Itô's formula, the stochastic exponential and change of measure on general time scales

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Time Scales Seminar

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Definition 1 (Brownian Motion indexed by time scale)

A Brownian Motion indexed by a time scale $\mathbb{T}$ is an adapted process $\{W_t\}_{t\in\mathbb{T}}$ on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ such that (1) $\mathbb{P}(W_0=0)=1$

(2) If $0\leq s<t\leq 1$ and $s,t\in\mathbb{T}$ then the increment W_t-W_s is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and variance $t-s$

(3). W_t is almost surely continuous in $\mathbb{T}$

Definition 2 (Deterministic and Stochastic Integrals on time scales)

$$f : [0, \infty)_{\mathbb{T}} \times \Omega \longrightarrow \mathbb{R}$$

"extension" $\tilde{f} : [0, \infty) \times \Omega \longrightarrow \mathbb{R}$

$$\tilde{f}(t, \omega) = f(\sup [0, t]_{\mathbb{T}}, \omega)$$

for all $t \in [0, \infty)$

$$\int_0^T f(t, \omega) \Delta t = \int_0^T \tilde{f}(t, \omega) \, dt$$

$$\int_0^T f(t, \omega) \Delta W_t = \int_0^T \tilde{f}(t, \omega) \, dW_t$$

Definition 3 (Δ - stochastic differential equation)

$$\Delta X_t = b(t, X_t) \Delta t + \sigma(t, X_t) \Delta W_t$$

$$\Leftrightarrow \quad X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} b(t, X_t) \Delta t + \int_{t_1}^{t_2} \sigma(t, X_t) \Delta W_t$$

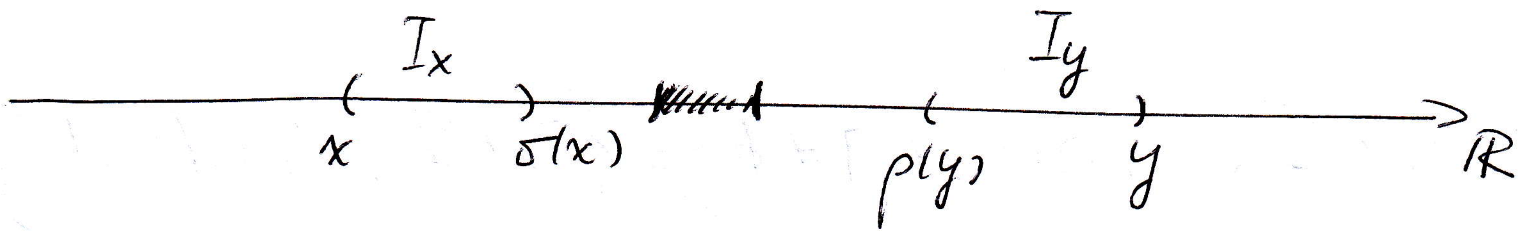
$t_1, t_2 \in \mathbb{T}$.

Geometric structure of a time scale.

$I_x = (x, \sigma(x))$ for right-scattered point $x \in \mathbb{T}$

$I_x = (\rho(x), x)$ for left-scattered point $x \in \mathbb{T}$

$\rho(x) \in \mathbb{T}$ $\quad \sigma(x) \in \mathbb{T}$



either $I_x \cap I_y = \phi$ or $I_x = I_y$

$$\underbrace{\text{"continuous part" + "jump part"}}_{\text{main philosophy}}$$

$\mathcal{I} = \{I_1, I_2, \cdots\}$ $\quad C =$ (at most) countable set of all left/right scattered points

(1) for any $k \neq l$, $I_k \cap I_l = \phi$

(2) either the left endpoint or right endpoint or both endpoints of any of the I_k's are in $\mathbb{T}$, and are left or right scattered

(3) $I_k \cap \mathbb{T} = \phi$ for any $k = 1, 2, \cdots$

(4) any point in C is a left or right endpoint of one of the I_k's.

