

"dimension of U" = $N \gg d$

$\langle \nu AU, U \rangle \geq \alpha |U|^2$, A — linear dissipative

B is bilinear, $\langle B(V, U), U \rangle = 0$, $V, U \in \mathbb{R}^N$
model Navier-Stokes nonlinearity

$\sigma_1 \ldots \sigma_d$ are fixed elements in $\mathbb{R}^N$

$$F(U) = \nu AU + B(U, U)$$

— vector fields in $\mathbb{R}^N$

Define recursively

$$\mathcal{V}_0 = \text{span}\{\sigma_k : k = 1 \ldots d\}$$

...

$$\mathcal{V}_n = \text{span}\{E, [E, F], [E, \sigma_k], k = 1 \ldots d, \; E \in \mathcal{V}_{n-1}\}$$

where $[G, H] = \nabla H\, G - \nabla G\, H$

$G, H: \mathbb{R}^N \to \mathbb{R}^N$ are vector fields

parabolic Hörmander's condition $\bigcup_n \mathcal{V}_n(U) = \mathbb{R}^N$ for all $U \in \mathbb{R}^N$

<u>Conclusion</u>

<u>Theorem 3</u> There exists a unique ergodic invariant measure μ for (5) s.t. μ is mixing and strong law of large numbers hold for μ.

Proposition 1 (Well posedness)

For every $U_0 \in \mathbb{R}^N$ there exists a unique $U(\cdot, U_0)$
$\Omega \times [0, \infty) \to \mathbb{R}^N$ such that

$$U(\cdot, U_0) \in C([0,\infty); \mathbb{R}^N) \text{ almost surely and is } \mathcal{F}_t\text{-adapted}$$

such that

$$U(t) + \int_0^t (\nu AU + B(U,U))\,ds = U_0 + \sum_{k=1}^{d} \sigma_k W^k(t)$$

for every $t \geq 0$. Moreover solutions depend on intial data in a continuously way, i.e. $U_0^k \to U_0$ implies $U(t, U_0^k) \to U(t, U_0)$ a.s. for $t \geq 0$

Markovian framework

$$P_t(U_0, A) = \mathbb{P}(U(t, U_0) \in A)$$

$$P_t \phi(U_0) = \int_{\mathbb{R}^N} \phi(U)\, P_t(U_0, dU) = \mathbb{E}\,\phi(U(t, U_0))$$

Existence of invariant measure; moment estimates

$$d|U|^2 + 2\langle \nu AU + B(U,U), U\rangle = |\sigma|^2 + 2\langle \sigma, U\rangle dW$$

$$\Rightarrow |U(t)|^2 + 2\alpha \int_0^T |U|^2 dt \leq |U_0|^2 + |\sigma|^2 T + 2\sum_{k=1}^{d} \int_0^T \langle \sigma_k, U\rangle dW^k$$

Let $\mu_T(A) = \frac{1}{T}\int_0^T \mathbb{P}(U(t)\in A)\,dt$

$$\mu_T(B(0,R)) = 1 - \frac{1}{T}\int_0^T \mathbb{P}(|U(t)|^2 \geq R^2)\,dt$$

$$\geq 1 - \frac{1}{TR^2}\int_0^T \mathbb{E}|U(t)|^2\,dt$$

$$\geq 1 - \frac{|\sigma|^2 + |U_0|^2 T^{-1}}{2\alpha R^2}$$

Thus $\{\mu_T\}_{T\geq 0}$ is tight and hence it is a weakly compact sequence. Take any sub-sequence limit $\mu = \lim_{T_j\to\infty} \mu_{T_j}$.

$$\langle P_t^* \mu, \phi\rangle = \lim_{T_j\to\infty} \frac{1}{T_j}\int_0^{T_j} P_{t+s}\phi(U_0)\,ds$$

$$= \lim_{T_j\to\infty} \frac{1}{T_j}\int_t^{T_j+t} P_s\phi(U_0)\,ds$$

$$= \lim_{T_j\to\infty}\Big(\langle \mu_{T_j}, \phi\rangle + \frac{1}{T_j}\int_{T_j}^{T_j+t} P_s\phi(U_0)\,ds - \frac{1}{T_j}\int_0^t P_s\phi(U_0)\,ds\Big)$$

$$= \langle \mu, \phi\rangle.$$ —— μ is invariant measure

Irreducibility

Proposition 2 For every $R>0$ and every $\varepsilon>0$ there exists $T_* = T_*(R, \varepsilon, \alpha)$ s.t. $\inf_{U_0 \in B_R} P_t(U_0, B_\varepsilon) > 0$

for every $t > T_*$ and $B_R = \{U : |U| \le R\}$

Crucial Final Step Smoothing = Strong Feller

Proposition 3 $\forall t>0 \quad \forall \phi \in C_b^1(\mathbb{R}^N)$

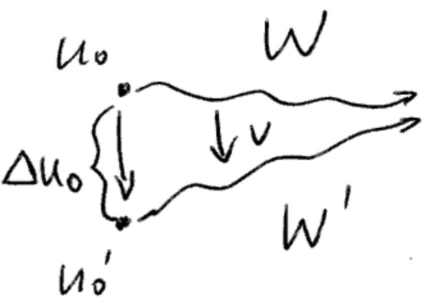
$$\| \nabla P_t \phi(U_0) \| \le C \left(\sup_{U \in \mathbb{R}^N} |\phi(U)| \right) \tag{6}$$

where $C = C(|U_0|, t)$ is independent of ϕ

so $\{P_t\}_{t \ge 0}$ is strong Feller

What is the meaning of the smoothing estimate (6) ?

Smoothing of P_t = "Loss of memory"



shift initial condition $u_0 \to u_0' = u_0 + \Delta u_0$

= shift noise $W \to W' = W + v$

$\|u(t) - u(t')\| \to 0$

Apply Girsanov $\Rightarrow$ two initial conditions induce equivalent measures on the infinite future

Infinitesimal version of the above picture

$$
\begin{tikzcd}
\Delta u_0 \arrow[r] \arrow[d, leftrightarrow] & \text{infinitesimal perturbation } \xi \text{ of } u_0 \arrow[r] \arrow[d, leftrightarrow] & D_\xi u_t \arrow[d, leftrightarrow] \\
V \arrow[r] & \text{infinitesimal perturbation } h \text{ of } W \arrow[r] & \mathcal{D}_h u_t
\end{tikzcd}
$$

Malliavin derivative

$$\mathbb{E}\,\| D_\xi u_t - \mathcal{D}_h u_t \| \to 0 \quad \text{as} \quad t \to \infty$$

$$\mathcal{D}_h u_t = \frac{d}{d\varepsilon} u_t(W + \varepsilon V), \quad V = \int_0^t h(s)\,ds$$

Let $\xi \in \mathbb{R}^N$ be a unit vector

$$\nabla P_t \phi(U_0)\xi = \mathbb{E}\left(\nabla\phi(U(t, U_0))\, \mathcal{J}_{0,t}\,\xi\right) \tag{7}$$

First derivative of (5)

Here $\mathcal{J}_{s,t}\,\xi$ solves the system

$$\frac{d\rho}{dt} + \nu A\rho + \nabla B(U)\rho = 0, \quad \rho(s) = \xi \tag{8}$$

with $\nabla B(U)\rho = B(U,\rho) + B(\rho, U)$, $\quad U$ — fixed as solution of (5)

Malliavin Calculus on $H = L^2([0,t]; \mathbb{R}^d)$
$\times \Omega$

$$F = F(W) = f\left(\int_0^t g_1 dW, \dots, \int_0^t g_n dW\right)$$

$$(\mathcal{D}F)_s = \mathcal{D}_s F = \sum_{k=1}^n \partial_{x_k} f\left(\int_0^t g_1 dW, \dots, \int_0^t g_n dW\right) g_k(s)$$

$\uparrow$
Malliavin derivative

<u>Fact</u> • if $F \in \mathcal{F}_r$ then $\mathcal{D}_s F = 0$ for $s > r$

• $$\langle \mathcal{D}F(W), v\rangle_{L^2([0,t];\mathbb{R}^d)} = \lim_{\varepsilon \to 0} \frac{F(W+\varepsilon V) - F(W)}{\varepsilon}$$

$$V(s) = \int_0^s v(r)\,dr, \quad v \in L^2([0,t]; \mathbb{R}^d)$$

Malliavin chain rule: $\mathcal{D}\phi(F) = \nabla\phi(F)\,\mathcal{D}F$

Integration by parts: $\mathbb{E}\langle \mathcal{D}F, v\rangle_{L^2([0,t];\mathbb{R}^d)} = \mathbb{E}(F\mathcal{D}^* v)$

$\mathcal{D}^* v = \int_0^t v\,dW$ — skorokhod integral

Apply Malliavin Calculus to (5): $\mathcal{A}_{0,t} : L^2([0,t];\mathbb{R}^d) \to \mathbb{R}^N$

$$\mathcal{A}_{0,t} v = \langle \mathcal{D}U(t, U_0, W), v\rangle_{L^2([0,t];\mathbb{R}^d)}$$

$$= \int_0^t \mathcal{J}_{s,t}\,\sigma\, v(s)\,ds = \lim_{\varepsilon \to 0} \frac{U(t,U_0,W+\varepsilon V) - U(t,U_0,W)}{\varepsilon}$$

where $V(t) = \int_0^t v(s)\,ds$

Let $\bar{\rho} = \mathcal{A}_{0,t} v$ satisfies

$$\frac{d\bar{\rho}}{dt} + \nu A \bar{\rho} + \nabla B(U)\,\bar{\rho} = \sum_{k=1}^{d} \sigma_k v_k \,, \quad \bar{\rho}(0) = 0 \qquad (9)$$

Thus $\nabla P_t \phi(U_0)\xi$

$$= \mathbb{E}\left(\nabla\phi(U(t,U_0))\mathcal{J}_{0,t}\xi \right)$$

$$= \mathbb{E}\langle \mathcal{D}\phi(U(t,U_0)), v\rangle_{L^2([0,t];R^d)}$$

$$+ \mathbb{E}\Big(\nabla\phi(U(t,U_0))(\mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}v)\Big)$$

$$= \mathbb{E}\left(\phi(U(t,U_0))\int_0^t v\,dW\right) + \mathbb{E}\Big(\nabla\phi(U(t,U_0))\underbrace{(\mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}v)}_{\text{Want to drive to } 0}\Big)$$

<u>Goal</u> : To solve $\mathcal{J}_{0,t}\xi = \mathcal{A}_{0,t} v(\xi)$ (10)

So that $\nabla P_t\phi(U_0)\xi = \mathbb{E}\left(\phi(U(t,U_0))\int_0^t v(\xi)\,dW\right)$ (11)

and we need to estimate

$$\sup_{|\xi|=1} \mathbb{E} \left| \int_0^t v(s)\, dW \right|^2 < \infty \qquad (12)$$

Thus $\| \nabla P_t \phi(U_0) \xi \| \le C \sup_{U \in \mathbb{R}^N} |\phi(U)|$ which is (6)!

How to solve (10)?

Malliavin matrix $(M_{0,t})_{ij} = \langle \mathcal{D}^i U(t, U_0, W), \mathcal{D}^j U(t, U_0, W) \rangle_{L_2([0,t]; \mathbb{R}^d)}$

$$M_{0,t} = ((M_{0,t})_{ij})_{1 \le i, j \le N}$$

Alternating representation: if $\eta = (\eta_1 \cdots \eta_N) \in \mathbb{R}^N$

$$(M_{0,t}\, \eta)_i = \sum_{j=1}^N (M_{0,t})_{ij}\, \eta_j$$

$$= \sum_{j=1}^N \langle \mathcal{D}^i U(t, U_0, W), \mathcal{D}^j U(t, U_0, W) \rangle_{L_2([0,t]; \mathbb{R}^d)} \eta_j$$

$$= \langle \mathcal{D}^i U(t, U_0, W), \sum_{j=1}^N \mathcal{D}^j U(t, U_0, W) \eta_j \rangle_{L_2([0,t]; \mathbb{R}^d)}$$

$$= \mathcal{A}^i_{0,t} \left(\sum_{j=1}^N \mathcal{D}^j U(t, U_0, W) \eta_j \right) \stackrel{\text{def}}{=} \mathcal{A}^i_{0,t} \mathcal{A}^*_{0,t}\, \eta$$

where $(\mathcal{A}_{0,t}^* \eta)(s) = \sigma^* \mathcal{J}_{s,t}^* \eta$

$$\mathcal{A}_{0,t}^* \eta^{(s)} = \left(\sum_{j=1}^{N} D\hat{J}U(t, v_0, w)\, \eta_j \right)^{(s)} : \mathbb{R}^N \to L^2([0,t]; \mathbb{R}^d)$$

$$\sigma^* : \mathbb{R}^N \to \mathbb{R}^d$$

$$= \sigma^* \mathcal{J}_{s,t}^* \eta, \quad \text{where } J_{s,t}^* : \mathbb{R}^N \to \mathbb{R}^N$$

$\mathcal{J}_{s,t}^* \eta = \rho^*(s)$ satisfy

$$\left. \begin{aligned} -\frac{d}{dt} \rho^* + \nu A \rho^* + (\nabla B(U))^* \rho^* = 0 \\ \rho^*(t) = \eta \end{aligned} \right\} \quad (13)$$

How to solve (10) ?

$v = \mathcal{A}_{0,t}^* \eta$ —— ansatz

$$M_{0,t}\, \eta = \mathcal{A}_{0,t} \mathcal{A}_{0,t}^* \eta = \mathcal{J}_{0,t} \xi$$

so $\quad v(\xi) = \mathcal{A}_{0,t}^* M_{0,t}^{-1} \mathcal{J}_{0,t} \xi \qquad (14)$

<u>Proposition 4</u>. For all $q \geq 1$ there exists $\varepsilon_0 = \varepsilon_0(q, \nu, |\sigma|) > 0$ and $C = C(q, \nu, |\sigma|, |U_0|)$

such that

$$\mathbb{P}\left(\inf_{\eta \in \mathbb{R}^N \setminus \{0\}} \frac{\langle M_{0,t}\, \eta, \eta \rangle_{\mathbb{R}^N}}{|\eta|^2} \geq \varepsilon \right) \geq 1 - C\varepsilon^q \tag{15}$$

for all $0 < \varepsilon < \varepsilon_0$

<u>Corollary 1</u> $\forall p \geq 2 \quad \exists K = K(p, \nu, |\sigma|, |\sigma_0|) < \infty$

s.t. $\mathbb{E}\, \| M_{0,t}^{-1} \|^p \leq K < \infty$ (16)

Alternating representation of $M_{0,t}$

$$\langle M_{0,t}\, \eta, \eta \rangle_{\mathbb{R}^N} = \langle \mathcal{A}_{0,t} \mathcal{A}_{0,t}^* \eta, \eta \rangle_{\mathbb{R}^N}$$

$$= \langle \mathcal{A}_{0,t}^* \eta, \mathcal{A}_{0,t}^* \eta \rangle_{L_2([0,t]; \mathbb{R}^d)}$$

$$= \int_0^t |\sigma^* \mathcal{J}_{s,t}^* \eta|_{\mathbb{R}^d}^2 \, ds$$

$$= \sum_{j=1}^d \int_0^t \langle \mathcal{J}_{s,t}^* \eta, \sigma_j \rangle_{\mathbb{R}^N}^2 \, ds$$

Why bracket condition? Very similar to classical Malliavin's proof of Hörmander's hypoellipticity (see Appendix A)!

