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Nonlinear Schrödinger Equations at Non-Conserved Critical Regularity

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by

Jason Carl Murphy

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Abstract of the Dissertation

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We study the initial-value problem for defocusing nonlinear Schrödinger equations of the form

$$\begin{cases} (i\partial_t + \Delta)u = |u|^p u, \\ u(0) = u_0. \end{cases}$$

Here $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space. We choose initial data u_0 from the critical Sobolev space $\dot{H}_x^{s_c}(\mathbb{R}^d)$, where $s_c := \frac{d}{2} - \frac{2}{p}$.

We adapt techniques that were originally developed to treat the mass- and energy-critical equations to the case of 'non-conserved' critical regularity. In particular, we follow the minimal counterexample approach to the induction on energy technique of Bourgain.

For a range of (d, s_c), we prove that any solution that remains bounded in the critical Sobolev space $\dot{H}_x^{s_c}$ must exist globally in time, obey spacetime bounds, and scatter to a free solution. In certain cases, the main result applies only to radial solutions. An equivalent formulation of the main result is the statement that any solution that fails to scatter must blow up its $\dot{H}_x^{s_c}$-norm.

The dissertation of Jason Carl Murphy is approved.

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To my family, for all of their love and support.

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Publications

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The radial defocusing nonlinear Schrödinger equation in three space dimensions. Submitted to Comm. PDE.

CHAPTER 1

Introduction

We study the initial-value problem for defocusing power-type nonlinear Schrödinger equations (NLS):

$$\begin{cases} (i\partial_t + \Delta)u = |u|^p u, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space and $p > 0$.

From a physical perspective, nonlinear Schrödinger equations serve as simple models for a variety of wave-like behaviors, including light in nonlinear media [30, 60], Bose–Einstein condensates [24], and even rogue water waves in the ocean [23, 27]. Mathematically, the simple structure of NLS makes it well-suited for rigorous analysis, while the variety of behaviors exhibited by solutions makes it a good model for more general dispersive systems.

The equation (1.1) enjoys a scaling symmetry, namely

$$u(t,x) \mapsto \lambda^{2/p} u(\lambda^2 t, \lambda x),$$

which defines an important notion of *criticality* for the equation. In particular, the only homogeneous L_x^2-based Sobolev space of initial data whose norm is left invariant by this rescaling is $\dot{H}_x^{s_c}(\mathbb{R}^d)$, where the *critical regularity* is defined by $s_c := d/2 - 2/p$. In this thesis, we study the *critical* problem for NLS, that is, we choose $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$. (Choosing $u_0 \in \dot{H}^s(\mathbb{R}^d)$ for $s > s_c$ results in the well-understood *subcritical* problem, while choosing $u_0 \in \dot{H}^s(\mathbb{R}^d)$ for $s < s_c$ results in the famously intractable *supercritical* problem.)

We begin by discussing two special cases of NLS that have received the most attention in the past, as they feature a conserved quantity at the level of the critical regularity. These are the *mass-critical NLS*, which corresponds to choosing $p = 4/d$, and the *energy-critical NLS*, which corresponds to choosing $p = 4/(d-2)$ (in dimensions $d \geq 3$). For the mass-critical NLS, the critical regularity is $s_c = 0$, and the conserved quantity is the *mass*, defined by

$$\mathcal{M}[u(t)] := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx.$$

For the energy-critical NLS, the critical regularity is $s_c = 1$, and the conserved quantity is the *energy*, defined by

$$E[u(t)] := \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{1}{p+2}|u(t,x)|^{p+2} \, dx.$$

Thanks to the work of many mathematicians over the last 10–15 years, it is now known that for the defocusing mass- and energy-critical NLS, arbitrary data in the critical Sobolev space lead to global solutions that obey spacetime bounds and scatter to free solutions. The energy-critical case was handled first by Bourgain [3], Grillakis [26], and Tao [62] for radial data, and subsequently by Colliander–Keel–Staffilani–Takaoka–Tao [15], Ryckman–Visan [52], and Visan [68, 69] for arbitrary data. The mass-critical case was treated first by Tao–Visan–Zhang [65], Killip–Visan–Zhang [44], and Killip–Tao–Visan [37] for radial data, and subsequently by Dodson [18, 19, 20] for arbitrary data. Killip–Visan [43] and Visan [70] have also revisited the energy-critical problem in light of new techniques developed to treat the mass-critical NLS.

A key ingredient in the defocusing mass- and energy-critical cases is a certain degree *a priori* control over solutions that follows from the conservation laws. In particular, one gets uniform in time $\dot{H}_x^{s_c}$-bounds. However, due to the critical nature of the problems, these bounds alone are not enough even to deduce global existence (much less spacetime bounds or scattering). A further significant difficulty then stems from the fact that none of the known *a priori* monotonicity formulae (that is, Morawetz estimates) scale like the mass or energy, and hence cannot be used directly. It was Bourgain's *induction on energy* technique that showed how one can ultimately move beyond this difficulty: by finding solutions that concentrate on a characteristic length scale, one can 'break' the scaling symmetry of the problem and hence bring the available estimates back into play, despite their non-critical scaling. All subsequent techniques developed to treat NLS at critical regularity have built upon this fundamental idea. We discuss some of these techniques in more detail below.

We next turn to the case of 'non-conserved' critical regularity, that is, $s_c \notin \{0,1\}$. In this case, one loses the *a priori* control in the form of $\dot H_x^{s_c}$-bounds. However, the success of the techniques developed to treat the mass- and energy-critical problems suggests that in fact, this should be the *only* missing ingredient for a proof of a global well-posedness and scattering. We thus arrive at the following conjecture.

Conjecture 1.0.1 *Let $d \geq 1$ and $p > 0$ such that $s_c := \frac{d}{2} - \frac{2}{p} \geq 0$. Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to* (1.1) *such that $u \in L_t^\infty \dot H_x^{s_c}(I \times \mathbb{R}^d)$. Then u is global, obeys spacetime bounds, and scatters to a free solution.*

Conjecture 1.0.1 (in the case of non-conserved critical regularity) was first addressed by Kenig–Merle [32]. They treated the cubic NLS in three dimensions, in which case $s_c = 1/2$. Some cases of Conjecture 1.0.1 in the energy-supercritical regime (i.e. $s_c > 1$) were subsequently treated by Killip–Visan [40]. In particular, they dealt with the cubic NLS in dimensions $d \geq 5$, as well as some other cases for which $s_c > 1$ and $d \geq 5$. The author has also addressed the energy-supercritical regime by treating the cases $1 < s_c < 3/2$ in dimension $d = 4$ in collaboration with C. Miao and J. Zheng [48], as well as treating the cases $1 < s_c < 3/2$ in dimension $d = 3$ for the case of radial solutions [51].