$$I_k = (s_{I_k}^-, s_{I_k}^+) \qquad s_{I_k}^-, s_{I_k}^+ \in \mathbb{T}$$

$t_1, t_2 \in \mathbb{T}$

$$\{I_k \in \mathcal{I},\ I_k \cap [t_1, t_2] \neq \phi\} = \{I_k \in \mathcal{I} : I_k \subset (t_1, t_2)\}$$

Theorem 1 Let $f \in \mathbb{C}^{(2)}(\mathbb{R}_+, \mathbb{R})$ and any $t_1 \leq t_2$, $t_1, t_2 \in [0, \infty)_{\mathbb{T}}$, then we have

$$f(t_2, W_{t_2}) - f(t_1, W_{t_1})$$

$$= \int_{t_1}^{t_2} \frac{\partial f}{\partial t}(s, W_s)\Delta s + \int_{t_1}^{t_2} \frac{\partial f}{\partial x}(s, W_s)\Delta W_s$$

$$+ \frac{1}{2}\int_{t_1}^{t_2} \frac{\partial^2 f}{\partial x^2}(s, W_s)\Delta s + \sum_{\substack{I_k \in \mathcal{I} \\ I_k \subset (t_1, t_2)}} \left[\begin{array}{l} f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) \\ - \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \\ - \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) \\ - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \end{array} \right]$$

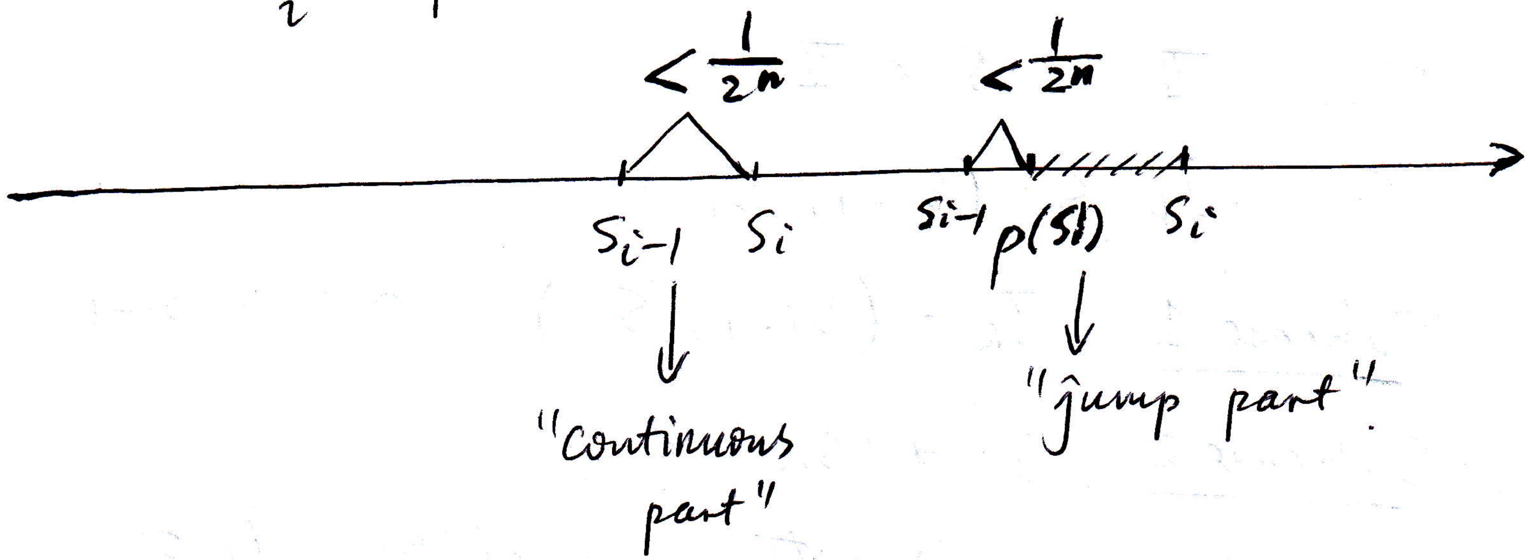
Basic Idea

$$[t_1, t_2]_{\mathbb{I}} = [t_1, t_2] \cap \mathbb{I}$$

$\pi^{(n)}$: $t_1 = s_0 < s_1 < \cdots < s_n = t_2$

(1) each $s_i \in \mathbb{I}$;