From (14) to (12): Itô's isometry for Skorokhod integral

$$\mathbb{E}\left(\int_0^t v(s)\,dW_s\right)^2 = \mathbb{E}\int_0^t |v|^2 dt + \mathbb{E}\int_0^t\int_0^t tr(\mathcal{D}_s v(r)\mathcal{D}_r v(s))\,dr\,ds \tag{17}$$

$v \in \mathbb{D}^{1,2}(L^2([0,t]; \mathbb{R}^d))$

$\mathcal{D}v \in L^2(\Omega; L^2([0,t]^2; \mathbb{R}^{d\times d}))$

First term in (17)

$$\mathbb{E}\int_0^t |\mathcal{A}_{0,t}^* M_{0,t}^{-1} \mathcal{J}_{0,t}\xi|^2 ds$$

$$\leq \left(\mathbb{E}\|M_{0,t}^{-1}\|^6\, \mathbb{E}\|\mathcal{J}_{0,t}\|^6\right)^{\frac{1}{3}} \int_0^t \left(\mathbb{E}\|\sigma^* \mathcal{J}_{s,t}^*\|^6\right)^{\frac{1}{3}} ds$$

Apply (8) the standard energy estimate

$$\frac{d}{dt}|\rho|^2 + \alpha|\rho|^2 \leq |\langle B(\rho, U), \rho\rangle| \leq C|\rho|^2 U$$

$$\Rightarrow \mathbb{E}\left(\sup_{s\in[0,t]} \|\mathcal{J}_{s,t}\|^p\right) \leq C\,\mathbb{E}\exp\left(\eta\int_0^t |U|^2 ds\right) \tag{18}$$

$$\leq C\exp(\eta|U_0|^2) < \infty$$

Here (18) is based on exponential moment estimate of (5). Actually

$$|U(t)|^2 + 2\alpha \int_0^t |U|^2 ds - 2\gamma \sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle^2 ds - |U(0)|^2 - |\sigma|^2 t$$

$$\leq 2\sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle dW^k - 2\gamma \sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle^2 ds$$

Since

$$\exp\left(2\sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle dW^k - 2\gamma \sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle^2 dt\right)$$

is a local martingale, we can apply Doob's inequality along sequence of stopping times and take limit so that

$$\mathbb{P}\left(\sup_{t\geq 0}\left(2\sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle dW^k - 2\gamma \sum_{k=1}^d \int_0^t \langle \sigma_k, U\rangle^2 dt\right) \geq k\right)$$

$$\leq e^{-\gamma k}$$

This provides us with the exponential estimates:

$\exists \gamma = \gamma(|\sigma|^2, \nu)$ s.t. $\forall K \geq 2|U_0|^2$

$$\mathbb{P}\left(\sup_{t\geq 0}\left(|U(t)|^2 + \alpha \int_0^t |U|^2 ds - |\sigma|^2 t\right) \geq \frac{K}{2}\right) \leq e^{-K\gamma} \quad (19)$$

and

$$\mathbb{E}\exp\left(\eta\left(\sup_{s\in[0,t]}\left(|U(s)|^2 + \alpha\int_0^t |U(s)|^2 ds\right)\right)\right), \quad \eta = \eta(|\sigma|^2, \nu) \text{ independent of } t > 0.$$

$$\leq \exp\left(\eta\left(|U_0|^2 + |\sigma|^2 t\right)\right) \quad (20)$$

Second term in (17)

Malliavin chain rule

$$\mathcal{D}v(\xi) = \mathcal{D}\mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} \mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} \mathcal{D}\mathcal{M}_{0,t} \mathcal{M}_{0,t}^{-1} \mathcal{J}_{0,t}\xi$$

$$+ \mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} \mathcal{D}\mathcal{J}_{0,t}\xi$$

$$= \mathcal{D}\mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} \mathcal{J}_{0,t}\xi$$

$$- \mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} (\mathcal{D}\mathcal{A}_{0,t} \mathcal{A}_{0,t}^{*} + \mathcal{A}_{0,t} \mathcal{D}\mathcal{A}_{0,t}^{*}) \mathcal{M}_{0,t}^{-1} \mathcal{J}_{0,t}\xi$$

$$+ \mathcal{A}_{0,t}^{*} \mathcal{M}_{0,t}^{-1} \mathcal{D}\mathcal{J}_{0,t}\xi \qquad (21)$$

$$\mathcal{D}_\tau \mathcal{J}_{s,t}\xi = \begin{cases} \mathcal{J}_{\tau,t}^{(2)}(\sigma, \mathcal{J}_{s,\tau}\xi) & \text{when } s < \tau \\ \mathcal{J}_{s,t}^{(2)}(\mathcal{J}_{\tau,s}\sigma, \xi) & \text{when } s \geq \tau \end{cases}$$

$\tilde{\rho} = \mathcal{J}_{s,t}^{(2)}(\xi, \xi')$ $\quad \xi, \xi' \in \mathbb{R}^N$ is the second derivative operator satisfy

$$\frac{d\tilde{\rho}}{dt} + \nu A\tilde{\rho} + B(U, \tilde{\rho}) + B(\tilde{\rho}, U) + B(\mathcal{J}_{s,t}\xi, \mathcal{J}_{s,t}\xi')$$

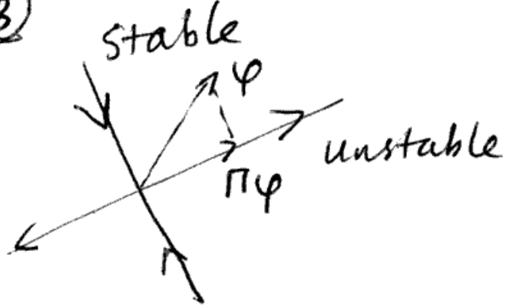
$$+ B(\mathcal{J}_{s,t}\xi', \mathcal{J}_{s,t}\xi) = 0 \quad , \tilde{\rho}(s) = 0 \qquad (22)$$

Apply Corollary 1 and exponential estimates, possible to arrive at (12). [redacted] (see H-M paper 2006. Section 4.8)

Let us finally mention difficulties in infinite dimensions

① Malliavin matrix becomes an operator

② "While it is possible to prove that the Malliavin matrix is almost surely non-degenerate it seems very difficult to charaterize its range"

③ 

We essentially only need the invertibility of the Malliavin matrix on the space spanned by the unstable directions, which is finite dimensional in our case

$$\mathbb{P}\Big(\inf_{\substack{\varphi \in \mathcal{H} \\ \|\Pi\varphi\| \geq \frac{1}{2}\|\varphi\|}} \langle M_t \varphi, \varphi \rangle < \varepsilon \|\varphi\|^2 \Big) = o(\varepsilon^p)$$

④ "truely hypoelliptic setting" replace $M_{0,t}$ by $\widetilde{M}_t \overset{\text{def}}{=} M_{0,t} + \beta$ for β very small and $\beta \to 0$

$\widetilde{M}_{0,t}^{-1}$ will be an inverse "up to a scale" depending on β

$\tilde{M}_{0,t}^{-1}$ is close to $M_{0,t}^{-1}$ on large scales and

$\tilde{M}_{0,t}^{-1}$ is very close to β^{-1} on small scales.

so $\tilde{M}_{0,t}^{-1}$ is effective on large scales

⑤ only control V at $[n, n+\frac{1}{2}]$

the Jacobian will contract the high modes before the low modes start to grow out of control.

<u>Remark</u> To apply the result we proved only need to check

$$W_0 = \{\sigma_j : j = 1 \dots d\}$$

$$\vdots$$

$$W_n = \{[[F, \psi], \sigma_j]\} = B(\psi, \sigma_j) + B(\sigma_j, \psi)$$

$$j = 1 \dots d, \ \psi \in W_{n-1}\} \cup W_{n-1}$$

so that $\mathrm{Span}(W_n) = \mathbb{R}^N$ for some n

Generalizations (Constantin- Vicol- GlattHoltz)

①. Fractionally dissipated Euler

$$d\omega + \left((-\Delta)^{\gamma/2}\omega + (u\cdot\nabla)\omega\right)dt = \sigma dW$$

$$u = K_2 * \omega$$

$$\omega(0) = \omega_0$$

as $\gamma \to 0$ need forced modes in $\sigma \uparrow$

"Essentially elliptic setting"

Critical estimate come from the control of exponential moments of ω which is a result of parabolic Smoothing

② Boussinesq System

(Foldes- GlattHoltz- Richards- Thomann)

$$du + (u\cdot\nabla)u\,dt = (-\nabla p + \nu_1 \Delta u + g\theta)\,dt, \quad \nabla\cdot u = 0$$

$$d\theta + (u\cdot\nabla)\theta\,dt = \nu_2 \Delta\theta + \sigma_\theta\, dW$$

θ — tempreature

Calculate brackets of form $[[[F, \sigma_{k_1}], F], \sigma_{k_2}]$

Observation of sharper condition

$$\langle \mathcal{Q}(\omega)\phi ; \phi \rangle \geq c\left(\alpha - \varepsilon\left(1 + \|u\|^p_{\tilde{H}}\right)\right)\|\phi\|^2$$

for every $\phi \in \mathcal{S}_{\alpha,N}$

m-dimensional stochastic differential equation

$$X_t = x_0 + \sum_{j=1}^{d} \int_0^t A_j(X_s)\, dW_s^j + \int_0^t B(X_s)\, ds$$

$A_j,\ B: [0,T] \times \mathbb{R}^m \to \mathbb{R}^m,\ 1 \le j \le d$ are measurable functions.

Existence of unique solution $X = \{X_t : t \in [0,T]\}$

Lipschitz condition

$$\max\left(|A_j(x) - A_j(y)|,\ |B(x) - B(y)|\right) \le K|x-y|$$

$$Y_t = I + \sum_{\ell=1}^{d} \int_0^t \partial A_\ell(X_s)\, Y_s\, dW_s^\ell + \int_0^t \partial B(X_s)\, Y_s\, ds$$

$$(\partial A_\ell)_j^i = \partial_j A_\ell^i$$

$$D_r X_t = Y_t Y_r^{-1} A(X_r)$$

Malliavin matrix of X_t

$$Q_t \doteq \int_0^t D_r X_t\, D_r^T X_t\, dr = Y_t \left\{ \sum_{\ell=1}^{d} \int_0^t Y_s^{-1} A_\ell(X_s) A_\ell^T(X_s) (Y_s^{-1})^T ds \right\} Y_t^T$$

$$= Y_t C_t Y_t^T$$

$$C_t = \sum_{l=1}^{d} \int_0^t Y_s^{-1} A_l(X_s) A_l^T(X_s) (Y_s^{-1})^T ds$$

$$= \int_0^t Y_s^{-1} \sigma(X_s)(Y_s^{-1})^T ds$$

$$A_j = \sum_{i=1}^{m} A_j^i(x) \frac{\partial}{\partial x_i}$$

$$B = \sum_{i=1}^{m} B^i(x) \frac{\partial}{\partial x_i}$$

$$A_j \overset{\nabla}{} A_k = \sum_{i,l=1}^{m} A_j^l \partial_l A_k^i \frac{\partial}{\partial x_i}$$

$$[A_j, A_k] = A_j \overset{\nabla}{} A_k - A_k \overset{\nabla}{} A_j$$

$$A_0 = B - \frac{1}{2} \sum_{l=1}^{d} A_l \overset{\nabla}{} A_l$$

Hörmander's condition:

Vector space spanned by the vector fields

$A_1, \ldots, A_d, [A_i, A_j], 0 \le i \le d, 0 \le j \le d,$

$[A_i, [A_j, A_k]], 0 \le i, j, k \le d$

$\ldots$ at point x_0 is $\mathbb{R}^m$

<u>Theorem 7.4</u> Assume that Hörmander's condition holds, then for any $t>0$ the random vector X_t has an infinitely differentiable density.

<u>Lemma 7.5.</u> Let $\{Z_t : t \geq 0\}$ be a real-valued, adapted continuous process such that $Z_0 = z_0 \neq 0$. Suppose $\exists\, \alpha > 0$ s.t. $\forall p \geq 1$ $\forall t \in [0,T]$

$$\mathbb{E}\Big(\sup_{0 \leq s \leq t} |Z_s - z_0|^p \Big) \leq C_{p,T}\, t^{p\alpha}$$

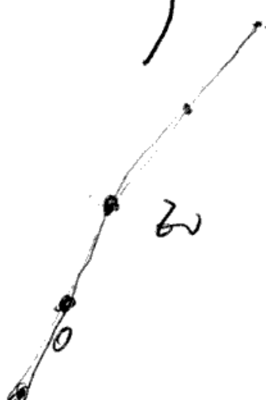
Then for all $p \geq 1$ and $t \in [0,T]$

$$\mathbb{E}\Big[\Big(\int_0^t |Z_s|\, ds \Big)^{-p} \Big] < \infty.$$

<u>Proof of Lemma 7.5.</u> $\forall\, 0 < \varepsilon < \frac{t|z_0|}{2}$ we have

$$\mathbb{P}\Big(\int_0^t |Z_s|\, ds < \varepsilon \Big) \leq \mathbb{P}\Big(\int_0^{\frac{2\varepsilon}{|z_0|}} |Z_s|\, ds < \varepsilon \Big)$$

~~If $\sup_{0 \le s \le \frac{2\varepsilon}{|z_0|}} |Z_s - z_0| \leq \frac{1}{2}|z_0|$ then $\int_0^{2\varepsilon/|z_0|} |Z_s|\, ds \geq \varepsilon$~~



$$\cancel{\left\{ \int_0^{\frac{2\varepsilon}{|z_0|}} |z_s|\, ds < \varepsilon \right\} \subset \left\{ \sup_{0 \le s \le \frac{2\varepsilon}{|z_0|}} |z_s| \ge 3|z_0| \right\}^c}$$

If ε is small then for k small

$$\mathbb{P}\left(\int_0^{\frac{2\varepsilon}{|z_0|}} |z_s|\, ds < \varepsilon \right) \le \mathbb{P}\left(\sup_{0 \le s \le 2\varepsilon/|z_0|} |z_s - z_0| > k|z_0| \right)$$

$$\le \frac{C_{p,T}}{k^p |z_0|^p} \left(\frac{2\varepsilon}{|z_0|} \right)^{p\alpha}$$

$$\Rightarrow \quad \mathbb{E}\left(\left(\int_0^t |z_s|\, ds \right)^{-p} \right) < \infty.$$

Lemma 7.6 (Norris Lemma)

Consider a continuous semimartingale of the form

$$Y_t = y + \int_0^t a(s)\, ds + \sum_{i=1}^d \int_0^t u_i(s)\, dW_s^i$$

where

$$a(t) = \alpha + \int_0^t \beta(s)\, ds + \sum_{i=1}^d \int_0^t \gamma_i(s)\, dW_s^i$$

Suppose the processes a, u_i, β, γ_i are adapted and satisfy

$$c = \mathbb{E}\left(\sup_{0 \le t \le T} \left(|\beta(t)| + |\gamma(t)| + |a(t)| + |u(t)| \right)^p \right) < \infty$$

Fix $q > 8$. Then for all $r, \nu > 0$ s.t. $18r + 9\nu < q - 8$ there exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$ we have

$$\mathbb{P}\left(\int_0^T Y_t^2\, dt < \varepsilon^q,\ \int_0^T (|a(t)|^2 + |u(t)|^2)\, dt \geq \varepsilon \right)$$

$$\leq C_1 \varepsilon^{rp} + e^{-\varepsilon^{-\nu}}$$

from Hairer-Mattingly 2011 paper

"Most existing bounds on the inverse of the Malliavin matrix in a hypoelliptic situation make use of some version of Norris' lemma. In its form taken from [Norris 1986], it states that if a semimartingale $Z(t)$ is small and one has some control on the roughness of both its bounded variation part $A(t)$ and its quadratic variation process $Q(t)$, then both A and Q taken separately can be small

Extension: If a process is composed of

a "regular" part and "irregular" part

& then these two parts cannot cancel each other

Hairer - Mattingly work

$$Z(t) = A_\phi(t) + \sum_{\ell=1}^{M} \sum_{|\alpha|=\ell} A_\alpha(t) W_{\alpha_1}(t) \cdots W_{\alpha_\ell}(t)$$

Wiener polynomials

Proof of Theorem 7.4.