In this thesis, we present a range of cases of Conjecture 1.0.1 in the *intercritical* regime, that is, $0 < s_c < 1$. The results presented appear originally in [49, 50, 51].

We first extend the result of Kenig–Merle [32] by treating the cases

$$s_c = \tfrac{1}{2}, \quad d \geq 4. \tag{1.2}$$

As we will see, the $\dot H_x^{1/2}$-critical problem is greatly simplified by the presence of a Morawetz estimate with critical scaling, namely the Lin–Strauss Morawetz estimate. The results concerning (1.2) appear originally in [50].

The second set of cases that we consider is the following:

$$\begin{cases} s_c \in (\frac{1}{2}, \frac{3}{4}] & \text{if } d = 3, \\ s_c \in (\frac{1}{2}, 1) & \text{if } d \in \{4, 5\}. \end{cases} \tag{1.3}$$

The results addressing these cases appear originally in [49]. As we will see below, the restrictions on (d, s_c) in (1.3) arise from the intersection of several issues, some of which are technical, but some of which hint at deeper obstacles.

We also address the cases

$$s_c \in (\tfrac{3}{4}, 1), \quad d = 3, \tag{1.4}$$

that is, the cases 'missing' from (1.3). In order to treat these cases, however, we will need to restrict to the case of radial solutions. These cases appear originally in [51].

Finally, we will treat the cases

$$s_c \in (0, \tfrac{1}{2}), \quad d = 3. \tag{1.5}$$

These cases also appear originally in [51]. They represent the first work on Conjecture 1.0.1 in the regime $0 < s_c < 1/2$. As for the previous cases, however, treating these cases will require that we restrict to the radial setting.

Before proceeding to a more detailed discussion of our results, we pause here to mention some related problems. For results concerning the focusing mass- and energy-critical NLS, one can refer to [21, 31, 37, 39, 44]. These problems have also been studied via induction on energy and the developments thereof. For these problems there is a sharp threshold size for scattering, determined by so-called ground state solutions that arise from related elliptic problems. We also note that the conjecture analogous to Conjecture 1.0.1 for the nonlinear wave equation has also been studied. For results in this direction one can refer to [4, 5, 6, 22, 33, 41, 42, 54, 55].

1.1 Discussion of main results

The main result of this thesis is the following.

Theorem 1.1.1 *Let* (d, s_c) *satisfy* (1.2), (1.3), (1.4), *or* (1.5). *Suppose* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *is a maximal-lifespan solution to* (1.1) *such that* $u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^d)$. *In the case of* (1.4) *or* (1.5), *assume in addition that* u *is radial. Then* u *is global and obeys the spacetime bounds*

$$\iint_{\mathbb{R}\times\mathbb{R}^d} |u(t,x)|^{\frac{p(d+2)}{2}}\, dx\, dt \le C(\|u\|_{L_t^\infty \dot{H}_x^{s_c}}) \tag{1.6}$$

for some function $C : [0,\infty) \to [0,\infty)$. *Furthermore, the solution* u *scatters, that is, there exist* $u_\pm \in \dot{H}_x^{s_c}(\mathbb{R}^d)$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} = 0.$$

Standard arguments show that the scattering statement follows from the spacetime bounds (1.6). Thus the proof of Theorem 1.1.1 primarily consists in establishing these bounds.

Theorem 1.1.1 is a conditional result, as we do not know whether the assumed $\dot{H}^{s_c}$-bounds hold for all solutions. Unfortunately, proving such bounds still remains well beyond the reach of existing technology (although numerical results at least support the belief that such bounds do hold [16]). Nonetheless, Theorem 1.1.1 makes a definitive statement about the possible dynamics for (1.1): solutions cannot fail to scatter without blowing up their $\dot{H}_x^{s_c}$-norm.

To prove Theorem 1.1.1, we will follow the concentration compactness (or 'minimal counterexample') approach to induction on energy. Minimal counterexamples were introduced over the course of several papers in the context of the mass-critical NLS [1, 2, 7, 34, 35, 47]. They were first used to establish a global well-posedness result by Kenig–Merle [31], who developed the technique in the focusing energy-critical setting. Since then, the approach has become one of the most powerful techniques available for treating critical problems in dispersive PDE.

We can now give a brief outline the proof of Theorem 1.1.1. We argue by contradiction and assume that Theorem 1.1.1 is false. As we can prove that global existence and spacetime bounds for sufficiently small initial data (see Chapter 3), we deduce the existence of a threshold size, below which the result holds but above which we can find (almost) counterexamples.

Using concentration compactness arguments, we can prove the existence of nonscattering solutions living exactly at the threshold, that is, *minimal counterexamples*; furthermore, as a consequence of their minimality, these solutions can be shown to possess a strong compactness property, namely *almost periodicity* (see Chapter 4). To complete the proof of Theorem 1.1.1, we then need to rule out the existence of such minimal counterexamples.

The best known tools available for carrying out this final step are certain *a priori* monotonicity formulae that hold for solutions to defocusing NLS, referred to as *Morawetz estimates*. In most cases, however, the available estimates do not have the critical scaling for the problem and hence cannot be used directly. This is where compactness plays a key role: minimal counterexamples can be shown to concentrate on some (possibly time-dependent) length scale. In this sense, they break the scaling symmetry of the problem and hence bring the available Morawetz estimates back into play. By making use of (possibly modified versions of) Morawetz estimates, we can ultimately rule out the existence of minimal counterexamples and thereby complete the proof of Theorem 1.1.1 (see Chapters 5–9).

We now discuss in more detail the cases of Conjecture 1.0.1 that we address in this thesis. As described in the outline above, there are two major steps for the proof of Theorem 1.1.1: the reduction to almost periodic solutions, and the preclusion of almost periodic solutions.

Step 1: The reduction to almost periodic solutions.

For the first step, we begin with the following theorem.

Theorem 1.1.2 (The reduction to almost periodic solutions) *If Theorem 1.1.1 fails, then there exists a maximal-lifespan solution* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *to* (1.1) *such that* u *is almost periodic and blows up forward and backward in time. In the cases* (1.4) *and* (1.5), *the solution* u *is radial.*

For the precise definitions of 'maximal-lifespan solution' and 'blowup', see Chapter 3. We will define 'almost periodic' and prove Theorem 1.1.2 in Chapter 4. In fact, we will prove that if Theorem 1.1.1 fails, then there exist solutions with minimal $L_t^\infty \dot{H}_x^{s_c}$-norm among all blowup solutions. Minimal counterexamples can then be shown to possess the property of almost periodicity.

