Let us consider flow over a heated vertical wall as shown in figure. A free convection flow induced by buoyancy forces is established close to the wall.
Boundary conditions:

At $t = 0$:

$u = v = 0$, $T = T_{\text{inf}}$ (everywhere in the solution domain)

At $t > 0$:

$y = 0$: $u = v = 0$, $T = T_{w}$ (heated wall condition)

$y = \text{inf}$: $u = 0$, $T = T_{\text{inf}}$ (far field condition)

$x = 0$: $u = v = 0$, $T = T_{\text{inf}}$ (bottom wall condition)
Governing equations

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] \hspace{1cm} (1)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + g\beta(T - T_\infty) + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \] \hspace{1cm} (2)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + g\beta(T - T_\infty) + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \] \hspace{1cm} (3)

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{\mu}{\rho c_p} \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right] \] \hspace{1cm} (4)

The Navier-Stokes equations derived in Chapter 2 can be simplified using the boundary layer assumptions.

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] \hspace{1cm} (1)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} \] \hspace{1cm} (5)

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \] \hspace{1cm} (6)
Note the neglected terms in the above equations

Eq. 2: 2nd derivative of u w.r.t x is neglected because it is small compared to the other term. Pressure gradient is assumed to be zero.

Eq. 3: This equation is not solved in the present formulation. All the terms on the RHS are assumed to be zero. Since p is assumed to be a constant, there is no need to use this equation.

Eq. 4: 2nd derivative of temperature w.r.t x and the viscous dissipation terms are neglected.

The simplified equations (continuity, x-momentum and energy) can be non-dimensionalized as follows:

\[ \xi = x \left( g\beta \Delta T / \nu^3 \right)^{\frac{1}{3}} \]
\[ \eta = y \left( g\beta \Delta T / \nu^2 \right)^{\frac{1}{3}} \]
\[ \tau = \tau \left( g\beta \Delta T \right)^{\frac{2}{3}} / \nu^{\frac{1}{3}} \]
\[ \bar{u} = u / (\nu g\beta \Delta T)^{\frac{1}{3}} \]
\[ \bar{v} = v / (\nu g\beta \Delta T)^{\frac{1}{3}} \]
\[ \theta = \frac{T - T_{\infty}}{T_1 - T_{\infty}} = \frac{T - T_{\infty}}{\Delta T} \]
Note that $g_x$ is the magnitude of the gravity component in the $x$-direction. $\beta$ is the volumetric coefficient of thermal expansion defined as

$$\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T}$$

Further, we assume that the Prandtl number $Pr (= \mu c_p/k = \nu/\alpha) = 1$. The equations will now be obtained in a form that is easy to solve numerically.

Note $\beta$ has units of $(1/K)$

If the assumption $Pr = 1$ is not made, the RHS of the energy equation will have a $Pr$ factor.

The energy equation for non-unity Prandtl number can be differenced using the explicit formulation as follows

$$\frac{\theta^{n+1}_{i,j} - \theta^n_{i,j}}{\Delta \tau} + \overline{u}^n_{i,j} \frac{\theta^n_{i,j} - \theta^n_{i-1,j}}{\Delta \xi} + \overline{v}^n_{i,j} \frac{\theta^n_{i,j+1} - \theta^n_{i,j}}{\Delta \eta} =$$

$$\frac{1}{Pr} \left( \frac{\theta^n_{i,j+1} - 2\theta^n_{i,j} + \theta^n_{i,j-1}}{(\Delta \eta)^2} \right)^{n+1}_{i,j}$$

Note the use of BD for the second term on the LHS and FD for the third term and CD for the RHS term.
The x-momentum equation can now be written in the finite difference form as

\[
\frac{\bar{u}_{i,j}^{n+1} - \bar{u}_{i,j}^n}{\Delta \tau} + \frac{\bar{u}_{i,j}^n - \bar{u}_{i-1,j}^n}{\Delta \xi} + \frac{\bar{u}_{i,j}^n - \bar{u}_{i+1,j}^n}{\Delta \eta}
\]

\[
= \theta_{i,j}^{n+1} + \frac{\bar{u}_{i,j+1}^n - 2\bar{u}_{i,j}^n + \bar{u}_{i,j-1}^n}{(\Delta \eta)^2}
\] .................................\(8\)

The continuity equation becomes

\[
\frac{\bar{u}_{i,j}^{n+1} - \bar{u}_{i,j+1}^{n+1}}{\Delta \xi} + \frac{\bar{v}_{i,j}^{n+1} - \bar{v}_{i,j-1}^{n+1}}{\Delta \eta} = 0
\] .................................\(9\)

Equations 7, 8 and 9 are solved sequentially for each time step. Note that \(\theta^{n+1}\) is known once the energy equation is solved.

Therefore, in the momentum equation, it can be treated as a known quantity.

Since \(u\) is known from solving the x-momentum equation, \(v^{n+1}\) are the only unknowns in Eq. (9).

Thus these set of equations can be solved explicitly.
Program Completed

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