Step 1. We want to show that for all $t > 0$ and all $p \geq 2$ we have $\mathbb{E}[(\det Q_t)^{-p}] < \infty$

Q_t is the Malliavin matrix of X_t

$\mathbb{E}\left(|\det Y_t^{-1}|^p + |\det Y_t|^p\right) < \infty$

Suffices to show $\mathbb{E}\left[(\det C_t)^{-p}\right] < \infty$ for all $p \geq 2$

Step 2 Fix $t>0$ and then the problem is reduced to show that for all $p\geq 2$ we have

$$\sup_{|v|=1} \mathbb{P}\left(v^T C_t v \leq \varepsilon \right) \leq \varepsilon^p$$

for any $0<\varepsilon\leq \varepsilon_0(p)$

$$v^T C_t v = \sum_{j=1}^{d} \int_0^t \langle v, Y_s^{-1} A_j(X_s) \rangle^2 ds$$

Step 3 Fix smooth V, Apply Itô's formula to compute differential of $Y_t^{-1} V(X_t)$

$$d(Y_t^{-1} V(X_t)) = Y_t^{-1} \sum_{k=1}^{d} [A_k, V](X_t)\, dW_t^k$$

$$+ Y_t^{-1} \left\{ [A_0, V] + \frac{1}{2} \sum_{k=1}^{d} [A_k,[A_k, V]] \right\} (X_t) dt$$

Actually

$$X_t = x_0 + \sum_{j=1}^{d} \int_0^t A_j(X_s) dW_s^j + \int_0^t B(X_s) ds$$

$$Y_t = I + \sum_{\ell=1}^{d} \int_0^t \partial A_\ell(X_s) Y_s dW_s^\ell + \int_0^t \partial B(X_s) Y_s ds$$

$$Z_t = I - \sum_{\ell=1}^{d} \int_0^t Z_s \partial A_\ell(X_s)\, dW_s^\ell - \int_0^t Z_s \left[\partial B(X_s) - \sum_{\ell=1}^{d} \partial A_\ell(X_s)\partial A_\ell(X_s)\right] ds$$

$$Z_t = Y_t^{-1}$$

$$d(Y_t^{-1} V(X_t)) = d(Z_t V(X_t))$$

$$= (dZ_t) V(X_t) + Z_t\, dV(X_t) + \frac{1}{2} (dZ_t)(dV(X_t))$$

$$dZ_t = -\sum_{\ell=1}^{d} Z_t \partial A_\ell(X_t)\, dW_t^\ell - Z_t \left[\partial B(X_t) - \sum_{\ell=1}^{d} \partial A_\ell(X_t)\partial A_\ell(X_t)\right] dt$$

$$dV(X_t) = \sum_{\ell=1}^{d}\sum_{j=1}^{d} \frac{\partial V}{\partial x^j}(X_t) A_\ell^j(X_t)\, dW_t^\ell + \sum_{j=1}^{d} \frac{\partial V}{\partial x^j}(X_t) B^j(X_t)\, dt$$

$$+ \frac{1}{2} \sum_{i,j,\ell=1}^{d} \frac{\partial^2 V}{\partial x^j \partial x^i}(X_t) A_\ell^j A_\ell^i\, dt$$

$$(dZ_t)(dV(X_t)) = -\sum_{\ell=1}^{d}\sum_{j=1}^{d} Z_t \frac{\partial V}{\partial x^j}(X_t) A_\ell^j(X_t) \partial A_\ell(X_t)\, dt$$

So $$d(Y_t^{-1} V(X_t)) = Y_t^{-1} \sum_{k=1}^{d} [A_k, V](X_t)\, dW_t^k$$

$$+ Y_t^{-1}\left\{[A_0, V] + \frac{1}{2} \sum_{k=1}^{d} [A_k, [A_k, V]]\right\} dt$$

Step 4 We introduce the following sets of vector fields

$$\Sigma_0 = \{A_1, \ldots, A_d\}$$

$$\Sigma_n = \{[A_k, V],\ k=0,\ldots,d,\ V \in \Sigma_{n-1}\} \text{ if } n \geq 1$$

$$\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$$

and

$$\Sigma'_0 = \Sigma_0$$

$$\Sigma'_n = \Big\{ [A_k, V],\ k=0,\ldots,d,\ V \in \Sigma'_{n-1}$$

$$[A_0, V] + \frac{1}{2}\sum_{j=1}^{d} [A_j, [A_j, V]],\ V \in \Sigma'_{n-1} \Big\}$$

$$\text{if } n \geq 1$$

$$\Sigma' = \bigcup_{n=0}^{\infty} \Sigma'_n$$

We denote by $\Sigma_n(x)$ (resp. $\Sigma'_n(x)$) the subset of $\mathbb{R}^m$ obtained by freezing the variable x in the vector fields of Σ_n (resp. Σ_n')

Clearly the vector spaces spanned by $\Sigma(x_0)$ or by $\Sigma'(x_0)$ coincide, and under Hörmander's condition this vector space is $\mathbb{R}^m$.

$\Rightarrow \exists R > 0 \quad c > 0$ s.t.

$$\sum_{j=0}^{j_0} \sum_{v \in \Sigma_j'} \langle v, V(y) \rangle^2 \geq c$$

for all v and y with $|v| = 1$ and $|y - x_0| < R$

<u>step 5</u> For any $j = 0, 1, \dots, j_0$ we put $m(j) = 2^{-4j}$ and we define the set

$$E_j = \left\{ \sum_{v \in \Sigma_j'} \int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \leq \varepsilon^{m(j)} \right\}$$

Notice that $\{ v^T C_t v \leq \varepsilon \} = E_0$ because $m(0) = 1$.

Consider the decomposition

$$E_0 \subset (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{j_0-1} \cap E_{j_0}^c) \cup F$$

where $F = E_0 \cap E_1 \cap \cdots \cap E_{j_0}$

Then for any unit vector v we have

$$\mathbb{P}\left\{ v^T C_t v \le \varepsilon \right\} = \mathbb{P}(E_0) \le \mathbb{P}(F) + \sum_{j=0}^{j_0} \mathbb{P}(E_j \cap E_{j+1}^c)$$

<u>Step 6</u> $$\mathbb{P}(F) \le \mathbb{P}\left(\sum_{j=0}^{j_0} \sum_{V \in \Sigma_j'} \int_0^t \langle v, Y_s^{-1} \cancel{V}(X_s) \rangle^2 ds \le (j_0+1)\, \varepsilon^{m(j_0)} \right)$$

$$Z_s = \sum_{j=0}^{j_0} \sum_{V \in \Sigma_j'} \langle v, Y_s^{-1} V(X_s) \rangle^2$$

Lemma 7.5

$$\Rightarrow \mathbb{E}\left(\left| \sum_{j=0}^{j_0} \sum_{V \in \Sigma_j'} \int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \right|^{-p} \right) < \infty.$$

Thus for any $p \ge 1$ there exists ε_0 such that

$$\mathbb{P}(F) \leq \varepsilon^{p} \quad \text{for any } \varepsilon < \varepsilon_0$$

<u>Step 7</u> For any $j = 0, \dots, j_0$ the probability of the event $E_j \cap E_{j+1}^c$ is bounded by the sum with respect to $V \in \Sigma_j'$ of the probability that the two following event happens

$$\int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \leq \varepsilon^{m(j)}$$

and
$$\sum_{k=1}^{d} \int_0^t \langle v, Y_s^{-1} [A_k, V](X_s) \rangle^2 ds$$

$$+ \int_0^t \left\langle v, Y_s^{-1} \left([A_0, V] + \frac{1}{2} \sum_{j=1}^{d} [A_j, [A_j, V]] \right)(X_s) \right\rangle^2 ds$$

$$\geq \frac{\varepsilon^{m(j+1)}}{n(j)}$$

where $n(j) = \# \Sigma_j'$.

Consider the continuous semimartingale $\{\langle v, Y_s^{-1} V(X_s) \rangle\ s \geq 0\}$. The quadratic variation is

$$\sum_{k=1}^{d} \int_0^s \langle v, Y_\sigma^{-1} [A_k, V](X_\sigma) \rangle^2 d\sigma$$

bounded variation is

$$\int_0^s \left\langle v,\ Y_\sigma^{-1} \left\{ [A_0, V] + \frac{1}{2} \sum_{j=1}^{d} [A_j, [A_j, V]] \right\} (X_\sigma) \right\rangle d\sigma$$

Apply Norris Lemma

$$\mathbb{P}\,(E_j \cap E_{j+1}^c) \leq \varepsilon^p$$

for all $\varepsilon \leq \varepsilon_0$.

Lecture 3

On inviscid limit and related problems in turbulence

2-d Navier Stokes equations

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p &= \nu\Delta u + f \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ on } \mathbb{T}^2$$

ν —— viscosity

inviscid limit : letting $\nu \to 0$

asymptotic problems

Vorticity forumlation

$$\frac{\partial w}{\partial t} + u\cdot\nabla w - \nu\Delta w = \operatorname{curl} f$$

$u = Kw$ $\qquad$ $u = Kw = \nabla^{\perp}\Delta^{-1}w$ — Biot-Savart operator

We assume $\int_{\mathbb{T}^2} f(x,t)\,dx = 0$ for each t

<u>Classical literature</u>

"Kolmogorov seminar on selected questions of analysis 1958–1959" by Arnold, V. I. & Meshchalkin, L. D

Russian Mathematical Surveys <u>15</u>, 1, pp. 247–250 (1960)

Kolmogorov forcing $f = \begin{pmatrix} \gamma \sin y \\ 0 \end{pmatrix}$

<u>Conjecture</u>: "for small ν there must be a structually stable solution, that is, there is a non-trivial invariant measure μ_ν in $(x,y) \to (u,v)$ space such that $\mu_\nu \to 0$ as $\nu \to 0$, where limiting measure is concentrated on <u>continuous functions</u>"

Modern formulation of this problem

Let f be Random

$$f(x,t) = \alpha \sum_k b_k e_k(x) \dot{W}^k(t)$$

$$e_k(x) = \left(-\frac{ik_2}{|k|}, \frac{ik_1}{|k|} \right) e^{ikx}$$

divergence-free

Kuksin's scaling — inviscid scaling : $\alpha = c\sqrt{\nu}$

$$du + (u \cdot \nabla u + \nabla p - \nu \Delta u) dt = c\sqrt{\nu} \sum_k b_k \vec{e}_k(x) dW^k(t)$$

Vorticity equation

$$dw + (u \cdot \nabla w - \nu \Delta w) dt = c\sqrt{\nu} \sum_k g_k dW^k(t) e^{ikx}$$

$$g_k = |k| b_k$$

Slogan Construct invariant measure $\{\mu_\nu\}_{\nu > 0}$

letting $\nu \to 0$ limiting measure μ_0

$$\boxed{\int \|w\|_{L_\infty} d\mu_0(w) < +\infty.}$$

Kuksin measure

$$\boxed{\int \|w\|_{C^0}\, d\mu_0(w) < \infty \quad ?? \quad \text{unknown}}$$

corresponding to
Kolmogorov's conjecture

Apply Itô's formula

$$d\int_{\mathbb{T}^2} w^2\, dx = \int_{\mathbb{T}^2} \left(w\, dw + \frac{1}{2}\, dw\, dw \right) dx$$

so

$$\mathbb{E}\left(\int_{\mathbb{T}^2} |\nabla w|^2 dx \right) = \int \|\nabla w\|_{L_2}^2\, d\mu_\nu(w)$$

$$\overset{(4)}{=\!=} \frac{c^2}{2} \sum_k |g_k|^2$$

Take $H^1 = \left\{ w \in H^1(\mathbb{T}^2) ;\ \int_{\mathbb{T}^2} w\, dx = 0 \right\}$

as $\nu \to 0+$, the family of measures μ_ν has a subsequence converging weakly to a limit μ_0 supported on H^1, s.t.

$$\mathbb{E}\left(\|\nabla w\|_{L_2}^2\right) = \int \|\nabla w\|_{L_2}^2 \, d\mu_0(w) \leq \frac{c^2}{2} \sum_k |g_k|^2$$

$\mu_0(w)$ — Kuksin measure [Kuksin 2004]

Set up and Notations

$$\begin{aligned} dw + (u \cdot \nabla w - \nu \Delta w)\, dt &= \sqrt{\nu}\, \sigma \, dW \\ &= \sqrt{\nu} \sum_k \sigma_k \, dW^k \end{aligned} \qquad (1)$$

$$w(0) = w_0$$

$$\underbrace{\int_{\mathbb{T}^2} w_0 \, dx = 0 \ , \quad \int_{\mathbb{T}^2} \sigma \, dx}_{\text{Solution always has mean } 0} = 0$$

Sobolev spaces

$$H^k = \left\{ w \in H^k_{per} : \int_{\mathbb{T}^2} w_0 \, dx = 0 \right\}$$

$H^0 = L^2_{2,per}$, $\|\cdot\|_k$, $(\cdot,\cdot)_k$

L_p , $p \geq 1$, $\|\cdot\|_{L_p}$

Solution of 15) $w^{\nu}(t, w_0) = w(t, w_0)$

$$\sum_{\ell} \|\sigma_\ell\|_{L_2}^2 < \infty \Rightarrow w(\cdot, w_0)$$

$$\in C([0,\infty); H^0) \cap L_{2,loc}([0,\infty); H^1)$$

for any $w_0 \in H^0$

Higher regularity result also available

$$\|\sigma\|_{L_2} = \left(\sum_{\ell} \|\sigma_\ell\|_{L_2}^2\right)^{\frac{1}{2}}$$

$$\|\sigma\|_{L_p} = \left(\int_{\mathbb{T}^2} \left(\sum_{\ell} |\sigma_\ell(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$

$$\|\sigma\|_{L_\infty} = \sup_{x \in \mathbb{T}^2} \sum_{\ell} |\sigma_\ell(x)|^2$$

Markovian framework

$$P_t(w_0, \Gamma) = \mathbb{P}(w(t, w_0) \in \Gamma) \qquad t \geq 0, \quad w_0 \in H^0$$

$\Gamma \in B(H^k)$
$\uparrow$ Borel sets in H^k

$C_b(H^k)$ $\downarrow$ bounded continuous

$M_b(H^k)$ $\downarrow$ bounded Borel measurable

Markov semigroup

$$P_t\phi(w_0) = \mathbb{E}\,\phi(w(t,w_0)) = \int_{H^k} \phi(w)\, P_t(w_0, dw)$$

$$P_t : M_b(H^k) \longrightarrow M_b(H^k)$$

$w(t, w_0)$ depends continuously on $w_0 \in H^k$

$\Rightarrow P_t$ is Feller i.e. $P_t: C_b(H^k) \circlearrowleft$

$Pr(H^k)$ is the set of Borel probability measures on H^k

$\mu \in Pr(H^0)$ is an invariant measure if and only

if $$\int_{H^k} \phi(w_0)\, d\mu(w_0) = \int_{H^k} P_t\phi(w_0)\, d\mu(w_0)$$

for every $t \geq 0$

Stationary solution $w_s^\nu(\cdot)$

$$\mu_\nu(\cdot) = \mathbb{P}(w_s^\nu(t) \in \cdot)$$

Balance relations

$$\mathbb{E}\|w_s^\nu\|_{L_2}^2 = \int \|w_0\|_{L_2}^2 \, d\mu_\nu(w_0) = \frac{1}{2}\sum_k \|\rho_k\|_{L_2}^2$$

$$\mathbb{E}\|w_s^\nu\|_{H^1}^2 = \int \|w_0\|_{H^1}^2 \, d\mu_\nu(w_0) = \frac{1}{2}\sum_k \|\sigma_k\|_{L_2}^2$$

Further

$$\mathbb{E}\exp\left(\delta \|w_s^\nu\|_{L_2}^2\right) \leq C < \infty$$

where C is independent of ν

$$\mathbb{E}\|w_s^\nu\|_{H^{k+1}}^2 = \int \|w_0\|_{H^{k+1}}^2 \, d\mu_\nu(w_0) \leq C(\nu)$$

$C(\nu)$ is finite.

Theorem [Vicol - Glatt-Holtz - Sverak '12]

$$\mathbb{E}\|w_s^\nu\|_{L_\infty} \leq C < \infty, \quad C = C(\sigma)$$

is independent of ν

$\{\mu_\nu\}_{\nu>0}$ $\exists \mu_0$ and $j \to \infty$

$\mu_{\nu_j} \to \mu_0$ weakly in $Pr(H^0)$ s.t. $\mu_0(L_\infty) = 1$

Proof.

$$\tilde{t} = \frac{t}{\nu}$$

$$d\tilde{\omega} + \left(\frac{1}{\nu}\tilde{u}\cdot\nabla\tilde{\omega} - \Delta\tilde{\omega}\right)dt = \sigma\, d\tilde{W}\,,\quad \tilde{\omega}(0) = \omega_s(0)$$

$$\tilde{\omega}(t, x) = \omega(\tilde{t}, x)$$

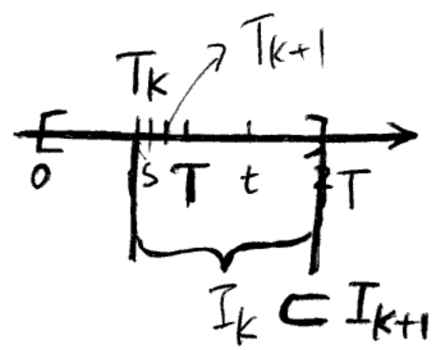
$$\tilde{u}(t, x) = u(\tilde{t}, x)$$

$$\tilde{W}(t) = \sqrt{\nu}\, W(\tilde{t})$$

Fix $T > 0$, $p > 1$, $\{T_k\}$ is an increasing sequence of times

$$T_0 = 0\,,\quad T_k \nearrow T \text{ as } k \to \infty$$

$$I_k = [T_k, 2T] \longrightarrow [T, 2T]$$



L_p - Itô lemma

$$d\|\omega\|_{L_p}^p = \left(\frac{p}{\nu}T_{1,p} + p\,T_{2,p} + \frac{p(p-1)}{2}T_{3,p}\right)dt + p\sum_m S_{m,p}\, dW^m \qquad (2)$$

Obtained by applying Itô's formula to $|\omega|^p$ and then integrate on $\mathbb{T}^2$.