The reduction to almost periodic solutions is now fairly standard in the field of nonlinear dispersive PDE, especially in the setting of NLS. Keraani [35] originally established the existence of minimal blowup solutions to NLS, while Kenig–Merle [32] were the first to use them as a tool to prove global well-posedness. This technique has since been used in a variety of settings and has proven to be extremely effective. One can refer to [28, 32, 37, 39, 40, 43, 44, 48, 49, 50, 65, 70] for some examples in the case of NLS. See also [38] for a good introduction to the method. The general approach is well-understood; however, we will see that to carry out this reduction in the cases we consider will require some new ideas and careful analysis.

The proof of Theorem 1.1.2 requires three main ingredients: (i) a profile decomposition for the linear Schrödinger propagator $e^{it\Delta}$, (ii) a stability result for the nonlinear equation, and (iii) a decoupling statement for nonlinear profiles. The first profile decompositions established for $e^{it\Delta}$ were adapted to the mass- and energy-critical settings [1, 7, 34, 47]; the case of non-conserved critical regularity was addressed in [53]. For the cases we consider in this thesis, we will be able to import the profile decomposition that we need directly from [53] (see Lemma 4.3.3).

Ingredients (ii) and (iii) are closely related, in that the decoupling must be established in a space that is dictated by the stability result. Most often, stability results require errors to be small in a space with the scaling-critical number of derivatives, that is, $|\nabla|^{s_c}$. We will prove such a result in Chapter 3 for the cases (1.3), (1.4), and (1.5). In [34], Keraani showed how to establish the decoupling in such a space for the energy-critical problem ($s_c = 1$). The argument relies on pointwise estimates, and hence it is also applicable to the mass-critical problem ($s_c = 0$). However, for the case of non-conserved critical regularity (that is, $s_c \notin \{0, 1\}$) the nonlocal nature of $|\nabla|^{s_c}$ prevents the direct use of this argument.

In some cases for which $s_c \notin \{0, 1\}$, it is nonetheless possible to adapt the arguments of [34] in order to establish the decoupling in a space with s_c derivatives. Kenig–Merle [32] were able to succeed in the case $(d, s_c) = (3, 1/2)$ (in which case the nonlinearity is cubic) by exploiting the algebraic nature of the nonlinearity and making use of a paraproduct estimate. Killip–Visan [40] handled some cases for which $s_c > 1$ by utilizing a square function of Strichartz [58] that shares estimates with $|\nabla|^{s_c}$. Their approach relies strongly on the fact that $s_c > 1$. The cases that we consider feature both fractional derivatives ($0 < s_c < 1$) and non-polynomial nonlinearities, and hence they present a new technical challenge.

In Chapter 4, we will present two approaches for establishing the necessary decoupling statement. The first (appearing originally in [49]) allows us to adapt the arguments of [34] in the presence of both fractional derivatives and non-polynomial nonlinearities. The approach relies on a careful reworking of the proof of the fractional chain rule, in which the Littlewood–Paley square function and other tools from harmonic analysis allow us to work at the level of individual frequencies. As we will see, this approach requires that $p > 1$. We use this approach to deal with cases (1.3), (1.4), and (1.5).

The second approach (appearing originally in [50]) is inspired by the work of Holmer–Roudenko [28] on the focusing $\dot{H}_x^{1/2}$-critical NLS in dimension $d = 3$. It relies on the observation that for $s_c = 1/2$, one can develop a stability theory that only requires errors to be small in a space without derivatives. In Chapter 3 we will prove such a stability result for the cases (1.2). To establish the decoupling in a space without derivatives, we can then simply rely on pointwise estimates and apply the arguments of [34] directly. For the cases (1.2), we thus avoid the technical issues related to fractional differentiation altogether.

After giving the proof of Theorem 1.1.2 in Chapter 4, we will carry out a few further reductions to the class of solutions that we consider (see Theorems 4.5.1 through 4.5.4). Thus the proof of Theorem 1.1.1 is reduced to ruling out the existence of solutions as in Theorems 4.5.1 through 4.5.4.

Step 2: The preclusion of almost periodic solutions.

We now turn to the second step of the proof of Theorem 1.1.1. As alluded to above, the best tools available for this step are monotonicity formulae that hold for defocusing NLS known as *Morawetz estimates*. In this thesis, we will use versions of the Lin–Strauss Morawetz (introduced originally in [46]) and the interaction Morawetz (introduced originally in [14]). For a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to defocusing NLS in dimensions $d \geq 3$, the *Lin–Strauss Morawetz* is given by

$$\iint_{I\times\mathbb{R}^d} \frac{|u(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim \||\nabla|^{1/2}u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}^2, \tag{1.7}$$

while the *interaction Morawetz* reads

$$\begin{aligned} -\int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} &|u(t,x)|^2\Delta(\tfrac{1}{|x-y|})|u(t,y)|^2\,dx\,dy\,dt \\ &\lesssim \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}^2 \||\nabla|^{1/2}u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}^2. \end{aligned} \tag{1.8}$$

The estimate (1.7) has the scaling of $\dot H_x^{1/2}$. The estimate (1.8) has the scaling of $\dot H_x^{1/4}$, but requires control over both the L_x^2-norm and the $\dot H_x^{1/2}$-norm. Because of the weight $\frac{1}{|x|}$, the estimate (1.7) is well-suited for preventing concentration near the origin, and hence it is most effective in the radial setting. The estimate (1.8), on the other hand, controls the degree to which mass can interact with itself throughout all of $\mathbb{R}^d$, and hence it is useful even in the non-radial setting. Roughly speaking, we use the interaction Morawetz if possible and 'retreat' to the Lin–Strauss Morawetz otherwise. In these latter cases, we will only be able to treat the case of radial solutions. The exception to this rule is the case (1.2), for which $s_c = 1/2$. In this case the Lin–Strauss Morawetz is the right tool to use, even in the non-radial setting. Indeed, it has the critical scaling for the problem.

We employ the interaction Morawetz inequality for the cases (1.3). Note, however, that we cannot use the estimate directly, as the solutions we consider need only belong to $L_t^\infty \dot H_x^{s_c}$ and hence the right-hand side of (1.8) need not be finite. One solution to this issue, first implemented by Colliander–Keel–Staffilani–Takaoka–Tao [15] in their pioneering work on the energy-critical NLS, is to prove a frequency-localized version of (1.8). By now, this approach has been adapted to many different settings [18, 19, 20, 21, 43, 48, 49, 52, 68, 69, 70].