(2). $\max_i (\rho(s_i) - s_{i-1}) \le \frac{1}{2^n}$, $i = 1, 2, \ldots, n$



Class (a): all intervals (s_{i-1}, s_i) such that for all $I_k \in \mathcal{I}$ we have $I_k \cap (s_{i-1}, s_i) = \phi$

Class (b): all intervals (s_{i-1}, s_i) such that there exist some $I_k \in \mathcal{I}$ with $(s_{i-1}, s_i) \cap I_k \neq \phi$

Class (a) Characterization

$p(s_i) = s_i$

because otherwise $(p(s_i), s_i)$ will be one of the I_k's

so $s_i - s_{i-1} < \frac{1}{2^n}$

Class (b) Characterization

$s_{i-1} \in \mathbb{I}$ $s_i \in \mathbb{I}$

So $I_k \subseteq (s_{i-1}, s_i)$

Subcase 1 $I_k = (s_{i-1}, s_i)$, $p(s_i) = s_{i-1}$

Subcase 2 $I_k \neq (s_{i-1}, s_i)$

$(p(s_i), s_i) \in \mathcal{I}$ is one of the I_k's

$p(s_i) - s_{i-1} < \frac{1}{2^n}$

all I_k's are contained in intervals (s_{i-1}, s_i) that belong to class (b)

$$\sum_{\substack{(s_{i-1}, s_i) \in (b) \\ s_{i-1} < p(s_i) < s_i}} (p(s_i) - s_{i-1}) < \frac{n}{2^n}$$

$$f(t_2, W_{t_2}) - f(t_1, W_{t_1})$$

$$= \sum_{i=1}^{n} \left[f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) \right]$$

$$= \sum_{(a)} \left[f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) \right] + \sum_{(b)} \left[f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) \right]$$

$$= (\mathrm{I}) + (\mathrm{II})$$

$$(\mathrm{I}) = \sum_{(a)} \left[f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) \right]$$

$$= \sum_{(a)} \left[\frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) \right.$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2$$

$$\left. + R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) \right]$$

$$= \underbrace{\sum_{i=1}^{n} \left[\frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) \right]}_{(\mathrm{III})_1}$$

$$+ \underbrace{\left(\sum_{(a)} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2 + \sum_{(b)} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right)}_{(\mathrm{III})_2},$$

$$\underbrace{- \sum_{(b)} \left[\frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right]}_{(\mathrm{IV})}$$

$$+ \underbrace{\sum_{(a)} R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i})}_{(\mathrm{V})}$$

- Auxiliary Result on convergence of Δ-deterministic and Stochastic Integrals

$f \in \mathbb{C}([t_1, t_2])$

partition $\pi^{(n)}: t_1 = s_0 < s_1 < \dots < s_n = t_2$

$$\max_{i=1,2,\dots,n} (p(s_i) - s_{i-1}) < \frac{1}{2^n}$$

then $$\mathbb{P}\left(\lim_{n\to\infty} \sum_{i=1}^{n} f(s_{i-1}, w)(s_i - s_{i-1}) = \int_{t_1}^{t_2} f(s,w)\Delta s\right) = 1$$

$$\mathbb{P}\left(\lim_{n\to\infty} \sum_{i=1}^{n} f(s_{i-1}, w)(W_{s_i} - W_{s_{i-1}}) = \int_{t_1}^{t_2} f(s,w)\Delta W_s\right) = 1$$