$$T_{1,p}(t) = -\int_{\mathbb{T}^2} u(t,x) \cdot \nabla w(t,x)\, w(t,x) \left| w(t,x) \right|^{p-2} dx$$

$$T_{2,p}(t) = \int_{\mathbb{T}^2} \Delta w(t,x)\, w(t,x) \left| w(t,x) \right|^{p-2} dx$$

$$T_{3,p}(t) = \sum_m \int_{\mathbb{T}^2} \left| w(t,x) \right|^{p-2} \sigma_m^2(x)\, dx$$

$$S_{m,p}(t) = \int_{\mathbb{T}^2} \sigma_m(x)\, w(t,x) \left| w(t,x) \right|^{p-2} dx$$

$$p \geq 2$$

Let $s \in [T_k, T_{k+1}]$, $t > s$

$$\|w(t)\|_{L_p}^p - p \int_s^t T_{2,p}(\tau)\, d\tau$$

$$= \|w(s)\|_{L_p}^p + \frac{p(p-1)}{2} \int_s^t T_{3,p}(\tau)\, d\tau + p \sum_m \int_s^t S_{m,p}(\tau)\, dW_\tau^m$$

$$= \|w(s)\|_{L_p}^p + \frac{p(p-1)}{2} \int_s^t T_{3,p}(\tau)\, d\tau + p \sum_m \int_{T_k}^t S_{m,p}(\tau)\, dW_\tau^m$$

$$- p \sum_m \int_{T_k}^s S_{m,p}(\tau)\, dW_\tau^m$$

Take supremum over all $t \in I_{k+1}$ we obtain

$$\|w\|^p_{L_\infty(I_{k+1};L_p)} - p\int_{I_{k+1}} T_{2,p}(\tau)\,d\tau$$

$T_{2,p} \leq 0$ will be proved later

$$\leq \|w(s)\|^p_{L_p} + \frac{p(p-1)}{2}\int_{I_k} |T_{3,p}(\tau)|\,d\tau$$

$$+ p \sup_{t\in I_{k+1}} \left| \sum_m \int_{T_k}^{t} S_{m,p}(\tau)\,dW^m_\tau \right|$$

$$+ p \left| \sum_m \int_{T_k}^{s} S_{m,p}(\tau)\,dW^m_\tau \right|$$

$$\leq \|w(s)\|^p_{L_p} + \frac{p(p-1)}{2}\int_{I_k} |T_{3,p}(\tau)|\,d\tau + 2p \sup_{t\in I_k} \left| \sum_m \int_{T_k}^{t} S_{m,p}(\tau)\,dW^m_\tau \right|$$

Notations: ~~For~~ any $0 \leq r \leq t$

$$M_{[r,t],p} = \sum_m \int_r^t S_{m,p}(\tau)\,dW^m_\tau$$

$$M^*_{[r,t],p} = \sup_{s\in[r,t]} \left| \sum_m \int_r^s S_{m,p}(\tau)\,dW^m_\tau \right|$$

$\langle M_{[r,\cdot]},p \rangle_t$ is the quadratic variation of $M_{[r,t]},p$

$$\langle M_{[r,\cdot]},p \rangle_t = \int_r^t \sum_m S_{m,p}^2 \, ds$$

Burkholder - Davis - Gundy inequality

$Z \geqslant 0$ r.v. $r<t$ $\delta>0$

$$\mathbb{E}\left(M^*_{[r,t]},p \vee Z \right)^{\delta} \leq C_{BDG}(\delta) \cdot \mathbb{E}\left(\langle M_{[r,\cdot]},p \rangle_t^{\frac{1}{2}} \vee Z \right)^{\delta}$$

The constant $C_{BDG}(\delta)$ is universal which depends only on δ and is independent of the form of the martingale $M^*_{[r,t)},p$ or Z, also, $\exists\ \delta_0 > 0$

s.t. $C_{BDG}(\delta) \leq 2\,\delta^{1/2}$ whenever $\delta < \delta_0$

Reference. Denis, L., Matoussi, A., Stoica, L.

L_p estimates for the uniform norm of solutions of quasilinear SPDEs, PTRF, 133(4), pp. 437-463

2005

So

$$\|w\|^p_{L_\infty(I_{k+1};L_p)} - p\int_{I_{k+1}} T_{2,p}(\tau)\,d\tau$$

$$\leq \frac{1}{T_{k+1}-T_k}\int_{T_k}^{T_{k+1}} \|w_{(s)}\|^p_{L_p}\,ds + \frac{p(p-1)}{2}\int_{I_k} |T_{3,p}(\tau)|\,d\tau$$

$$+ 2p\, M^*_{[T_k,2T],p}$$

$$\leq \frac{1}{(T_{k+1}-T_k)^{\frac{1}{2}}}\|w\|^p_{L_{2p}(I_k;L_p)} + \frac{p(p-1)}{2}\int_{I_k} |T_{3,p}(\tau)|\,d\tau$$

$$+ 2p\, M^*_{[T_k,2T],p} \qquad (3)$$

Moser iteration technique

$$v = |w|^{\frac{p}{2}} \qquad \|w\|^p_{L_p} = \|v\|^2_{L_2}$$

$$|\nabla v|^2 = \frac{p^2}{4}|\nabla w|^2 |w|^{p-2}$$

$$-p\,T_{2,p} = p(p-1)\int_{\mathbb{T}^2} |\nabla w|^2 |w|^{p-2}\,dx = \frac{4(p-1)}{p}\|\nabla v\|^2_{L_2} \geq 2\|\nabla v\|^2_{L_2}$$

$$-p T_{2,p} = -p \int_{\mathbb{T}^2} \Delta w \cdot w |w|^{p-2} \, dx$$

$$= +p \int_{\mathbb{T}^2} \nabla w \cdot \nabla (w |w|^{p-2}) \, dx$$

$$\nabla (w |w|^{p-2}) = \nabla w \, |w|^{p-2} + w \, \nabla (|w|^{p-2})$$

$$= \nabla w \, |w|^{p-2} + \frac{p-2}{2} |w|^{p-4} \cdot 2 \sum_{i,j} w_j \frac{\partial w_j}{\partial x_i} w_i$$

$$= (p-1) \, \nabla w \, |w|^{p-2}$$

Sobolev embedding in dimension 2 $\quad H^1 \hookrightarrow L_{2^*}$

$$\frac{1}{2C_S} \| v \|_{L_{2^*}}^2 \leq \| \nabla v \|_{L_2}^2 + \| v \|_{L_2}^2$$

$$W^{k,p} \hookrightarrow W^{\ell,q}$$

$$\frac{1}{q} = \frac{1}{p} - \frac{k-\ell}{n}$$

$$\frac{1}{2^*} = \underbrace{\frac{1}{2} - \frac{1}{2}}_{=0} \qquad 2^* \in [2, \infty)$$

$C_S > 0$ depends only on the size of the box and the choice of 2^*

We chose $2^* = 4$

Thus we get

$$\|w\|^p_{L_\infty(I_{k+1};\, L_p)} - p\int_{I_{k+1}} T_{2,p}(\tau)\, d\tau$$

$$\geq \|v\|^2_{L_\infty(I_{k+1};\, L_2)} + 2\,\|\nabla v\|^2_{L_2(I_{k+1};\, L_2)}$$

$$= \|v\|^2_{L_\infty(I_{k+1};\, L_2)} - 2\,\|v\|^2_{L_2(I_{k+1};\, L_2)}$$

$$+ 2\left(\|v\|^2_{L_2(I_{k+1};\, L_2)} + \|\nabla v\|^2_{L_2(I_{k+1};\, L_2)}\right) \tag{4}$$

Assume

$$4\,|I_{k+1}| = 2(2T - T_{k+1}) \leq 1 \iff \boxed{T \leq \frac{1}{8}}$$

So that

$$\|v\|^2_{L_\infty(I_{k+1};\, L_2)} + 2\,\|\nabla v\|^2_{L_2(I_{k+1};\, L_2)}$$

$$\geq \|v\|^2_{L_\infty(I_{k+1};\, L_2)}\left(1 - 2\,|I_{k+1}|\right) + \frac{1}{C_S}\,\|v\|^2_{L_2(I_{k+1};\, L_4)}$$

$$\geq \frac{1}{2}\,\|v\|^2_{L_\infty(I_{k+1};\, L_2)} + \frac{1}{C_S}\,\|v\|^2_{L_2(I_{k+1};\, L_4)} \tag{5}$$

$L_t^p L_x^q$ interpolation inequality

$1 \le p_1, p_2, q_1, q_2, r_1, r_2 \le \infty$, $0 \le \gamma \le 1$

$$\frac{1}{r_1} = \frac{\gamma}{p_1} + \frac{1-\gamma}{q_1} \quad , \quad \frac{1}{r_2} = \frac{\gamma}{p_2} + \frac{1-\gamma}{q_2}$$

$$g \in L_{p_1}(I; L_{p_2}) \cap L_{q_1}(I; L_{q_2})$$

$$\|g\|_{L_{r_1}(I; L_{r_2})} \le \|g\|^{\gamma}_{L_{p_1}(I; L_{p_2})} \|g\|^{1-\gamma}_{L_{q_1}(I; L_{q_2})}$$

$r_1 = 5$, $r_2 = \frac{5}{2}$, $p_1 = \infty$, $p_2 = 2$, $q_1 = 2$, $q_2 = 4$, $\gamma = \frac{3}{5}$

Combined with ε- Young inequality $\frac{a^p}{p} + \frac{b^q}{q} \ge ab$

$$\frac{1}{2C_5^{2/5}} \|v\|^2_{L_5(I_{k+1}; L_{5/2})}$$

$$\le \frac{1}{2C_5^{2/5}} \|v\|^{6/5}_{L_\infty(I_{k+1}; L_2)} \|v\|^{4/5}_{L_2(I_{k+1}; L_4)}$$

$$\le \frac{1}{2} \|v\|^2_{L_\infty(I_{k+1}; L_2)} + \frac{1}{C_5} \|v\|^2_{L_2(I_{k+1}; L_4)} \qquad (16)$$

$$p = \frac{10}{6} \quad q = \frac{10}{4}$$

(4) (5) (6) $\Rightarrow$

LHS of (3)

$$\geq \frac{1}{C_S'} \|v\|^2_{L_5(I_{k+1};\, L_{5/2})}$$

$$= \frac{1}{C_S'} \|\omega\|^p_{L^{5p/2}(I_{k+1};\, L^{5p/4})}$$

$$= \frac{1}{C_S'} \|\omega\|^p_{L_{2\lambda p}(I_{k+1};\, L_{\lambda p})}$$

Here $4|I_{k+1}| = 2(2T - T_{k+1}) \leq 1$

$C_S' = 2C_S^{2/5} \vee 1$

$\lambda = 5/4$

Now we consider RHS of (3)

$$\frac{p(p-1)}{2} T_{3,p} \leq \frac{p(p-1)}{2} \|\omega\|^{p-2}_{L_p} \|\sigma\|^2_{L_p}$$

$$\Rightarrow \frac{p(p-1)}{2} \int_{I_k} T_{3,p}(\tau)\, d\tau \leq \frac{p(p-1)}{2} \|\sigma\|^2_{L_p} |I_k|^{\frac{p+2}{2p}} \|\omega\|^{p-2}_{L_{2p}(I_k;\, L_p)}$$

Hölder inequality

So finally we obtain the following inequality

$$\frac{1}{C_s'}\left(\|w\|_{L_{2\lambda p}(I_{k+1};\,L_{\lambda p})} \vee \|\sigma\|_{L_\infty}\right)^p$$

$$\leq \frac{1}{C_s'}\left(\|w\|^p_{L_{2\lambda p}(I_{k+1};\,L_{\lambda p})} + \|\sigma\|^p_{L_\infty}\right)$$

$$\leq \frac{1}{(T_{k+1}-T_k)^{\frac{1}{2}}}\|w\|^p_{L_{2p}(I_k;\,L_p)}$$

$$+ \operatorname{Vol}(\mathbb{T}^2)^{\frac{2}{p}} p^2 |I_k|^{\frac{p+2}{2p}} \|\sigma\|^2_{L_\infty} \|w\|^{p-2}_{L_{2p}(I_k;\,L_p)}$$

$$+ \|\sigma\|^p_{L_\infty} + 2p\, M^*_{[T_k,\,2T],\,p}$$

Here $\|\sigma\|_{L_p} \leq |\mathbb{T}^2|^{\frac{1}{p}} \|\sigma\|_{L_\infty}$

Let us now define

$$k(p,T) := 4C_s'\left(\frac{1}{(T_{k+1}-T_k)^{\frac{1}{2}}} + \operatorname{Vol}(\mathbb{T}^2)^{\frac{2}{p}} p^2 |I_k|^{\frac{p+2}{2p}} + 1 + 2p\operatorname{Vol}(\mathbb{T}^2)^{\frac{1}{p}} |I_k|^{\frac{1}{2p}}\right)$$

Thus

$$\mathbb{E}\left(\|w\|_{L_{2\lambda p}(I_{k+1};\, L_{\lambda p})} \vee \|\sigma\|_{L_\infty} \right)$$

$$\leq (\kappa(p,T))^{\frac{1}{p}} \, \mathbb{E}\left(\left(\|w\|_{L_{2p}(I_k;\, L_p)} \vee \|\sigma\|_{L_\infty} \right)^p \vee \left(Vol(\mathbb{T}^2)\right)^{-\frac{1}{p}} \right.$$

$$\left. |I_k|^{-\frac{1}{2p}} \, M^*_{[T_k,\, 2T],\, p} \right)^{\frac{1}{p}}$$

$$\leq (\kappa(p,T))^{\frac{1}{p}} \, C_{BDG}(p^{-1}) \, \mathbb{E}\left(\left(\|w\|_{L_{2p}(I_k;\, L_p)} \vee \|\sigma\|_{L_\infty} \right)^p \right.$$

$$\left. \vee \, Vol(\mathbb{T}^2)^{-\frac{1}{p}} \, |I_k|^{-\frac{1}{2p}} \langle M_{[T_k,\, \cdot],\, p} \rangle_{2T}^{1/2} \right)^{\frac{1}{p}}$$

quadratic variation estimate

$$\langle M_{[T_k,\, \cdot],\, p} \rangle_{2T}^{1/2} = \left(\int_{T_k}^{2T} \sum_m S_{m,p}^2 \, dt \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{I_k} \left(\int_{\mathbb{T}^2} \left(\sum_m \sigma_m^2 \right)^{1/2} |w|^{p-1} dx \right)^2 dt \right)^{1/2}$$

$$\leq \left(\int_{I_k} \left(\int_{\mathbb{T}^2} \left(\sum_m \sigma_m^2 \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{T}^2} |w|^p dx \right)^{\frac{2(p-1)}{p}} dt \right)^{\frac{1}{2}}$$

$$= \int_{I_k} \|\sigma\|_{L_p}^2 \, \|w\|_{L_p}^{2(p-1)} \, dt.$$

$$\leq |I_k|^{\frac{1}{2p}} \|\sigma\|_{L_p} \|w\|^{p-1}_{L_{2p}(I_k;\, L_p)}$$

$$\leq \mathrm{Vol}(\mathbb{T}^2)^{\frac{1}{p}} |I_k|^{\frac{1}{2p}} \|\sigma\|_{L_\infty} \|w\|^{p-1}_{L_{2p}(I_k;\, L_p)}$$

$$\leq \mathrm{Vol}(\mathbb{T}^2)^{\frac{1}{p}} |I_k|^{\frac{1}{2p}} \left(\|\sigma\|_{L_\infty} \vee \|w\|_{L_{2p}(I_k;\, L_p)} \right)^{p}$$

Final Estimate: Moser iteration

$$\mathbb{E}\left(\|w\|_{L_{2\lambda p}(I_{k+1};\, L_{\lambda p})} \vee \|\sigma\|_{L_\infty} \right)$$

$$\leq \left(\kappa(p,T)\right)^{\frac{1}{p}} C_{BDG}(p^{-1})\, \mathbb{E}\left(\|w\|_{L_{2p}(I_k;\, L_p)} \vee \|\sigma\|_{L_\infty} \right)$$

for all $p \geq 2$

$p = p_k = 2\lambda^k$, $k \geq 0$, $\lambda = \frac{5}{4}$

$$A_k = \mathbb{E}\left(\|w\|_{L_{2p_k}(I_k,\, L_{p_k})} \vee \|\sigma\|_{L_\infty} \right)$$

$$a_k = \kappa(p_k, T)^{\frac{1}{p_k}}\, C_{BDG}(p_k^{-1}).$$

$$A_{k+1} \leq a_k A_k$$

$$\mathbb{E} \sup_{t \in [T, 2T]} \|\omega(t,\cdot)\|_{L_\infty} \leq A_\infty$$

$$\leq \left(\prod_{k \geq 0} a_k\right) \mathbb{E}\left(\|\omega\|_{L_4([0,2T]; L_2)} \vee \|v\|_{L_\infty}\right)$$

So $\prod_{k \geq 0} a_k \leq C \exp\left(\sum_{k \geq 0} \frac{\log \varkappa(p_k, T)}{p_k} \right)$

C is ν and T- independent constant

Remember $T \leq \frac{1}{8}$

Set $T_k = T(1-\lambda^{-k})$

$$T_{k+1} - T_k = T\lambda^{-k}(1+\lambda^{-1}) \geq T\lambda^{-k}/2 = T p_k^{-1}$$

Recall the definition of $\varkappa(p_k, T)$

$$\varkappa(p_k, T) \leq 4C_s'\left(T^{-\frac{1}{2}} p_k^{\frac{1}{2}} + \mathrm{Vol}(\mathbb{T}^2)^{\frac{2}{p_k}} p_k^2 (2T)^{\frac{p_k+2}{2p_k}} \right.$$

$$\left. + 1 + 2p_k \mathrm{Vol}(\mathbb{T}^2)^{\frac{1}{p_k}} (2T)^{\frac{1}{2p_k}} \right)$$

$(p_k > 1)$ $\leq C p_k^2 (T^{-\frac{1}{2}} + 1)$

We used $|I_k| \leq 2T \leq 1/4$

C is a sufficiently large T-independent constant.