As $s_c > 1/2$ for the cases (1.3), we work with the *high* frequency component of solutions to guarantee that the right-hand side of (1.8) is finite. To arrive at a useful estimate requires that we control the error terms that arise in the standard interaction Morawetz after we apply a frequency cutoff to a solution. For $s_c > 3/4$ in dimension $d = 3$, we cannot control one of the error terms unless we also impose a spatial truncation; this is the approach taken in the energy-critical setting, for example [15, 43]. However, this spatial truncation results in additional error terms that require control over the solution at the level of $\dot{H}^1_x$. Thus in the energy-critical case one can therefore push the arguments through, while in the case $3/4 < s_c < 1$ we were unable to control the resulting error terms.

Thus for the cases $3/4 < s_c < 1$ in dimension $d = 3$ (that is, case (1.4)), we instead use a frequency-localized Lin–Strauss Morawetz. As the scaling of the Lin–Strauss Morawetz is closer to the critical scaling, it turns out to be easier to control the error terms that arise from the frequency cutoff. Again we work with the high frequency component of solutions to guarantee that the right-hand side of the estimate is finite. The drawback is that using the Lin–Strauss Morawetz is only 'strong' enough to deal with radial solutions.

We also employ the Lin–Strauss Morawetz to deal with the cases $0 < s_c < 1/2$ in dimension $d = 3$ (that is, case (1.5)). In this setting, we need to work with the *low* frequency components to guarantee that the right-hand side of (1.7) is finite. Again, the use of the Lin–Strauss Morawetz ultimately leads to the restriction to radial solutions for the cases (1.5). It would be more difficult to prove a frequency-localized interaction Morawetz for these cases, as one would need to truncate both the low and high frequencies in order to control both the L^2_x- and $\dot{H}^{1/2}_x$-norms of the solution. While this approach could ultimately prove to be tractable, we did not pursue it here.

We now describe in some detail how we preclude the existence of almost periodic solutions.

Let us first discuss the simplest case, namely (1.2). This case, which we address in Chapter 5, corresponds to $s_c = 1/2$ in dimensions $d \geq 4$. We treat separately the cases of finite- and infinite-time blowup. Using the conservation of mass, we find that finite-time blowup solutions must have zero mass, which contradicts the fact that they blowup in the first place. For infinite-time blowup, we use the critically-scaling Lin–Strauss Morawetz. On one hand, the quantity appearing in (1.7) is bounded (as the solution is bounded in $\dot{H}^{1/2}_x$).

On the other hand, we can show that for almost periodic solutions this quantity diverges as the time interval grows. Thus we get a contradiction in this case as well.

We now turn to the remaining cases, that is, (1.3), (1.4), and (1.5). Before proceeding, we will need at least a heuristic description of almost periodicity. Leaving the exact definition to Chapter 4, we can say that a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ is almost periodic if it concentrates at each $t \in I$ around some spatial center $x(t)$ and at some frequency scale $N(t)$. The solution thus concentrates at a spatial scale of $N(t)^{-1}$, and in order to belong to $\dot{H}^{s_c}_x(\mathbb{R}^d)$ it should have amplitude $\sim N(t)^{d/2-s_c}$. With these heuristics in mind, we can use scaling arguments to approximate the size of the quantities appearing in the Morawetz estimates. In particular, we expect that

$$Q_{LS}(u; I) := \iint_{I\times\mathbb{R}^d} \frac{|u(t,x)|^{p+2}}{|x|}\,dx\,dt \sim_u \int_I N(t)^{3-2s_c}\,dt,$$

at least if $x(t)$ is small, and

$$Q_{IM}(u; I) := -\int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(t,x)|^2 \Delta(\tfrac{1}{|x-y|})|u(t,y)|^2\,dx\,dy\,dt \sim_u \int_I N(t)^{3-4s_c}\,dt,$$

regardless of the behavior of $x(t)$.

We therefore expect the Lin–Strauss Morawetz to preclude the existence of almost periodic solutions for which $\int N(t)^{3-2s_c}\,dt = \infty$ (and $x(t)$ is is small), while the interaction Morawetz should preclude almost periodic solutions for which $\int N(t)^{3-4s_c}\,dt = \infty$ (regardless of the behavior of $x(t)$). We refer to these solutions as *quasi-solitons*, and indeed, we will use Morawetz estimates to rule out this type of solution.

Let us first consider the case (1.3), for which we will employ the interaction Morawetz. We have already seen that we will need to prove a version of this estimate for the high frequencies of a solution, say $u_{>N}$, which will introduce errors. By choosing N small enough, we hope to capture 'most' of the solution and arrive at an estimate of the form

$$Q_{IM}(u_{>N}; I) \lesssim \|u_{>N}\|^2_{L^\infty_t L^2_x}\|u_{>N}\|^2_{L^\infty_t \dot{H}^{1/2}_x} + errors.$$

Note that by using Bernstein estimates (see Lemma 2.2.1), we can control the first term by $N^{1-4s_c}\|u\|^4_{L^\infty_t \dot{H}^{s_c}_x}$. To bound the errors will require good control over the low frequencies, but it remains to see exactly what estimates we will need.

To this end, we make the scaling arguments described above precise and establish the *lower* bound

$$Q_{IM}(u_{>N};I) \gtrsim_u \int_I N(t)^{3-4s_c}\,dt$$

for N sufficiently small (see Proposition 4.2.2). Thus we see that if we can make the errors above much smaller than $\int_I N(t)^{3-4s_c}\,dt$, we will be able combine the upper and lower bounds for $Q_{IM}(u_{>N};I)$ to get an estimate of the form

$$\int_I N(t)^{3-4s_c}\,dt \lesssim_u N^{1-4s_c} + \eta \int_I N(t)^{3-4s_c}\,dt.$$

For η small, this implies an upper bound on $\int_I N(t)^{3-4s_c}\,dt$, and hence prevents the quasi-soliton scenario.

The key to carrying out this argument in detail is to get control over the low frequencies of almost periodic solutions in terms of the quantity $\int_I N(t)^{3-4s_c}\,dt$. We prove just such an estimate (Proposition 6.3.1) in Chapter 6, and in Chapter 8 we use it to control the error terms that arise in the frequency-localized interaction Morawetz (Proposition 8.3.1).

Estimates such as Proposition 6.3.1 go by the name of *long-time Strichartz estimates*. Such estimates first appeared in the work of Dodson [18] in the setting of the mass-critical NLS. They have since appeared in the energy-critical setting [43, 70], the energy-supercritical setting [48], and the intercritical setting [49, 51].

It remains to describe how we preclude the existence of almost periodic solutions for which $\int N(t)^{3-4s_c}\,dt < \infty$, which we refer to as *frequency-cascades*. This case, which we treat in Chapter 7, turns out to be relatively simple: the long-time Strichartz estimates give such good control over the low frequencies that we can show that frequency-cascades have zero mass, contradicting the fact that they blow up.