Details skipped

This settles $(\text{III})_1 \longrightarrow \int_{t_1}^{t_2} \frac{\partial f}{\partial t}(s, W_s)\Delta s + \int_{t_1}^{t_2} \frac{\partial f}{\partial x}(s, W_s)\Delta W_s$

- Auxiliary Result on Quadratic Variation of Brownian Motion

partition $\pi^{(n)}: t_1 = s_0 < s_1 < \dots < s_n = t_2$

$$\max_{i=1,2,\dots,n} (p(s_i) - s_{i-1}) < \frac{1}{2^n}$$

$\mathbb{E} f^2(t,w)$ bounded on $[t_1, t_2]$

$$\mathbb{P}\left(\lim_{n\to\infty}\left[\sum_{(a)} f(s_{i-1},w)(W_{s_i}-W_{s_{i-1}})^2 - \sum_{(a)} f(s_{i-1},w)(s_i - s_{i-1})\right] = 0\right) = 1$$

This settles

$$\boxed{\text{III}}_2 \longrightarrow \int_{t_1}^{t_2} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\Delta s$$

One can also show $(V) \to 0$

$$(IV) \longrightarrow \sum_{\substack{I_k \in \mathcal{I} \\ I_k \subset (t_1,t_2)}} \Big[\frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) + \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \Big]$$

Be careful with the $(s_{i-1}, p(s_i))$ part

Remember $\displaystyle\sum_{i \in (b)} (p(s_i) - s_{i-1}) < \frac{n}{2^n}$

Stochastic Exponential

$$\Delta X_t = A(t)\, X_t\, \Delta W_t$$

$$X(t_0) = 1 \qquad t \in \mathbb{I}$$

Denote $\quad X_\cdot = \mathcal{E}_A(\cdot, t_0)$

$$\mathcal{E}_A(t, t_0) = 1 + \int_{t_0}^{t} A(s)\, \mathcal{E}_A(s, t_0)\, \Delta W_s$$

for all $t \in \mathbb{I}$

Denote $\quad U(t, t_0) = \prod_{I_k \in \mathcal{I},\ I_k \subset (t_0, t)} \left[1 + A(s_{I_k}^-)\left(W_{s_{I_k}^+} - W_{s_{I_k}^-}\right)\right]$

$$V(t, t_0) = \exp\left(\int_{t_0}^{t} A(s)\Delta W_s - \frac{1}{2}\int_{t_0}^{t} A^2(s)\Delta s - D(t, t_0)\right)$$

$$D(t, t_0) = \sum_{\substack{I_k \in \mathcal{I} \\ I_k \subset (t_0, t)}} A(s_{I_k}^-)\left(W_{s_{I_k}^+} - W_{s_{I_k}^-}\right) - \frac{1}{2}\sum_{\substack{I_k \in \mathcal{I} \\ I_k \subset (t_0, t)}} A^2(s_{I_k}^-)\left(s_{I_k}^+ - s_{I_k}^-\right)$$

Change of Measure

$$B_t = W_t - \int_0^t A(s) \Delta s$$

same as

$$\tilde{B}_t = \tilde{W}_t - \int_0^t \tilde{A}(s)\, ds$$

where $\tilde{A}(s, \omega) = A(\sup[0, s]_{\mathbb{T}}, \omega)$

$$\mathcal{G}_A(t, t_0) = \exp\left(\int_0^t \tilde{A}(s)\, dW_s - \frac{1}{2}\int_0^t \tilde{A}^2(s)\, ds\right)$$

$$= \exp\left(\int_0^t A(s) \Delta W_s - \frac{1}{2}\int_0^t A^2(s) \Delta s\right)$$

$$\frac{d\mathbb{P}^B}{d\mathbb{P}^W} = \mathcal{G}_A(T, t_0)$$

Option Pricing on time scales.