Use $\displaystyle\sum_{k\geq 0} \frac{1}{p_k} = \frac{5}{2}$

$$\sum_{k\geq 0} p_k^{-1} \log p_k < \infty$$

so $\displaystyle\prod_{k\geq 0} a_k \leq C\left(T^{-\frac{1}{2}}+1\right)^{\frac{5}{2}} \leq C\left(T^{-\frac{5}{4}}+1\right)$

for sufficiently large ν - and T- independent constant C

<u>Conclusion</u>

$$\mathbb{E} \sup_{t\in[T,2T]} \|\tilde{\omega}(t)\|_{L_\infty} \leq C\left(T^{-\frac{5}{4}}+1\right) \mathbb{E}\left(\|\tilde{\omega}\|_{L_4([0,2T];L_2)} \vee \|\sigma\|_{L_\infty}\right)$$

for any $T \leq \frac{1}{8}$, C is independent of T and ν

Rescaling to the original variable $\omega(t) = \tilde{\omega}(\nu t)$

$$\mathbb{E} \sup_{t\in[\frac{T}{\nu},\frac{2T}{\nu}]} \|\omega(t)\|_{L_\infty} \leq C\left(T^{-\frac{5}{4}}+1\right)\left(\left(\int_0^{T/\nu} \nu\, \mathbb{E}\left(\|\omega(s)\|_{L_2}^4\right) ds\right)^{1/4} + \|\sigma\|_{L_\infty}\right)$$

for any $T \leq \frac{1}{8}$

$\omega = \omega_S^\nu$ Stationary solution corresponding to

μ_ν, $T = \frac{1}{8}$

$$\mathbb{E}\,\|\omega_S^\nu\|_{L_\infty} \leq \mathbb{E} \sup_{t \in [\frac{1}{8\nu}, \frac{1}{4\nu}]} \|\omega_S^\nu(t)\|_{L_\infty}$$

$$\leq C\left(\left(\int_0^{\frac{1}{8\nu}} \nu\, \mathbb{E}\left(\|\omega_S^\nu(s)\|_{L_2}^4\right) ds\right)^{\frac{1}{4}} + \|\sigma\|_{L_\infty}\right)$$

$$\leq C\left(\left(\int_0^{\frac{1}{8\nu}} \frac{2\nu}{\delta^2}\, \mathbb{E} \exp\left(\delta \|\omega_S^\nu(s)\|_{L_2}^2\right) ds\right)^{1/4} + \|\sigma\|_{L_\infty}\right)$$

$$\leq C_\sigma$$

for a constant C_σ independent of ν $\square$

Lecture 4

The 2-d Euler equation and related topics

2-d Euler $\quad u = (u_1, u_2) \qquad \omega \in \mathbb{R}$

$$\left.\begin{aligned} &\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0 \\ &\omega(x, 0) = \omega_0(x) \end{aligned} \qquad x \in D \subseteq \mathbb{R}^2, \quad t \geq 0 \right\} \quad (1)$$

$$\nabla^{\perp} = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right)$$

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

ψ — stream function, $\nabla^{\perp}\psi = u$

so $\quad \Delta \psi = \omega$

Domain $D \subset \mathbb{R}^2$, boundary condition $u(x,t) \cdot n(x) = 0$ for $x \in \partial D$

We mainly consider in this lecture the case when

$D = \mathbb{R}^2$ or $D = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

Biot - Savart law: $u = (\nabla^{\perp} \Delta^{-1}) \omega$

$$\psi(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \, \omega(y,t) \, dy$$

$$\Rightarrow \quad u(x,t) = \int_{\mathbb{R}^2} K_2(x-y) \, \omega(y,t) \, dy \qquad (2)$$

where $K_2(x) = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)^T$

(Biot - Savart kernel)

∇K_2

[redacted] is standard singular integral operator

— harmonic analysis needed

for general domain D we have

$$u(x,t) = \nabla^{\perp} \int_D G_D(x,y) \, \omega(y,t) \, dy$$

G_D — Green's function for Dirichlet problem in D

$$K_D(x,y) = \nabla^{\perp} G_D(x,y)$$

$$u(x,t) = \int_D K_D(x,y) \, \omega(y,t) \, dy$$

K_D behaves very similarly as K_2

What can we say from (2)? Potential Estimates & Harmonic Analysis

This is crucial point!

Distributional Derivative $\overset{\mathcal{D}}{\partial_{x_j}} K_2$

$$K_2(x) = \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right) \in L^1_{loc}(\mathbb{R}^2)$$

For any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} K_2(x)\, \partial_{x_j} \varphi(x)\, dx \xlongequal[\text{convergence}]{\text{dominated}} \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} K_2(x)\, \partial_{x_j} \varphi(x)\, dx$$

$$\xlongequal{\text{Green's formula}} \lim_{\varepsilon \downarrow 0} \left\{ -\int_{|x| \geq \varepsilon} \partial_{x_j} K_2(x)\, \varphi(x)\, dx + \int_{|x| = \varepsilon} K_2(x) \varphi(x) \frac{x_j}{|x|}\, dx \right\}$$

$$= -\text{P.V.} \int_{\mathbb{R}^2} \partial_{x_j} K_2(x)\, \varphi(x)\, dx + \lim_{\varepsilon \downarrow 0} \int_{|x| = \varepsilon} K_2(x)\, \varphi(x) \frac{x_j}{|x|}\, dx$$

We have $\displaystyle\int_{|x| = \varepsilon} K_2(x)\, \varphi(x) \frac{x_j}{|x|}\, dx$

$$= \int_{|x| = 1} K_2(\varepsilon x)\, \varphi(\varepsilon x) \frac{\cancel{\varepsilon} x_j}{\cancel{\varepsilon} \cancel{|x|}}\, d(\varepsilon x) = \int_{|x| = 1} K_2(x)\, \varphi(\varepsilon x)\, x_j\, dx$$

Thus

$$\int_{\mathbb{R}^2} \partial_{x_j} K_2(x)\varphi(x)\,dx = \text{P.V.}\int_{\mathbb{R}^2} \partial_{x_j} K_2(x)\varphi(x)\,dx$$

$$- \int_{|x|=1} \varphi(0) K_2(x)\, x_j\, dx \tag{3}$$

We estimate

$$\left| \partial_{x_j} K_2(x) \right| \leq \frac{C}{|x|^2} \tag{4}$$

and we calculate

$$\int_{|x|=1} K_2(x)\, x_j\, dx = \int_{|x|=1} \frac{1}{2\pi}(-x_2, x_1)\, x_j\, dx$$

$$= \left(- \int_{|x|=1} x_2 x_j\, ds\,, \int_{|x|=1} x_1 x_j\, dx \right)$$

so that

$$\nabla u(x) = \text{P.V.}\int_{\mathbb{R}^2} \nabla K_2(x,y)\,\omega(y)\,dy + c\,\omega(x) \tag{5}$$

What are the consequences?

Lemma 1 (A-priori bound)

$$\|\nabla u(x,t)\|_{L_\infty} \le C(\alpha)\left(1 + \|w_0\|_{L_\infty}(1+\log_+ \|w(x,t)\|_{C^\alpha})\right) \tag{6}$$

Proof. By (5) we see that

$$\nabla u(x,t) = P.V. \int_{\mathbb{R}^2} \nabla K_2(x,y)\, w(y,t)\, dy + c\, w(x,t)$$

We will have $\|w(y,t)\|_{L_\infty} \le \|w_0\|_{L_\infty}$ by particle trajectory method that we mention later soon.

Thus by using (4) we get

$$\left| \int_{B_\delta^c(x)} \nabla K_2(x,y)\, w(y,t)\, dy \right| \le C\|w_0\|_{L_\infty} \int_{B_\delta^c(x)} \frac{1}{|x-y|^2}\, dy$$

$$\le C\|w_0\|_{L_\infty}(1+\log \delta^{-1})$$

$$\left| P.V. \int_{B_\delta(x)} \nabla K_2(x,y)\, w(y,t)\, dy \right|$$

$$= \left| \int_{B_\delta(x)} \nabla K_2(x,y)\,(w(y,t) - w(x,t))\, dy \right| \le C\|w(x,t)\|_{C^\alpha} \int_0^\delta r^{-1+\alpha}\, dr$$

$$\le C(\alpha)\, \delta^\alpha \|w(x,t)\|_{C^\alpha} \le C(\alpha)$$

Choose $\delta = \|\omega(x,t)\|_{C^\alpha}^{-1/\alpha}$ we get the bound $\square$

<u>Lemma 2</u>. Suppose $\omega(\cdot,t) \in L_{\infty,0}(\mathbb{R}^2)$ have compact support, then

$$\| \nabla u(x,t) \|_{L_p(\mathbb{R}^2)} \le C(\|\omega_0\|_{L_\infty})\, p \qquad (7)$$

Proof of (7) uses again (5) and singular integral operators (Calderon – Zygmund theory). We will skip it and refer to pp. 322-325 of [Bertozzi, A., Majda, A. Vorticity and incompressible flows. Cambridge University Press, 2002.]

<u>Particle trajectory method</u>

View (1) as a <u>transport equation</u>, with flow map $\Phi_t(x)$

$$\frac{d\,\Phi_t(x)}{dx} = u(\Phi_t(x), t), \qquad \Phi_0(x) = x \qquad (8)$$

Then $\omega(x,t) = \omega_0(\Phi_t^{-1}(x))$

$$\Rightarrow \|\omega(x,t)\|_{L_\infty} \le \|\omega_0\|_{L_\infty}, \quad \|\omega(x,t)\|_{L_1} \le \|\omega_0\|_{L_1}$$

More important consequenses: regularity a-priori bounds

Lemma 3 (Double exponential growth of $\|\nabla w\|_{L_\infty}$)

We have the a-priori estimate

$$\|\nabla w(x,t)\|_{L_\infty(\mathbb{R}^2)} \le \left(C\,\|\nabla w_0\|_{L_\infty}\right)^{\exp(C\|w_0\|_{L_\infty} t)} \tag{9}$$

Proof. By (8) we get

$$\left|\frac{\partial_t\,|\Phi_t(x)-\Phi_t(y)|}{|\Phi_t(x)-\Phi_t(y)|}\right| \le \|\nabla u\|_{L_\infty}$$

$$\le c\left(1+\|w_0\|_{L_\infty}\left(1+\log_+\|\nabla w(x,t)\|_{L_\infty}\right)\right)$$

Here we use Lemma 1

This after integration gives

$$f(t)^{-1} \le \frac{|\Phi_t(x)-\Phi_t(y)|}{|x-y|} \le f(t)$$

where $f(t) = \exp\left(c\int_0^t \left(1+\|w_0\|_{L_\infty}\left(1+\log_+\|\nabla w(x,s)\|_{L_\infty}\right)\right)ds\right)$

Similarly

$$f(t)^{-1} \le \frac{|\Phi_t^{-1}(x)-\Phi_t^{-1}(y)|}{|x-y|} \le f(t)$$

$$\|\nabla w(x,t)\|_{L^\infty} = \sup_{x,y} \frac{|w_0(\Phi_t^{-1}(x)) - w_0(\Phi_t^{-1}(y))|}{|x-y|}$$

$$\leq \|\nabla w_0\|_{L^\infty} \sup_{x,y} \frac{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}{|x-y|}$$

So

$$\|\nabla w(x,t)\|_{L^\infty} \leq \|\nabla w_0\|_{L^\infty} \exp\left(c \int_0^t (1+\|w_0\|_{L^\infty}(1+\log_+\|\nabla w(x,s)\|_{L^\infty}) \, ds \right)$$

i.e.

$$\log\|\nabla w(x,t)\|_{L^\infty} \leq \log\|\nabla w_0\|_{L^\infty} + c\int_0^t \left(1+\|w_0\|_{L^\infty}(1+\log_+\|\nabla w(x,s)\|_{L^\infty})\right) ds$$

Then one apply Gronwall to get (9).

<u>Lemma 4</u>. (Regularity for particle trajectory)

For any $T>0$, $\exists\ c>0$ and $\beta(t) = \exp\left(-c(\|w_0\|_{L^\infty} + \|w_0\|_{L^1})t\right)$

s.t.
$$|\Phi^{\pm t}_{(x_1)} - \Phi^{\pm t}_{(x_2)}| \leq c|x_1 - x_2|^{\beta(t)} \qquad (10a)$$

$$|\Phi^{\pm t_1}_{x} - \Phi^{\pm t_2}_{x}| \leq c|t_1 - t_2|^{\beta(t)} \qquad (10b)$$

for any $x_1, x_2 \in \mathbb{R}^2$ close enough.

<u>Proof</u>. First similar consideration as in (6) will give us

$$\sup_{0\le t\le T} |u(x^1,t)-u(x^2,t)| \le c(\|w_0\|_{L_\infty}+\|w_0\|_{L_1})|x_1-x_2|(1-\log^-|x_1-x_2|) \quad (10c)$$

(for details see [Majda-Bertozzi book] p.315 Lemma 8.1)

Let $\rho(t) = |\Phi^t(x_1) - \Phi^t(x_2)|$

Then $\frac{d\rho(t)}{dt} \le c(\|w_0\|_{L_\infty}+\|w_0\|_{L_1})\rho(t)(1-\log^-\rho(t))$

by (10c), so that $\rho(t) \le e\rho(0)^{\exp(-c(\|w_0\|_{L_\infty}+\|w_0\|_{L_1})t)}$

provided $\rho(0)\le 1$. This is (10a) in forward direction. Other estimates are similar. □

Lemmas 2 and 4 are crucial in proving existence and uniqueness of weak solutions.

Let us talk more about Lemma 3. Recent result of [Kiselev-Sverak, Ann. Math 2014] gives the following Theorem.

<u>Theorem 1.</u> Consider 2-d Euler in a unit disk D. There exists a smooth initial data w_0 with $\|Dw_0\|_{L_\infty}>1$ s.t. the corresponding solution $w(x,t)$

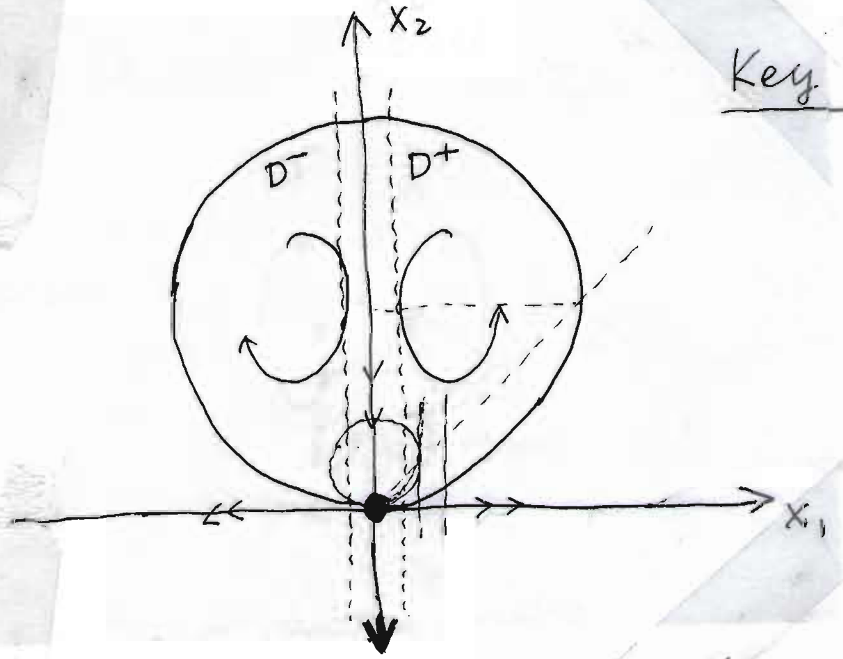
satisfies

$$\| \nabla \omega(x,t) \|_{L^\infty} \geq \| \nabla \omega_0 \|_{L^\infty}^{c \exp(c \|\omega_0\|_{L^\infty} t)}$$

for some $c>0$ and all $t \geq 0$.

Double exponential upper bound is in general optimal!