Finally, we discuss the cases (1.4) and (1.5). The arguments to treat these cases are analogous to the ones used for the case (1.3), except we use the Lin–Strauss Morawetz inequality instead of the interaction Morawetz inequality, and hence the relevant quantity is $\int_I N(t)^{3-2s_c}\,dt$. It turns out that we cannot prove the necessary lower bounds for $Q_{LS}(u;I)$

unless we have control over $x(t)$. Our solution to this issue is to restrict ourselves to the radial setting, in which case $x(t) \equiv 0$.

For the case (1.4), we once again need to work with $u_{>N}$ in the Morawetz estimate. We prove a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz (Proposition 6.2.1) in Chapter 6. This estimate allows us to prove a frequency-localized Lin–Strauss Morawetz (Proposition 8.2.1), which we use to rule out quasi-solitons in Chapter 9. We treat the frequency-cascade scenario in Chapter 7; once again, in this case the long-time Strichartz estimate gives such good control over the low frequencies that we can show the solutions have zero mass, giving a contradiction.

For the case (1.5), we instead need to work with $u_{\leq N}$ in the Morawetz estimate. We therefore prove a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz for the *high* frequencies of solutions (Proposition 6.1.1). This estimate allows us to prove a frequency-localized Lin–Strauss Morawetz (Proposition 8.1.1), which we use to rule out quasi-solitons in Chapter 9. We treat the frequency-cascade scenario in Chapter 7. This time, the long-time Strichartz estimate gives such good control over the *high* frequencies that we can show the solutions have zero energy, giving a contradiction.

The remainder of the thesis is organized as follows.

In Chapter 2, we collect notation and useful lemmas, including tools from harmonic analysis and estimates related to the linear Schrödinger equation.

In Chapter 3, we develop the local theory for (1.1). This includes local well-posedness results, as well as stability results that play an important role in the reduction to almost periodic solutions.

In Chapter 4, we introduce the notion of almost periodic solutions. We collect some standard results concerning such solutions and prove lower bounds for the quantities appearing in the Morawetz inequalities. We also carry out the proof of the reduction to almost periodic solutions, as well as some further reductions. In the end, we see that to prove Theorem 1.1.1 it suffices to rule out the existence of almost periodic solutions as in Theorems 4.5.1 through 4.5.4.

In Chapter 5, we treat the $\dot{H}_x^{1/2}$-critical case, that is, case (1.2). Specifically, we rule out solutions as in Theorem 4.5.1.

In Chapter 6, we prove three long-time Strichartz estimates. The first is a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz for high frequencies, which we use for case (1.5). The second is a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz for low frequencies, which we use for case (1.4). The third is a long-time Strichartz estimate adapted to the interaction Morawetz, which we use for case (1.3).

In Chapter 7, we preclude the existence of frequency-cascades. We first rule out the existence of almost periodic solutions as in Theorem 4.5.4 for which $\int_0^\infty N(t)^{3-2s_c}\,dt < \infty$. We then rule out the existence of almost periodic solutions as in Theorem 4.5.3 for which $\int_0^{T_{max}} N(t)^{3-2s_c}\,dt < \infty$. Finally, we rule out the existence of almost periodic solutions as in Theorem 4.5.2 for which $\int_0^{T_{max}} N(t)^{3-4s_c}\,dt < \infty$.

In Chapter 8, we prove three frequency-localized Morawetz estimates. The first is a frequency-localized Lin–Strauss Morawetz for the case (1.5); the second is a frequency-localized Lin–Strauss Morawetz for the case (1.4); the third is a frequency-localized interaction Morawetz for the case (1.3).

In Chapter 9, we preclude the existence of quasi-solitons. We first rule out the existence of almost periodic solutions as in Theorem 4.5.4 for which $\int_0^\infty N(t)^{3-2s_c}\,dt = \infty$. We then rule out the existence of almost periodic solutions as in Theorem 4.5.3 for which $\int_0^{T_{max}} N(t)^{3-2s_c}\,dt = \infty$. Finally, we rule out the existence of almost periodic solutions as in Theorem 4.5.2 for which $\int_0^{T_{max}} N(t)^{3-4s_c}\,dt = \infty$.

CHAPTER 2

Notation and useful lemmas

In this chapter we set notation, collect some useful lemmas from harmonic analysis, and describe some results pertaining to the linear Schrödinger equation.

2.1 Notation and basic estimates

For nonnegative quantities X and Y, we write $X \lesssim Y$ to denote the inequality $X \leq CY$ for some constant $C > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. The dependence of implicit constants on parameters will be indicated by subscripts, e.g. $X \lesssim_u Y$ denotes $X \leq CY$ for some $C = C(u)$. The dependence of constants on the ambient dimension d or the power of the nonlinearity p will not be explicitly indicated.

We use the expression $\emptyset(X)$ to denote a finite linear combination of terms that resemble X up to Littlewood–Paley projections, maximal functions, and complex conjugation.

For a time interval I, we write $L_t^q L_x^r(I \times \mathbb{R}^d)$ for the Banach space of functions $u : I \times \mathbb{R}^d \to \mathbb{C}$ equipped with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} := \left(\int_I \|u(t)\|_{L_x^r}^q \, dt \right)^{1/q},$$

with the usual adjustments when q or r is infinity. We will at times use the abbreviations $\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} = \|u\|_{L_t^q L_x^r}$ and $\|u\|_{L_x^r(\mathbb{R}^d)} = \|u\|_{L_x^r} = \|u\|_r$. Given $1 \leq r \leq \infty$, we write r' for the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

Throughout we will denote the nonlinearity $|u|^p u$ by $F(u)$. We record here some basic pointwise estimates related to the nonlinearity.

First, we have the following basic pointwise estimates:

$$\big||u+v|^p - |u|^p u\big| \lesssim |v|^{p+1} + |v|\,|u|^p,$$
$$\big||u+v|^{p+2} - |u|^{p+2} - |v|^{p+2}\big| \lesssim |u|\,|v|^{p+1} + |u|^{p+1}|v|.$$

Next, we have the following estimate.

Lemma 2.1.1 *Let $1 < p \leq 2$. Then*

$$\big|\,|a+c|^p - |a|^p - |b+c|^p + |b|^p\,\big| \lesssim |a-b|\,|c|^{p-1} \tag{2.1}$$

for all $a, b, c \in \mathbb{C}$.