Yudovich's prediction: boundaries are prone to generation of small scales in solutions to 2-d Euler.



double exponential growth

Key lemma Scaling symmet[ry] of Euler

$$\omega(x) \longrightarrow \omega\left(\frac{x}{\lambda}\right)$$

$$\psi(x) \longrightarrow \lambda^2 \psi\left(\frac{x}{\lambda}\right)$$

[Reference: Small scale creation for solutions of the incompressible 2-d Euler equation. Kiselev & Sverak Ann. Math, 2014, pp.1205-1220]

We have not talked about existence and uniqueness yet. Now we introduce the material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$ and we write equation (1) in its weak form: $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\frac{d}{dt}\int_{\mathbb{R}^2} \varphi w \, dx = \int_{\mathbb{R}^2} \frac{D\varphi}{Dt} w \, dx + \int_{\mathbb{R}^2} \varphi \frac{Dw}{Dt} \, dx$$

$$\Rightarrow \int_{\mathbb{R}^2} \varphi(x,T) w(x,T) \, dx - \int_{\mathbb{R}^2} \varphi(x,0) w(x,0) \, dx$$

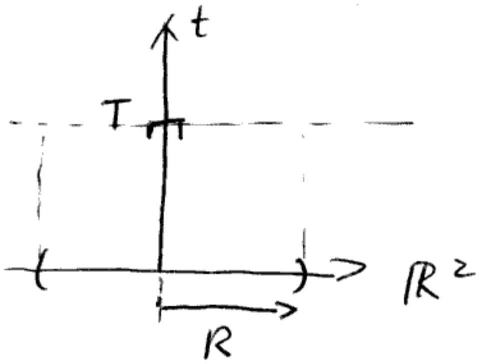
$$= \int_0^T \int_{\mathbb{R}^2} \frac{D\varphi}{Dt} w \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} \varphi \frac{Dw}{Dt} \, dx \, dt$$

But $\frac{Dw}{Dt} = 0$

So $$\int_{\mathbb{R}^2} \varphi(x,T) w(x,T) \, dx - \int_{\mathbb{R}^2} \varphi(x,0) w_0(x) \, dx$$

$$= \int_0^T \int_{\mathbb{R}^2} \frac{D\varphi}{Dt} w \, dx \, dt$$

In the weak formulation above we just need the test function $\varphi \in C^1([0,T]\times \mathbb{R}^2)$



$$\mathrm{Supp}(\varphi(\cdot,t)) \subset \{x : |x| \le R\}$$

<u>Definition 1</u>. Given $w_0 \in L_1(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)$ (u,w) is a weak solution to the vorticity-stream formulation of 2-D Euler equation with initial data $w_0(x)$, provided that

(i). $w \in L_\infty([0,T];\ L_1(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2))$

(ii). $u(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\,w(y,t)\,dy$

$\quad w = \operatorname{curl} u$

(iii). For any test function $\varphi \in C^1([0,T]\times\mathbb{R}^2)$ such that $\mathrm{supp}(\varphi(\cdot,t)) \subset \{x : |x|\le R\}$ we have

$$\int_{\mathbb{R}^2} \varphi(x,T)\,w(x,T)\,dx - \int_{\mathbb{R}^2}\varphi(x,0)\,w_0(x)\,dx = \int_0^T\int_{\mathbb{R}^2}\frac{D\varphi}{Dt}\,w\,dx\,dt \tag{11}$$

Theorem 2 (Existence of Weak Solutions)

Let the initial vorticity $\omega_0 \in L_1(\mathbb{R}^2) \cap L_\infty(\mathbb{R}^2)$ Then for all time there exists a weak solution (u, ω) to the vorticity-stream formulation of the 2-D Euler equation in the sense of Definition 1.

Remark First proof of the existence + uniqueness of weak solutions with $\omega_0 \in L_\infty$ in bounded domains by Yudovich 1963

[Yudovich, V.I., The flow of a perfect, incompressible liquid through a given region, Soviet Physics Doklady I, 1963, pp. 789-791]

Sketch in the proof.

Slogan ① Smooth the initial data and get the existence of a smooth solution for all time

② Pass to the limit in the regularization parameter

$$\omega_0^\varepsilon(x) = J_\varepsilon \omega_0 = \varepsilon^{-2} \int_{\mathbb{R}^2} \rho\left(\frac{x-y}{\varepsilon}\right) \omega_0(y)\,dy$$

$$\varepsilon > 0$$

$$\rho \in C_0^\infty(\mathbb{R}^2), \ \rho \geq 0, \ \int_{\mathbb{R}^2} \rho(x)\,dx = 1$$

$$\omega^\varepsilon(x,0) = \omega_0^\varepsilon(x), \qquad u^\varepsilon = K_2 * \omega^\varepsilon$$

$$\int_{\mathbb{R}^2} \varphi(x,T)\omega^\varepsilon(x,T)\,dx - \int_{\mathbb{R}^2} \varphi(x,0)\,\omega_0^\varepsilon(x)\,dx$$

$$= \int_0^T \int_{\mathbb{R}^2} \frac{D^\varepsilon \varphi}{Dt}\, \omega^\varepsilon \,dx\,dt$$

where $\dfrac{D^\varepsilon}{Dt} = \dfrac{\partial}{\partial t} + u^\varepsilon \cdot \nabla$, $\varphi \in C^1([0,T]; C_0^1(\mathbb{R}^2))$.

Extract subsequences $(\omega^{\varepsilon'})$ and $(u^{\varepsilon'})$, show their convergence to limits ω and u, and finally to show that these limits are a weak solution in Definition 1.

Compactness argument mainly based on Lemma 4 (10a-c). □

Theorem 3 (Uniqueness of weak solutions)

Let the initial vorticity $\omega_0 \in L^{\infty,0}(\mathbb{R}^2)$ have compact support $\operatorname{supp}\omega_0 \subset \{x : |x| \le R_0\}$. Then the weak solution $\omega \in L^\infty([0,\infty); L^{\infty,0}(\mathbb{R}^2))$ in the sense of Definition 1 is unique.

Proof. One more crucial techique: energy method

Yudovich's idea: ordinary doffenential inequalities with non-unique solutions.

First
$$\int_{\mathbb{R}^2} \omega(x,t)\,dx = \int_{\mathbb{R}^2} \omega_0(x)\,dx$$
by particle trajectory and weak formulation (11)

Suppose there are two weak solutions ω_j, $u_j = K_2 * \omega_j$ with the same initial vorticity $\omega_0 \in L^{\infty,0}(\mathbb{R}^2)$, $\operatorname{supp}\omega_0 \subset \{x : |x| \le R_0\}$, $j=1,2$

Velocities u_j solve the 2-D Euler in distributional sense:

$$\left.\begin{aligned} \frac{\partial u_j}{\partial t} + u_j \cdot \nabla u_j &= -\nabla p_j \\ \operatorname{div} u_j &= 0 \end{aligned}\right\}$$

<u>Claim</u> energy difference $w = u_1 - u_2$

$$E(t) = \int_{\mathbb{R}^2} (w(x,t))^2 \, dx < \infty$$

Actually particle trajectory gives us

$$w_j(\cdot, t) \subset \{x : |x| \leq R_j(T)\}$$

and then potential theory gives us

$$u_j(x,t) = \frac{c}{|x|} \int_{\mathbb{R}^2} w_j(y,t)\, dy + O(|x|^{-2})$$

$$\text{for } |x| \geq 2R_j(T)$$

So that

$$u_j(x,t) = \frac{c}{|x|} \int_{\mathbb{R}^2} w_0(x)\, dx + O(|x|^{-2})$$

$$\text{for } |x| \geq 2R_j(T)$$

and $\quad w(x,t) \sim \mathcal{O}(|x|^{-2})$ for $|x| \geq 2 \max_{j=1,2} R_j(T)$.

Now write equation (weak form) for $w = u_1 - u_2$

$$w_t + u_1 \cdot \nabla w + w \cdot \nabla u_2 = -\nabla(p_1 - p_2)$$

Do L_2 - inner product with w and integrating by parts

$$\frac{1}{2}\frac{d}{dt} E(t) - \int_{\mathbb{R}^2} w^2 \cdot \nabla u_1 \, dx + \int_{\mathbb{R}^2} (w \cdot \nabla u_2) w \, dx$$

$$= \int_{\mathbb{R}^2} (p_1 - p_2) \nabla \cdot w \, dx$$

By divergence free condition of u_1 and u_2

$$\frac{d}{dt} E(t) \leq 2 \int_{\mathbb{R}^2} w^2 |\nabla u_2| \, dx$$

(Then Hölder) $\leq 2 \|\nabla u_2\|_{L_p} \left(\int_{\mathbb{R}^2} |w|^{\frac{2p}{p-1}} dx \right)^{\frac{p-1}{p}}$

$$\leq 2 \|\nabla u_2\|_{L_p} \left(\|w(\cdot,t)\|_{L_\infty}^{\frac{2}{p-1}} \int_{\mathbb{R}^2} |w|^2 dx \right)^{\frac{p-1}{p}}$$

Now apply Lemma 2

$$\frac{d}{dt} E(t) \leq p M \, E(t)^{1-\frac{1}{p}} \tag{12}$$

$$M = c\left(\|w_0\|_{L^\infty}\right) \|w_0\|_{L^\infty}$$

Now comes the crucial step!

Equation (12) does not admit unique solution (trivial one) $E(t) \equiv 0$.

maximal solution $\bar{E}(t) = (Mt)^p$

any solution $E(t) \leq \bar{E}(t)$ (Exercise!)

Consider interval $[0, T^*]$, $MT^* \leq \frac{1}{2}$

Now $p \to \infty$

$$E(t) \leq \left(\frac{1}{2}\right)^p \to 0 \quad \text{as} \quad p \to \infty$$

so $E(t) \equiv 0$ for $0 \leq t \leq T^*$

Repeating $\Rightarrow$ $E(t) = 0 \quad \forall \, 0 \leq t \leq T$

so $u_1 = u_2$ almost everywhere $\square$

Finally we consider stochastic 2D Euler vorticity equation that has area-preserving flow maps:

$$\left.\begin{aligned} &d\omega + u\cdot\nabla\omega\, dt + \sum_{k=1}^{\infty} \sigma_k\cdot\nabla\omega\circ dW^k = 0 \\ &\omega(x,0) = \omega_0(x), \quad x\in\mathbb{T}^2 \\ &u = K_2 * \omega \end{aligned}\right\} \qquad (13)$$

Here σ_k's are bounded, regular enough, divergence-free vector fields, W^k's are independent Brownian motions

(13) can be written in Itô's form

$$d\omega + u\cdot\nabla\omega\, dt + \sum_{k=1}^{\infty}\sigma_k\cdot\nabla\omega\, dW^k - \frac{1}{2}\sum_{k=1}^{\infty}(\sigma_k\cdot\nabla)\sigma_k\cdot\nabla\omega\, dt$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \mathrm{tr}\left[\sigma_k\sigma_k^T D^2\omega\right] dt$$

Assume $\quad a(x) = \sum_{k=1}^{\infty}\sigma_k(x)\,\sigma_k^T(x) = c\, I_2$

Check $\displaystyle\sum_k\sum_i \sigma_{k,i}(x)\,\partial_i\sigma_{k,j} = \sum_i \partial_i\Big(\sum_k \sigma_{k,i}(x)\,\sigma_{k,j}(x)\Big)$

$$= \sum_i \partial_i a_{ij}(x) = 0$$

(13) reduces to

$$dw + u \cdot \nabla w \, dt + \sum_{k=1}^{\infty} \sigma_k \cdot \nabla w \, dW^k = \frac{1}{2} c \Delta w \, dt \qquad (14)$$

Particle trajectory equations

$$\Phi^t(x) = x + \int_0^t \int_{\mathbb{T}^2} K_2(\Phi^r(x) - \Phi^r(y)) \, w_0(y) \, dy \, dr$$
$$+ \sum_{k=1}^{\infty} \int_0^t \sigma_k(\Phi^r(x)) \, dW_r^k \qquad (15)$$

$w(x,t) = w_0(\Phi^{-t}(x))$, $\Phi^t(x)$ — area preserving

Apply Itô's formula (15) ⇒ (14)

(15) is non-local SDE : existence and uniqueness

Reference: [Brzeźniak, Z., Flandoli, F., Maurelli, M., Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity

arXiv: 1401.5938

Lecture 5 Motion of incompressible ideal fluid from group theoretic point of view

Classical Mechanics from group theoretic point of view

G $\quad$ n - dimensional Lie group

$\mathfrak{g} = T_eG$ $\quad$ Lie Algebra

Lie bracket $[\ ,\ ]$

$$[\xi, \eta] = \frac{\partial^2}{\partial s \partial t}\bigg|_{t=0} e^{t\xi} e^{s\eta} e^{-t\xi} e^{-s\eta}, \quad \xi, \eta \in T_eG$$

Left action: $\quad h \longrightarrow gh$

Right action: $\quad h \longrightarrow hg$

Left action on $\mathfrak{g}$: $\quad \xi \longrightarrow (g)_* \xi \quad : T_eG \longrightarrow T_gG$

Right action on $\mathfrak{g}$: $\quad \xi \longrightarrow \xi (g)_* \quad : T_eG \longrightarrow T_gG$

Adjoint action: $\quad Ad_g : \mathfrak{g} \rightarrow \mathfrak{g}$

$$Ad_g\, \xi = (g)_* \xi (g^{-1})_*$$

$$Ad_g\,[\xi, \eta] = [Ad_g\,\xi,\ Ad_g\,\eta]$$

$$Ad_{gh} = Ad_g\,Ad_h$$

$$ad = Ad_{*e} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}\ ;\quad ad_\xi = \frac{d}{dt}\Big|_{t=0} Ad_{e^{t\xi}}$$

One can show $ad_\xi\,\eta = [\xi, \eta]$

$\mathfrak{g} \longleftrightarrow \mathfrak{g}^*$, pairing $(\xi, \eta) \in \mathbb{R}$

$\xi \in T_g^*G \quad \eta \in T_g G$

$(Ad_g^*\,\xi, \eta) = (\xi, Ad_g\,\eta)$ Dual adjoint

Co-adjoint map: $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$

$$ad_\xi^* : \mathfrak{g}^* \to \mathfrak{g}^* \qquad ad_\xi^* = \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{t\xi}}$$

$(ad_\xi^*\,\eta, \zeta) = (\eta, ad_\xi\,\zeta) \quad \eta \in \mathfrak{g}^*, \xi \in \mathfrak{g}, \zeta \in \mathfrak{g}$

$ad_\xi^*\,\eta = \{\xi, \eta\}$, $\xi \in \mathfrak{g}$, $\eta \in \mathfrak{g}^*$

$(\{\xi, \eta\}, \zeta) = (\eta, [\xi, \zeta])$

Co-adjoint orbit : $\mathcal{O}^*(\eta) = \{Ad_g^* \eta \,;\, g \in G\}$

$\eta \in T_e^* G = \mathfrak{g}^*$

Symplectic Structure on $\mathcal{O}^*(\eta)$ — Kirillov form

ξ_1, ξ_2 tangent to $\mathcal{O}^*(\eta)$ at $x \in \mathfrak{g}^*$

$\xi_i = \{a_i, x\} \qquad a_i \in \mathfrak{g}, \; x \in \mathfrak{g}^*$

$\Omega(\xi_1, \xi_2) = (x, [a_1, a_2])$

<u>Exercise</u> Ω is well-defined 2-form

Ω is skew symmetric

Ω is non-degenerate closed 2-form

Ω — Kirollov form

What is the corresponding Hamiltonian dynamics?

G — Symmetry group of motion

Conservation of energy (kinetic) in free motion

Conservation of angular momentum

↳ restrict the phase trajectory to $\mathcal{O}^*(\eta)$

Free motion = geodesic flow on G

Recall theory of rigid body motion : $G = SO(3)$

Inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$

A is symmetric & positive definite, linear

$A_g : T_gG \to T_g^*G$ —— moment of inertia

$$A_g \xi = (g)^* A (g^{-1})_* \xi , \quad \xi \in T_gG$$

This makes G a Riemannian manifold !

$$\langle \xi, \eta \rangle_g = (A_g \xi, \eta) = (A_g \eta, \xi) = \langle \eta, \xi \rangle_g$$

left-invariant $\langle \xi, \eta \rangle_e = \langle L_{g*}\xi, L_{g*}\eta \rangle_g$

Recall : Another biinvariant measure — Haar measure

we will come back to it later

$\langle , \rangle$ makes $T_e^* G = \mathfrak{g}^*$ an inner product space also!

$$\langle \xi, \mu \rangle = \langle \xi, \mu \rangle_e = (\xi, A^{-1}\mu)$$

for $\xi, \mu \in T_e^* G = \mathfrak{g}^*$

$\mathcal{O}^*(\eta)$ is also a Riemmanian manifold!

Classical mechanics = geodesic $g(t)$ on G

w.r.t. $\langle , \rangle_g$

(Principle of least action)

Lagrangian = Kinetic energy

$$T(t) = E(t) = \frac{1}{2} \langle \dot{g}(t), \dot{g}(t) \rangle_{g(t)}$$

Action $S_{oT}(g) = \int_0^T \frac{1}{2} \langle \dot{g}(s), \dot{g}(s) \rangle_{g(s)} \, ds$

The geodesic $g(t)$ is s.t. the first variation of $S_{oT}(g)$ vanishes — principle of least action angular velocity $\omega = \dot{g}$

angular velocity in the body $\omega_c = (g^{-1})_* \dot{g}$

angular velocity in space $\omega_s = \dot{g}\,(g^{-1})_*$

angular momentum $M = A_g \dot{g}$

angular momentum in the body $M_c = (g^{-1})^* M$

angular momentum in space $M_s = M\,(g^{-1})^*$

Exercise Work those notions out for $G = SO(3)$

Kinetic energy

$$T = E = \frac{1}{2}\langle \dot{g}, \dot{g}\rangle_g = \frac{1}{2}\langle \omega_c, \omega_c\rangle = \frac{1}{2}(A\omega_c, \omega_c)$$

$$= \frac{1}{2}(A_g \dot{g}, \dot{g}) = \frac{1}{2}(M_c, \omega_c) = \frac{1}{2}(M, \dot{g})$$

Theorem 1. ① (Conservation of angular momentum in space)

$$\frac{dM_s}{dt} = 0$$

② (Euler's equation) $\dfrac{dM_c}{dt} = \{\omega_c, M_c\}$

③ (Conservation of angular momentum in the body)

$M_c(t) \in \mathcal{O}^*(y)$ for all $t \geq 0$

④ (Hamiltonian structure)

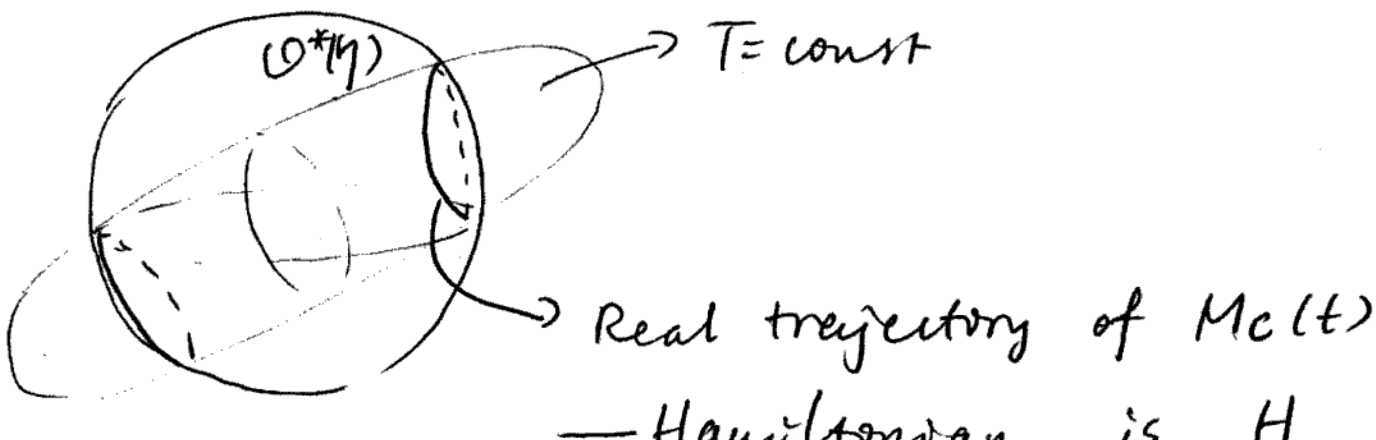
The **Euler's** equation for M_c is a Hamiltonian equation on $\mathcal{O}^*(y)$ with Hamiltonian H

$$T = \frac{1}{2}(M_c, A^{-1}M_c)$$

$$H : \mathcal{O}^*(y) \to \mathbb{R} \quad : \quad H(M_c) = \frac{1}{2}(M_c, A^{-1}M_c)$$

energy surface $\quad T = \text{const}$

co-adjoint orbit $\quad \mathcal{O}^*(y)$



$i_X \Omega = dH \qquad \Omega$ — Kirillov form

Proof of Theorem 1.

① and ②: Let $g(r,s)$ be a variation of $g(s)$

First variation of action

$$0 = \partial_r \int_0^t \frac{1}{2} \langle g_s(r,s), g_s(r,s) \rangle_{g(r,s)} ds$$

$$= \partial_r \int_0^t \frac{1}{2} \langle (g^{-1})_* g_s(r,s), (g^{-1})_* g_s(r,s) \rangle ds$$

$$= \int_0^t \langle \partial_r ((g^{-1})_* g_s(r,s)), (g^{-1})_* g_s(r,s) \rangle ds$$

$$= \int_0^t \langle (g^{-1})_* g_{sr} - (g^{-1})_* g_r (g^{-1})_* g_s , (g^{-1})_* g_s \rangle ds$$

$$= \int_0^t \langle \partial_s ((g^{-1})_* g_r) + [(g^{-1})_* g_s, (g^{-1})_* g_r], (g^{-1})_* g_s \rangle ds$$

$$\left(\begin{array}{l} \partial_r((g^{-1})_* g_s) = (g^{-1})_* g_{sr} - (g^{-1})_* g_r (g^{-1})_* g_s \\ \partial_s((g^{-1})_* g_r) = (g^{-1})_* g_{rs} - (g^{-1})_* g_s (g^{-1})_* g_r \end{array} \right)$$

$$= \int_0^t \langle (g)_* \left(\partial_s ((g^{-1})_* g_r) + [(g^{-1})_* g_s, (g^{-1})_* g_r] \right), g_s \rangle_g ds$$

$$= \int_0^t \left((g)_* \left(\partial_s ((g^{-1})_* g_r) + [(g^{-1})_* g_s, (g^{-1})_* g_r] \right), A_g g_s \right) ds$$

$$= \int_0^t \left(\partial_s ((g^{-1})_* g_r) + [(g^{-1})_* g_s, (g^{-1})_* g_r], (g^{-1})^* A_g g_s \right) ds$$

$$= \int_0^t \left(\partial_s ((g^{-1})_* g_r) + [w_c, (g^{-1})_* g_r], M_c \right) ds$$

$$= -\left[\int_0^t ((g^{-1})_* g_r, \partial_s M_c) - ((g^{-1})_* g_r, \{w_c, M_c\}) \, ds \right]$$

$\Longrightarrow$ Euler's equation ② holds

integration by parts

Fix $w \in \mathfrak{g}$

$$\left(\frac{dM_s}{dt}, w \right) = \frac{d}{dt} (g^* M_c (g^{-1})^*, w)$$

$$= \frac{d}{dt} (M_c, (g^{-1})_* w (g)_*)$$

$$= (\{w_c, M_c\}, (g^{-1})_* w (g)_*) + (M_c, [(g^{-1})_* w (g)_*, w_c])$$

$$= (\{w_c, M_c\}, (g^{-1})_* w g_*) - (\{w_c, M_c\}, (g^{-1})_* w g_*)$$

$= 0 \quad \Longrightarrow \quad$ ① holds

③ $\frac{dM_s}{dt} = 0$ and $M_c(t) = Ad^*_{g(t)} M_s(t)$

$$\Rightarrow M_c(t) \in \mathcal{O}^*(\eta) \qquad \eta = M_c(0)$$

④ $X = \dot{M}_c(t) = \{w_c, M_c\}$

$T = T(M_c) = \frac{1}{2}(M_c, A^{-1}M_c)$

$dT(M_c) = A^{-1}M_c = w_c$

ξ — tangent to $\mathcal{O}^*(\eta)$, $M_c \in \mathcal{O}^*(\eta)$, $\xi = \{f, M_c\}$, $f \in \mathcal{G}$

$$(i_X\Omega)(\xi) = \Omega(\xi, X) = (M_c, [f, dT])$$
$$= (dT, \{f, M_c\}) = (dH, \xi)$$

$$\Rightarrow i_X \Omega = dH \qquad \square$$

H —— Hamiltonian

$\underbrace{\Omega \wedge \cdots \wedge \Omega}_{d\text{-times}}$ — Liouville measure

$\dim \mathcal{O}^*(\eta) = 2d \leq n$

The above process can be abstracted ~~to define~~ making use of symplectic reduction procedure.

$$T^*G = G \times \mathfrak{g}^*$$

$G \times G$ acts on G : $(g_1, g_2)g = g_1 g g_2^{-1}$

$\xrightarrow{\text{extension}}$ $G \times G$ acts on T^*G

$$(g_1, g_2)\cdot(g, \eta) = (g_1 g g_2^{-1}, \ Ad_{g_2}^* \eta)$$

This action is Hamiltonian

T^*G is a symplectic G- manifold with 2-form σ generates Haar measure on G

Reduction Moment map μ: $T^*G \to \mathfrak{g}^* \oplus \mathfrak{g}^*$

$$\mu(g, \eta) = \eta \oplus Ad_g^* \eta$$

Fiber of moment map over $\eta \in \mathfrak{g}^*$ is $G \times \{\eta\}$

Reduced Manifold $= G/Stab(\eta) \cong \mathcal{O}^*(\eta)$

Canonical projection: $p: \mu^{-1}(\mathcal{O}^*(\eta)) \to \mathcal{O}^*(\eta)$

$p^* \Omega = \sigma$

Ω — Kirillov form

This is the relationshop between Kirillov's measure and Haar measure on G.

Ref. [A. A. Kirillov Lectures on the orbit method Graduate Studies in Mathematics, 64 AMS, 2004. pp. 9-11 Appendix II. 3. 2]

What is the stochastic picture?

Pick a process $\Gamma(t)$ on $\mathcal{O}^*(\eta) \cap \{H = E\}$

invariant measure of $\Gamma(t)$ is given by

$$\underbrace{\Omega \wedge \cdots \wedge \Omega}_{d\text{-times}}$$

This can be achieved by using a Laplace-Beltrami operator Δ_h corresponding to a metric $h = (h_{ij})_{(2d-1)^2}$ on $\mathcal{O}^*(\eta) \cap \{H = E\}$

h induces the right measure

Let $\Gamma(t)$ be BM generated by $\frac{1}{2}\Delta_h$

$\Gamma(t)$ is a stochastic $Mc(t) = A(g^{-1})_* \dot{g}$

So $A(g^{-1})_* \dot{g} = \Gamma(t)$

Coupled process $\vec{g}(t) = (g(t), A(g^{-1})_* \dot{g}(t))$

lives on $G \times \mathfrak{g}^*$

$$(g \,,\, \Gamma)$$

generator
$$L = \nabla_{A(g^{-1})_* \Gamma} + \frac{1}{2}\Delta_h$$

Condition L is hypo-elliptic on $G \times \mathcal{O}^*(\eta, E)$

$$\mathcal{O}^*(\eta, E) = \mathcal{O}^*(\eta) \cap \{H = E\}$$

If G is compact then hypo-ellipticity leads to convergence to invariant measure.
exponential

<u>Check</u> invariant measure

$=$ Haar measure on G $\times$ induced Kirillov measure on $\mathcal{O}^*(\mathfrak{g}, E)$

How to check? Use Backward Kolmogorov Eq.

If G is not compact

Case by case $\quad G = SL_2(\mathbb{R})$

$G =$ affine transformation of a line

...

Relevancy in fluid mechanics

①. Hamiltonian formalism of Euler's equation

②. SU(n) geodesics flow approximation

of 2-d Euler dynamics — Open problems related to concentration of measures

For ② see References — we do not expand here

[1]. Zeitlin, V., Finite mode analogues of 2-d hydrodynamics: co-adjoint orbits and local canonical structure, Physica D. 49 (3) 1991 pp. 353-362

[2]. Gallagher, I., Mathematical analysis of a structure-preserving approximation of the 2-d vorticity equation, Numerische Mathematik, 91 pp. 223-236, 2002

Let us briefly talk about ①.

We consider Euler's equation on a <u>simply connected</u> domain $M \subset \mathbb{R}^2$ with boundary ∂M

$$\left.\begin{aligned} \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega &= 0 \\ u &= K\omega \\ \omega(0) &= \omega_0 \end{aligned}\right\} \quad x \in M$$

with boundary condition $u \cdot n = 0$ on ∂M

$G = SDiff^+(M)$ infinite dimensional group

$\mathfrak{g} = SVect(M)$ — divergence-free vector fields on M tangent to ∂M

$\updownarrow$ identify

stream function ψ s.t. $\nabla^{\perp}\psi = v$

$$-\Delta \psi = \omega$$

Commutator : $Ad_g\, v = (g)_* \, v \, (g^{-1})_*$

$\Leftrightarrow$ in $SDiff^+(M)$ it is a change of coordinates on M

How to calculate $ad_v\, w = \left.\frac{d}{dt}\right|_{t=0} Ad_{e^{tv}}\, w$?

Let $g(t)$: $x \mapsto x + tv(x) + o(t)$ $\quad t \to 0$

$h(s): \quad x \mapsto x + s w(x) + o(s) \qquad s \to 0$

Then $\quad g^{-1}(t) = \quad x \mapsto x - t v(x) + o(t)$

$h(s) g^{-1}(t) = \quad x \mapsto x - t v(x) + o(t)$

$$+ s w(x - t v(x) + o(t)) + o(s)$$

$$= x - t v(x) + o(t) + s\left(w(x) - t \frac{\partial w}{\partial x} v(x) + o(t)\right) + o(s)$$

$g(t) h(s) g^{-1}(t): \quad x \mapsto x + s\left(w(x) + t\left(\frac{\partial v}{\partial x} w(x) - \frac{\partial w}{\partial x} v(x)\right)\right)$

$$+ o(t) + o(s)$$

so $\quad ad_v w = -[v, w]$

where $\quad [a, b]_j = \sum_{i=1}^{2} a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \quad , j = 1, 2$

What is $\mathfrak{g}^*$ dual to $\mathfrak{g} = \mathrm{SVect}(M)$?

Kinetic energy $\quad E = \frac{1}{2} \iint_{T^2} |v|^2 \, d\mu.$

$v = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right) \quad , \quad w = -\Delta \psi.$

$SVect^*(M) = $ 1-form $\alpha = f_1 dx_1 + f_2 dx_2$

$$(\alpha, v) = \iint_{T^2} (\alpha, v)\, d\mu, \quad v = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$$

$$(\alpha, v) = f_1 v_1 + f_2 v_2$$

<u>Lemma 1</u> (1) If $\alpha = df$ then $(\alpha, v) = 0$, $\forall v \in SVect(M)$

(2) If $(\alpha, v) = 0$ for $\forall v \in SVect(M)$ then $\alpha = df$

(3) If $(\alpha, v) = 0$ for $\forall \alpha = df$ then $v \in SVect(M)$

Thus $SVect^*(M) = \Omega^1(M) / d\Omega^0(M)$

Recall that M is simply-connected!

<u>Lemma 2</u> $\Omega^1(M)/d\Omega^0(M) \cong$ space of functions on M

$$\begin{array}{c} \Omega^1(M)/d\Omega^0(M) \\ \| \\ SVect^*(M) \end{array}$$

$$\alpha \longmapsto f$$

defined via $d\alpha = f\, dx_1 dx_2$

This is because every function f is the image of some closed 1-form α, determined modulo the differential of a function, by simple-connectedness co-adjoint action

$$\mathrm{Ad}_g^* f = (g)_\# f (g^{-1})_\#$$

— change of coordinates of f

<u>Question</u> Angular momentum $\omega = \mathrm{curl}\, v$?

— vorticity ?

Euler's equation in Theorem 1 is

$$\frac{dM_c(t)}{dt} = \mathrm{ad}_{w_c}^* M_c$$

In case $\mathcal{G} = \mathrm{SVect}(M)$, $\dot{g} = v$

$$w_c = (g^{-1})_* v$$

$$\left\langle \frac{dw_c}{dt}, b \right\rangle = \left\langle A^{-1} \frac{dM_c}{dt}, b \right\rangle \quad b \in \mathrm{SVect}^{\infty}(M)$$

$$= \left(\frac{dM_c}{dt}, b \right)$$

$$= (\mathrm{ad}_{w_c}^* M_c, b)$$

$$= (M_c, [w_c, b])$$

$$= \langle [w_c, b], w_c \rangle \overset{\text{def}}{=\!=} \langle B(w_c, w_c), b \rangle$$

$$\langle B(v,v), b\rangle = \langle [v,b], v\rangle$$

$$= \frac{1}{2}\iint_{\mathbb{T}^2} [v,b]\cdot v \, dx_1 dx_2$$

$$= -\iint_{\mathbb{T}^2} (v\cdot\nabla)v\cdot b \, dx_1 dx_2$$

So $\dfrac{dv}{dt} = -(v\cdot\nabla)v - \nabla p$

∇p is s.t. $B(v,v) \in SVect(M)$

(Leray - Helmholtz decomposition)

We can then arrive at Euler!

Remarks and Corrections to the Lecture Notes.

Lecture 1.

p.17. "weak lower semi–continuity of the $\dot{H}^1(\mathbb{T}^2;\mathbb{R}^2)$–norm" is not very precise, but close to the truth.

Actually if $\{x_n\}$ is a sequence in a normed linear space X with norm $|\bullet|$ and $x_n \rightharpoonup x$ then $|x| \leq \liminf_{n\to\infty} |x_n|$. This is because there exist $\ell \in X'$ so that $|x| = |\ell(x)|$, $|\ell| = 1$. Thus $\ell(x) = \lim_{n\to\infty} \ell(x_n)$ which implies that $|\ell(x_n)| \leq |\ell||x_n| = |x_n|$. See [9, Chapter 10, Theorem 5].