Proof. Defining $G(z) := |z+c|^p - |z|^p$, the fundamental theorem of calculus gives

$$\text{LHS}(2.1) = \left|(a-b)\int_0^1 G_z(b+\theta(a-b))\,d\theta + \overline{(a-b)}\int_0^1 G_{\bar{z}}(b+\theta(a-b))\,d\theta\right|.$$

Thus, to establish (2.1), it will suffice to establish

$$|G_z(z)| + |G_{\bar{z}}(z)| \lesssim |c|^{p-1} \quad \text{uniformly for } z \in \mathbb{C}.$$

That is, we need to show

$$\big|\,|z+c|^{p-2}(z+c) - |z|^{p-2}z\,\big| \lesssim |c|^{p-1}$$

uniformly in z. If $c = 0$, this inequality is obvious. Otherwise, setting $z = c\,\zeta$ reduces the problem to showing

$$\big|\,|z+1|^{p-2}(z+1) - |z|^{p-2}z\,\big| \lesssim 1 \tag{2.2}$$

uniformly in z. For $|z| \lesssim 1$, we immediately get (2.2) from the triangle inequality. For $|z| \gg 1$, we can use the fundamental theorem of calculus and the fact that $p \leq 2$ to see

$$\big|\,|z+1|^{p-2}(z+1) - |z|^{p-2}z\,\big| \lesssim |z|^{p-2} \lesssim 1.$$

Thus, we see that (2.2) holds, which completes the proof of Lemma 2.1.1. ∎

2.2 Tools from harmonic analysis

We define the Fourier transform on $\mathbb{R}^d$ by

$$\widehat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

The fractional differentation operators $|\nabla|^s$ are then defined via $\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi)$. The corresponding homogeneous Sobolev norm is given by

$$\|u\|_{\dot{H}^s_x(\mathbb{R}^d)} := \||\nabla|^s f\|_{L^2_x(\mathbb{R}^d)}.$$

The standard Littlewood–Paley projection operators are defined as follows. Let φ be a radial bump function supported on $\{|\xi| \leq 11/10\}$ and equal to one on $\{|\xi| \leq 1\}$. For $N \in 2^{\mathbb{Z}}$, we define

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &:= \widehat{f_{\leq N}}(\xi) := \varphi(\xi/N)\widehat{f}(\xi),\\
\widehat{P_{> N} f}(\xi) &:= \widehat{f_{> N}}(\xi) := (1-\varphi(\xi/N))\widehat{f}(\xi),\\
\widehat{P_{N} f}(\xi) &:= \widehat{f_{N}}(\xi) := (\varphi(\xi/N)-\varphi(\xi/2N))\widehat{f}(\xi).
\end{aligned}$$

We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<K\leq N} P_K$$

for $M < N$. Throughout the thesis all such summations should be understood to be over $K \in 2^{\mathbb{Z}}$.

The Littlewood–Paley projection operators are Fourier multiplier operators, and hence they commute with the free Schrödinger propagator $e^{it\Delta}$, as well as differential operators such as $(i\partial_t + \Delta)$ and $|\nabla|^s$. They also obey the following standard estimates.

Lemma 2.2.1 (Bernstein estimates) *For $1 \leq r \leq q \leq \infty$ and $s \geq 0$,*

$$\begin{aligned}
\||\nabla|^{\pm s} P_N f\|_{L^r_x(\mathbb{R}^d)} &\sim N^{\pm s} \|P_N f\|_{L^r_x(\mathbb{R}^d)},\\
\||\nabla|^{s} P_{\leq N} f\|_{L^r_x(\mathbb{R}^d)} &\lesssim N^{s} \|P_{\leq N} f\|_{L^r_x(\mathbb{R}^d)},\\
\|P_{>N} f\|_{L^r_x(\mathbb{R}^d)} &\lesssim N^{-s} \||\nabla|^{s} P_{> N} f\|_{L^r_x(\mathbb{R}^d)},\\
\|P_{\leq N} f\|_{L^q_x(\mathbb{R}^d)} &\lesssim N^{\frac{d}{r}-\frac{d}{q}} \|P_{\leq N} f\|_{L^r_x(\mathbb{R}^d)}.
\end{aligned}$$

Lemma 2.2.2 (Littlewood–Paley square function estimates) *For* $1 < r < \infty$,

$$\big\| \big(\sum |P_N f(x)|^2 \big)^{1/2} \big\|_{L_x^r(\mathbb{R}^d)} \sim \|f\|_{L_x^r(\mathbb{R}^d)},$$
$$\big\| \big(\sum N^{2s} |f_N(x)|^2 \big)^{1/2} \big\|_{L_x^r(\mathbb{R}^d)} \sim \||\nabla|^s f\|_{L_x^r(\mathbb{R}^d)} \quad \textit{for } s > -d,$$
$$\big\| \big(\sum N^{2s} |f_{>N}(x)|^2 \big)^{1/2} \big\|_{L_x^r(\mathbb{R}^d)} \sim \||\nabla|^s f\|_{L_x^r(\mathbb{R}^d)} \quad \textit{for } s > 0.$$

Next we record some fractional calculus estimates that appear originally in [10]. For a textbook treatment, one can refer to [61].

Lemma 2.2.3 (Fractional product rule [10]) *Let* $s > 0$ *and let* $1 < r, r_j, q_j < \infty$ *satisfy* $\frac{1}{r} = \frac{1}{r_j} + \frac{1}{q_j}$ *for* $j = 1, 2$. *Then*

$$\||\nabla|^s (fg)\|_{L_x^r} \lesssim \|f\|_{L_x^{r_1}} \||\nabla|^s g\|_{L_x^{q_1}} + \||\nabla|^s f\|_{r_2} \|g\|_{L_x^{q_2}}.$$

Lemma 2.2.4 (Fractional chain rule [10]) *Let* $G \in C^1(\mathbb{C})$ *and* $s \in (0,1]$. *Let* $1 < r, r_2 < \infty$ *and* $1 < r_1 \leq \infty$ *be such that* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then*

$$\||\nabla|^s G(u)\|_{L_x^r} \lesssim \|G'(u)\|_{L_x^{r_1}} \||\nabla|^s u\|_{L_x^{r_2}}.$$

We will also make use of the following refinement of the fractional chain rule from [42].

Lemma 2.2.5 (Derivatives of differences [42]) *Let* $p > 1$ *and* $0 < s < 1$. *Then for* $1 < r, r_1, r_2 < \infty$ *such that* $\frac{1}{r} = \frac{1}{r_1} + \frac{p-1}{r_2}$, *we have*

$$\||\nabla|^s [|u+v|^p - |u|^p]\|_r \lesssim \||\nabla|^s u\|_{r_1} \|v\|_{r_2}^{p-1} + \||\nabla|^s v\|_{r_1} \|u+v\|_{r_2}^{p-1}.$$

Next, we prove a paraproduct estimate in the spirit of [70, Lemma 2.3]. (See also [48, 49, 51, 52, 68, 69].)

Lemma 2.2.6 (Paraproduct estimate) *Let* $d \geq 1$, $1 < r < r_1 < \infty$, $1 < r_2 < \infty$, *and* $0 < s < 1$ *satisfy* $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} + \frac{s}{d} < 1$. *Then*

$$\||\nabla|^{-s}(fg)\|_{L_x^r(\mathbb{R}^d)} \lesssim \||\nabla|^{-s} f\|_{L_x^{r_1}(\mathbb{R}^d)} \||\nabla|^s g\|_{L_x^{r_2}(\mathbb{R}^d)}.$$

Proof. We will prove the equivalent estimate

$$\||\nabla|^{-s}(|\nabla|^s f|\nabla|^{-s}g)\|_r \lesssim \|f\|_{r_1}\|g\|_{r_2}$$

by decomposing the left-hand side into low-high and high-low frequency interactions. We introduce the projections π_{lh} and π_{hl}, which are defined for any pair of functions ϕ, ψ by

$$\pi_{lh}(\phi,\psi) := \sum_{N\lesssim M} \phi_N\psi_M \quad \text{and} \quad \pi_{hl}(\phi,\psi) := \sum_{N\gg M} \phi_N\psi_M.$$

We first consider the low-high interactions. By Sobolev embedding, we have

$$\||\nabla|^{-s}\pi_{lh}(|\nabla|^s f, |\nabla|^{-s}g)\|_r \lesssim \|\pi_{lh}(|\nabla|^s f, |\nabla|^{-s}g)\|_{\frac{dr}{d+rs}}, \tag{2.3}$$

where we note that the constraint $\frac{1}{r}+\frac{s}{d}<1$ guarantees that $\frac{dr}{d+rs}>1$. We now note that the multiplier of the operator given by

$$T(f,g) := \pi_{lh}(|\nabla|^s f, |\nabla|^{-s}g), \quad \text{that is,} \quad \sum_{N\lesssim M} |\xi_1|^s\widehat{f_N}(\xi_1)|\xi_2|^{-s}\widehat{g_M}(\xi_2),$$

is a symbol of order zero with $\xi=(\xi_1,\xi_2)$. Thus, continuing from (2.3), we can use a theorem of Coifman–Meyer [11, 13] to conclude

$$\||\nabla|^{-s}\pi_{lh}(|\nabla|^s f, |\nabla|^{-s}g)\|_r \lesssim \|f\|_{r_1}\|g\|_{r_2}.$$

We next consider the high-low interactions. We note that the multiplier of the operator given by

$$S(f,h) := |\nabla|^{-s}\pi_{hl}(|\nabla|^s f, h), \quad \text{that is,} \quad \sum_{N\gg M} |\xi_1+\xi_2|^{-s}\widehat{f_N}(\xi_1)\widehat{h_M}(\xi)_2,$$

is also symbol of order zero. Thus we can use the result cited above together with Sobolev embedding to estimate

$$\||\nabla|^{-s}\pi_{hl}(|\nabla|^s f, |\nabla|^{-s}g)\|_r \lesssim \|f\|_{r_1}\||\nabla|^{-s}g\|_{\frac{r_1 r}{r_1-r}} \lesssim \|f\|_{r_1}\|g\|_{r_2}.$$

Combining the low-high and high-low interactions completes the proof of Lemma 2.2.6. ∎

We next recall Hardy's inequality.

Lemma 2.2.7 (Hardy's inequality) *For $0 < s < d$ and $1 < r < d/s$,*

$$\|\tfrac{1}{|x|^s} f\|_{L_x^r(\mathbb{R}^d)} \lesssim \||\nabla|^s f\|_{L_x^r(\mathbb{R}^d)}.$$

Using Hardy's inequality and interpolation, we can also derive the following estimate for $0 \leq s \leq 1$:

$$\||\nabla|^s(\tfrac{x-y}{|x-y|}u)\|_{L_x^2} \lesssim \||\nabla|^s u\|_{L_x^2} \tag{2.4}$$

uniformly for $y \in \mathbb{R}^d$.

We will also make use of the Hardy–Littlewood maximal function, which we denote by M. Along with the standard maximal function estimate (i.e. the fact that M is bounded on L_x^r for $1 < r \leq \infty$), we will use the fact that

$$\|\nabla M(f)\|_{L_x^r} \lesssim \|\nabla f\|_{L_x^r} \tag{2.5}$$

for $1 < r < \infty$ (see [36], for example). Finally, we will need the following inequalities in the spirit of [61, §2.3].

Lemma 2.2.8 *Let $\check{\psi}$ denote the convolution kernel of the Littlewood–Paley projection P_1. Define $\delta_y f(x) := f(x-y) - f(x)$. For $y \in \mathbb{R}^d$ and $N \in 2^{\mathbb{Z}}$, we have*

$$\textstyle\int_{\mathbb{R}^d} N^d|\check{\psi}(Ny)|\,|f(x-y)|\,dy \lesssim M(f)(x), \tag{2.6}$$

$$|\delta_y f_N(x)| \lesssim N|y|\left\{M(f_N)(x) + M(f_N)(x-y)\right\}, \tag{2.7}$$

$$\textstyle\int_{\mathbb{R}^d} N^d|y|\,|\check{\psi}(Ny)|\;dy \lesssim \tfrac{1}{N}. \tag{2.8}$$

Proof. We begin with (2.6). Note first that

$$\eta := N^d|\check{\psi}(Ny)|$$

is a spherically-symmetric, decreasing function of radius; thus, we can write

$$\eta(y) = \int_0^\infty \chi_{B(0,r)}(y)(-\eta'(r))\,dr,$$

where $\eta' := \frac{\partial \eta}{\partial r}$. We can then use the definition of the Hardy–Littlewood maximal function and integrate by parts to estimate

$$\begin{aligned}\text{LHS(2.6)} &\lesssim \int_0^\infty \left(\int_{|y|\leq r} |f(x-y)|\,dy \right) (-\eta'(r))\,dr \\ &\lesssim \left(\int_0^\infty \eta(r) r^{d-1}\,dr \right) M(f)(x) \\ &\lesssim_\psi M(f)(x).\end{aligned}$$

For (2.7), we begin by defining $\psi_0(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2)$, the 'fattened' Littlewood–Paley multiplier. Then we can write

$$|\delta_y f_N(x)| = \Big| \int [N^d \check{\psi}_0(N(z-y)) - N^d \check{\psi}_0(Nz)] f_N(x-z)\,dz \Big|. \tag{2.9}$$

If $N|y| \geq 1$, we can use the triangle inequality and argue as above to see that

$$|\delta_y f_N(x)| \leq M(f_N)(x-y) + M(f_N)(x),$$

giving (2.7) in this case. If instead $N|y| \leq 1$, we can use the fact that $\check{\psi}_0$ is Schwartz to estimate

$$|\check{\psi}_0(N(z-y)) - \check{\psi}_0(Nz)| \lesssim N|y|(1+N|z|)^{-100d}.$$

Then continuing from (2.9) and once again arguing as for (2.6), we find

$$|\delta_y f_N(x)| \lesssim N|y| M(f_N)(x),$$

which gives (2.7) in this case.