For any $0 \leq t_1 \leq t_2 \leq T$ we know that the family of functions $u^{(m)}(x,t)$ converges weakly to $u(x,t)$ in $L^2([t_1,t_2];\dot{H}^1(\mathbb{T}^2;\mathbb{R}^2))$. Thus we apply the above paragraph we get $\lim_{m\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u^{(m)}|^2 dx \geq \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u|^2 dx$. Then we combine this estimate with the energy identity

$$\frac{1}{2}\int_{\mathbb{T}^2} |u^{(m)}|^2(x,t_2)dx = \frac{1}{2}\int_{T^2} |u^{(m)}|^2(x,t_1)dx + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u^{(m)}|^2(x,t)dx \ ,$$

and we take limit on both sides. Let us assume that t_2 is such a point that $\lim_{m\to\infty}\int_{\mathbb{T}^2} |u^{(m)}|^2(x,t_2)dx = \int_{\mathbb{T}^2} |u|^2(x,t_2)dx$. Then we have

$$\frac{1}{2}\int_{\mathbb{T}^2} |u|^2(x,t_2)dx \geq \frac{1}{2}\int_{T^2} |u|^2(x,t_1)dx + \int_{t_1}^{t_2} dt \int_{\mathbb{T}^2} |\nabla u|^2(x,t)dx \ .$$

Actually we may extract a subsequence (also denoted $u^{(m)}$) so that for an exceptional set $E \subset [0,T]$ of measure 0 we have $u^{(m)}$ converges weakly to u for each fixed t in $\dot{H}^1(\mathbb{T}^2;\mathbb{R}^2)$. Thus the original statement is true modulo the set E.

Lecture 2.

p.7. The notion of "irreversibility" may not be very accurate, but depends on how we define irreversibility ([4]).

p.7. Any two distinct ergodic invariant measures must be mutually singular. This can be proved as in [3, Proposition 3.2.5].

p.8. The dynamical smoothing part ("Lasota–Yorke inequality") and the asymptotic strong Feller property is there to control the behavior of the distribution of the process "at infinity". This is best illustrated by the Examples 3.14 and 3.15 in [8].

p.17. The Malliavin matrix $(\mathcal{M}_{0,t})_{i,j} = \langle \mathcal{D}^i U(t, U_0, W), \mathcal{D}^j U(t, U_0, W)\rangle_{L^2([0,t];\mathbb{R}^d)}$, $1 \leq i, j \leq N$. The operator $\mathcal{A}_{0,t} : L^2([0,t];\mathbb{R}^d) \to \mathbb{R}^N$ is such that $\mathcal{A}_{0,t}v = \langle \mathcal{D}U(t, U_0, W), v\rangle_{L^2([0,t];\mathbb{R}^d)} = \int_0^t \mathcal{J}_{s,t}\sigma v(s)ds$. We define $\mathcal{A}^*_{0,t} : \mathbb{R}^N \to L^2([0,t];\mathbb{R}^d)$ to be such that $\langle \mathcal{A}^*_{0,t}\xi, v\rangle_{L^2([0,t];\mathbb{R}^d)} = \langle \xi, \mathcal{A}_{0,t}v\rangle_{\mathbb{R}^N}$. From here we can calculate

$$\begin{aligned}
\langle \mathcal{A}^*_{0,t}\xi, v\rangle_{L^2([0,t];\mathbb{R}^d)} &= \langle \xi, \mathcal{A}_{0,t}v\rangle_{\mathbb{R}^N} \\
&= \sum_{i=1}^N \xi^i \langle \mathcal{D}^i U(t, U_0, W), v\rangle_{L^2([0,t];\mathbb{R}^d)} \\
&= \langle \sum_{i=1}^N \xi^i \mathcal{D}^i U(t, U_0, W), v\rangle_{L^2([0,t];\mathbb{R}^d)} \ .
\end{aligned}$$

Thus we see that $(\mathcal{A}^*_{0,t}\xi)(s) = \left(\sum_{i=1}^N \xi^i \mathcal{D}^i U(t, U_0, W)\right)(s)$, $0 \leq s \leq t$. This justifies the fact that $\mathcal{M}_{0,t} = \mathcal{A}_{0,t}\mathcal{A}^*_{0,t}$. Moreover we have

$$\begin{aligned}
\langle \mathcal{A}^*_{0,t}\xi, v\rangle_{L^2([0,t];\mathbb{R}^d)} &= \int_0^t \langle (\mathcal{A}^*_{0,t}\xi)(s), v(s)\rangle_{\mathbb{R}^d} ds \\
&= \langle \xi, \mathcal{A}_{0,t}v\rangle_{\mathbb{R}^N} \\
&= \int_0^t \langle \xi, \mathcal{J}_{s,t}\sigma v(s)\rangle_{\mathbb{R}^N} ds \\
&= \int_0^t \langle \sigma^* \mathcal{J}^*_{s,t}\xi, v(s)\rangle_{\mathbb{R}^d} ds \ .
\end{aligned}$$

Here $\mathcal{J}^*_{s,t}$ is the adjoint of $\mathcal{J}_{s,t}$ with respect to $\mathbb{R}^N$ standard inner product (= taking transpose). This justifies $\mathcal{A}^*_{0,t}\xi = \sigma^* \mathcal{J}^*_{s,t}\xi$.

Let $\mathcal{J}_{s,t}\xi = \rho(t)$ satisfy the equation

$$\frac{d}{dt}\rho(t) + \nu A\rho(t) + B(U(t), \rho(t)) + B(\rho(t), U(t)) = 0 \ , \ \rho(s) = \xi \ .$$

Let $\mathcal{J}^*_{s,t}\eta = \rho^*(s)$. Then $\langle \mathcal{J}_{s,t}\xi, \eta\rangle_{\mathbb{R}^N} = \langle \xi, \mathcal{J}^*_{s,t}\eta\rangle_{\mathbb{R}^N}$ implies that $\mathcal{J}^*_{t,t}\eta = \eta$. Moreover, taking t–derivative on both sides we get

$$
\begin{aligned}
\langle \frac{d}{dt}\mathcal{J}_{s,t}\xi, \eta\rangle_{\mathbb{R}^N} &= \langle \xi, \frac{d}{dt}\mathcal{J}^*_{s,t}\eta\rangle_{\mathbb{R}^N} \\
&= \langle \xi, -\frac{d}{ds}\mathcal{J}^*_{s,t}\eta\rangle_{\mathbb{R}^N} \\
&= \langle -\nu A\mathcal{J}_{s,t}\xi - B(U(t), \mathcal{J}_{s,t}\xi) - B(\mathcal{J}_{s,t}\xi, U(t)), \eta\rangle_{\mathbb{R}^N} \\
&= \langle \xi, -\nu A\mathcal{J}^*_{s,t}\eta - B^*(U(t), \mathcal{J}^*_{s,t}\eta) - B^*(\mathcal{J}^*_{s,t}\eta, U(t))\rangle_{\mathbb{R}^N} ,
\end{aligned}
$$

yielding

$$
-\frac{d}{ds}\rho^*(s) + \nu A\rho^*(s) + B^*(U(t), \rho^*(s)) + B^*(\rho^*(s), U(t)) = 0 \ .
$$

p.18. We have defined via the ansatz that $v = \mathcal{A}^*_{0,t}\eta$. From the formula $(\mathcal{A}^*_{0,t}\eta)(s) = \left(\sum\limits_{i=1}^N \xi^i \mathcal{D}^i U(t, U_0, W)\right)(s)$ we see that $v(s)$ is not adapted. So we need Skorokhod integral.

p.21. In the estimate (18) the constant $C = C(p, \eta, |\sigma|, t)$.

p.22. The first derivative operator $\mathcal{J}_{s,t}\xi$ satisfies the equation

$$
\frac{d}{dt}\mathcal{J}_{s,t}\xi + \nu A\mathcal{J}_{s,t}\xi + B(U(t), \mathcal{J}_{s,t}\xi) + B(\mathcal{J}_{s,t}\xi, U(t)) = 0 \ , \mathcal{J}_{s,s}\xi = \xi \ .
$$

The Malliavin derivative operator $\mathcal{A}_{0,t}v = \langle \mathcal{D}U(t, U_0, W), v\rangle_{L^2([0,t];\mathbb{R}^d)}$ satisfies the equation

$$
\frac{d}{dt}\mathcal{A}_{0,t}v + \nu A\mathcal{A}_{0,t}v + B(U(t), \mathcal{A}_{0,t}v) + B(\mathcal{A}_{0,t}v, U(t)) = \sigma\frac{dv(t)}{dt} \ , \mathcal{A}_{0,0}v = 0 \ .
$$

By variation of constants formula this gives

$$
\mathcal{A}_{0,t}v = \int_0^t \mathcal{J}_{s,t}\sigma v(s) ds \ .
$$

We can perform this calculation similarly for the second derivative operators. Denoting by $\mathcal{J}^{(2)}_{s,t}(\xi, \eta)$ the second derivative of $U(t, U_0, W)$ with respect to initial condition U_0 in the directions ξ and η. This gives us the equation

$$
\left\{
\begin{array}{lll}
\frac{d}{dt}\mathcal{J}^{(2)}_{s,t}(\xi,\eta) + \nu A\mathcal{J}^{(2)}_{s,t}(\xi,\eta) & +B(U(t), \mathcal{J}^{(2)}_{s,t}(\xi,\eta)) + B(\mathcal{J}^{(2)}_{s,t}(\xi,\eta), U(t)) & \\
& +B(\mathcal{J}_{s,t}\xi, \mathcal{J}_{s,t}\eta) + B(\mathcal{J}_{s,t}\eta, \mathcal{J}_{s,t}\xi) & = 0 \ , \\
\mathcal{J}^{(2)}_{s,s}(\xi,\eta) = 0 \ . & &
\end{array}
\right.
$$

On the other hand if we let $\mathcal{A}^{(2)}_{0,t}h = \langle \mathcal{D}\mathcal{J}_{s,t}\xi, h\rangle_{L^2([0,t];\mathbb{R}^d)}$ we have the equation for $\mathcal{A}^{(2)}_{0,t}h$:

$$
\left\{
\begin{array}{lll}
\frac{d}{dt}\mathcal{A}^{(2)}_{0,t}h + \nu A\mathcal{A}^{(2)}_{0,t}h & +B(U(t), \mathcal{A}^{(2)}_{0,t}h) + B(\mathcal{A}^{(2)}_{0,t}h, U(t)) & \\
& +B(\mathcal{A}_{0,t}h, \mathcal{J}_{s,t}\xi) + B(\mathcal{J}_{s,t}\xi, \mathcal{A}_{0,t}h) & = 0 \ , \\
\mathcal{A}^{(2)}_{0,s}h = 0 \ . & &
\end{array}
\right.
$$

Since we have already $\mathcal{A}_{0,t}v = \int_0^t \mathcal{J}_{s,t}\sigma v(s)ds$ we can then get, by assuming that h vanishes outside $[s,t]$, and using the fact that $\mathcal{J}_{r,t}\mathcal{J}_{s,r} = \mathcal{J}_{s,t}$, that

$$\mathcal{A}_{0,t}^{(2)}h = \int_s^t \mathcal{J}_{r,t}^{(2)}(\sigma h(r), \mathcal{J}_{s,r}\xi)dr \ ,$$

which implies $\mathcal{D}_r\mathcal{J}_{s,t}\xi = \mathcal{J}_{r,t}^{(2)}(\sigma, \mathcal{J}_{s,r}\xi)$. Making then use the fact that if $F \in \mathbb{D}^{1,p}$ is $\mathcal{F}_r$–measurable then $\mathcal{D}_s F = 0$ for all $s > r$, we get then $\mathcal{D}_\tau\mathcal{J}_{s,t}\xi = \begin{cases} \mathcal{J}_{\tau,t}^{(2)}(\sigma, \mathcal{J}_{s,t}\xi) & \text{when } s < \tau \ ; \\ \mathcal{J}_{s,t}^{(2)}(\mathcal{J}_{\tau,s}\sigma, \xi) & \text{when } s \geq \tau \ . \end{cases}$

We also have $\mathcal{D}_r\mathcal{A}_{0,t}v = \int_0^t \mathcal{D}_r\mathcal{J}_{s,t}\sigma v(s)ds$. From the equation satisfied by $\mathcal{J}_{s,t}^{(2)}$ and exponential estimates of $U(t)$ we readily infer that we have the bound $\mathbf{E}\|\mathcal{J}_{s,t}^{(2)}\|^p \leq C\exp(\eta|U_0|^2)$. From here we can estimate $\mathbf{E}\|\mathcal{D}v\|^2$.

Remark: We have $\mathcal{D}_s U(t, U_0, W) = \mathcal{J}_{s,t}\sigma$ for $s < t$.

Lecture 3.

p.3. The addition of noise is physically relevant in "choosing" a specific invariant measure, especially in the unique ergodic case.

p.4. It is not very precise to say that the conjecture $\int \|\omega\|_{C^0} d\mu_0(\omega) < \infty$ corresponds exactly to Kolmogorov's conjecture (since he works with velocity). However what these conjectures mean is that there should be some "regularity" or "stability" for the final behavior of the flow at inviscid limit.

p.5. If we do the scaling $d\omega + (\mathbf{u} \cdot \nabla\omega - \nu\Delta\omega)dt = \nu^a \sigma dW$, where $a > 1/2$, then μ_ν converges to a Dirac mass at 0; if $a < 1/2$, then there is no convergence. See Theorem 5.2.17 in [7].

p.5. If the equation is not at inviscid scaling, but rather it looks like

$$d\omega + (\mathbf{u} \cdot \nabla\omega - \Delta\omega)dt = \sqrt{\nu}\sigma dW \ ,$$

with ν very small, then the result about behavior of limiting measure is more close to a large deviation setting, see [2].

p.7. In this lecture we do not require the measure μ_ν to be unique, nor is μ_0.

p.7. Problem about limits of invariant measure for randomly perturbed Hamiltonian systems subject to dissipation has a longstanding history, see [5]. In there they consider systems of the form

$$\dot{X}_t = \overline{\nabla} H(X_t) + \varepsilon b(X_t) + \sqrt{\varepsilon\kappa}\dot{W}_t \ ,$$

and $b(x)$ is assumed to be of classical friction type, i.e., $\text{div} b < 0$. If $\kappa = 1$, then the situation is more close to our inviscid setting, and the result is that one needs averaging principle and we reduce the invariant measure problem to a diffusion process on a graph. If $\kappa \ll 1$, then the situation is more close to a large deviation effect.

p.8. $|k|\rho_k = \sigma_k$.

p.12. The BDG constant $C_{BDG} \leq 2^{\delta^{1/2}}$ whenever $\delta < \delta_0$ can be found at [10, Exercise (IV.4.30)].

p.14. $\nabla(\omega|\omega|^{p-2}) = \nabla\omega|\omega|^{p-2} + \omega\nabla(|\omega|^{p-2}) = \nabla\omega|\omega|^{p-2} + \omega(p-2)|\omega|^{p-3}(\text{sign}\omega)\nabla\omega = (p-1)\nabla\omega|\omega|^{p-2}$.

Lecture 4.

p.2. It is $\nabla K_2(x)$ being a singular integral operator, not $K_2(x)$.

p.10. The Kiselev–Sverak example has initial data so chosen that $\omega_0(x) \geq 0$ for $x_1 \geq 0$ and $\omega_0(x) \leq 0$ for $x_1 \leq 0$. The origin is a hyperbolic fixed point of the flow. The function $\omega_0(x)$ is in L^∞, but specifically chosen. And it is very important that the construction is at the boundary.

Lecture 5.

p.11. One can define a canonical symplectic structure σ on T^*G so that the extension of the action of $G \times G$ on T^*G preserves σ (Hamiltonian). The projection of σ on the group direction shall give the left Haar measure. For details see [6, Chapter 1, Section 2.3]

p.17. The energy is $E = \frac{1}{2}\iint_{\mathbb{T}^2} |v|^2 d\mu$. The enstrophy is $I = \frac{1}{2}\iint_{\mathbb{T}^2} \omega^2 d\mu$.

p.18. It is generally true that for an n–dimensional compact manifold M with boundary ∂M we have $\mathrm{SVect}^*(M) = \Omega^1(M)/d\Omega^0(M)$. The co–adjoint action acts as changes of coordinates on the cosets of 1–form α. The proof of this fact is found at [1, Chapter I, Theorem 8.3]. However in the case when $M = \mathbb{T}^2$ one cannot identify $\Omega^1(\mathbb{T}^2)/d\Omega^0(\mathbb{T}^2)$ with space of functions on $\mathbb{T}^2$. If M is 2–dimensional simply connected then one can identify $\Omega^1(\mathbb{T}^2)/d\Omega^0(\mathbb{T}^2)$ with space of functions on M.

p.19. Actually in the case when M is simply–connected the dual element of $v = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}$ is $\alpha = v_1 dx_1 + v_2 dx_2$ and $d\alpha = \left(-\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) dx_1 dx_2$ gives the vorticity.

p.19. The calculation cannot be done on $\mathbb{T}^2$, but it should be on a simply connected–domain M in $\mathbb{R}^2$ with boundary ∂M.

p.19. $b \in \mathrm{SVect}^*(M)$ is identified with a vector $v \in \mathrm{SVect}(M)$.

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