Finally, we note that since $\check{\psi}$ is Schwartz, we have

$$\int_{\mathbb{R}^d} N^d |Ny|\, |\check{\psi}(Ny)|\,dy \lesssim 1,$$

which immediately gives (2.8). $\blacksquare$

2.3 Strichartz estimates

In this section, we record some estimates related to the linear Schrödinger equation.

We denote the free Schrödinger propagator by $e^{it\Delta}$. This operator is defined via the Fourier transform: $\widehat{e^{it\Delta}f}(\xi) = e^{-it|\xi|^2}\widehat{f}(\xi)$. The function $v(t,x) = (e^{it\Delta}f)(x)$ solves the free Schrödinger equation $(i\partial_t + \Delta)v = 0$ with $v(0) = f$.

The definition of $e^{it\Delta}$ and Plancherel's theorem immediately imply that

$$\|e^{it\Delta}f\|_{L_x^2} = \|f\|_{L_x^2}$$

for all t. On the other hand, we have an explicit formula for $e^{it\Delta}$, namely

$$[e^{it\Delta}f](x) = (4\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}f(y)\,dy \qquad \text{for } t \neq 0,$$

which implies the dispersive estimate

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}\|f\|_{L_x^1(\mathbb{R}^d)} \qquad \text{for } t \neq 0.$$

Interpolating between the two estimates for $e^{it\Delta}$ yields

$$\|e^{it\Delta}f\|_{L_x^r(\mathbb{R}^d)} \lesssim |t|^{-(\frac{d}{2}-\frac{d}{r})}\|f\|_{L_x^{r'}(\mathbb{R}^d)} \tag{2.10}$$

for $t \neq 0$ and $2 \leq r \leq \infty$. This estimate can be used to prove the standard Strichartz estimates, which we state below. First we need the following definition.

Definition 2.3.1 (Strichartz spaces) *Let $d \geq 3$. We call a pair of exponents (q,r)* Schrödinger admissible *if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \textit{and} \quad 2 \leq q, r \leq \infty.$$

For an interval I and $s \geq 0$, we define the Strichartz norm by

$$\|u\|_{\dot{S}^s(I)} := \sup\Big\{\||\nabla|^s u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} : (q,r) \text{ Schrödinger admissible}\Big\}.$$

We define the Strichartz space $\dot{S}^s(I)$ to be the closure of the test functions under this norm, and denote the dual of $\dot{S}^s(I)$ by $\dot{N}^s(I)$. We note

$$\|F\|_{\dot{N}^s(I)} \lesssim \||\nabla|^s F\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^d)}$$

for any Schrödinger admissible (q,r).

We can now state the standard Strichartz estimates in the form that we will need them.

Lemma 2.3.2 (Strichartz estimates) *Let $d \geq 3$ and $s \geq 0$ and let I be a time interval. Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to the forced Schrödinger equation*

$$(i\partial_t + \Delta)u = F.$$

Then

$$\|u\|_{\dot{S}^s(I)} \lesssim \|u(t_0)\|_{\dot{H}^s_x(\mathbb{R}^d)} + \|F\|_{\dot{N}^s(I)}$$

for any $t_0 \in I$.

As mentioned above, the key ingredient in the proof of Lemma 2.3.2 is (2.10). For the endpoint $(q, r) = (2, \frac{2d}{d-2})$, see [29]. For the non-endpoint cases, see [25, 59].

The free propagator also obeys some local smoothing estimates (see [17, 56, 67] for the original results). We will make use of the following result, which appears in [38, Proposition 4.14].

Lemma 2.3.3 (Local smoothing) *For any $f \in L^2_x(\mathbb{R}^d)$ and any $\varepsilon > 0$,*

$$\iint_{\mathbb{R}\times\mathbb{R}^d} \big|[|\nabla|^{\frac{1}{2}} e^{it\Delta} f](x)\big|^2 \langle x\rangle^{-1-\varepsilon}\, dx\, dt \lesssim_\varepsilon \|f\|^2_{L^2_x(\mathbb{R}^d)}.$$

Finally, we record a bilinear Strichartz estimate. The following lemma can be deduced from [38, Corollary 4.19].

Lemma 2.3.4 (Bilinear Strichartz) *Let $0 < s_c < \frac{d-1}{2}$. For any interval I and any frequencies $M, N > 0$, we have*

$$\|u_{\leq M} v_{>N}\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim M^{\frac{d-1}{2}-s_c} N^{-\frac{1}{2}} \||\nabla|^{s_c} u\|_{S^*(I)} \|v_{>N}\|_{S^*(I)},$$

where

$$\|u\|_{S^*(I)} := \|u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^d)} + \|(i\partial_t + \Delta)u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)}.$$

CHAPTER 3

Local well-posedness and stability

In this chapter, we will develop the local theory for (1.1). For a more general introduction, one can refer to the textbooks [9, 63]. See also [38], which emphasizes critical problems and includes detailed discussions of issues related to stability.

3.1 Local well-posedness

In this section we describe standard local well-posedness results for (1.1). We begin by making the notion of solution precise.

Definition 3.1.1 (Solution) A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ on a non-empty time interval $I \ni 0$ is a *solution* to (1.1) if it belongs to $C_t \dot{H}_x^{s_c}(K \times \mathbb{R}^d) \cap L_{t,x}^{p(d+2)/2}(K \times \mathbb{R}^d)$ for every compact $K \subset I$ and obeys the Duhamel formula

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta}(|u|^p u)(s)\, ds$$

for all $t \in I$. We call I the *lifespan* of u; we say u is a *maximal-lifespan solution* if it cannot be extended to any strictly larger interval. If $I = \mathbb{R}$, we say u is *global*.

Definition 3.1.2 (Scattering size and blowup) We define the *scattering size* of a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) by

$$S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{p(d+2)}{2}}\, dx\, dt. \tag{3.1}$$

If there exists $t_0 \in I$ so that $S_{[t_0, \sup I)}(u) = \infty$, then we say u *blows up (forward in time)*. If there exists $t_0 \in I$ so that $S_{(\inf I, t_0]}(u) = \infty$, then we say u *blows up (backward in time)*.

We now state the main local well-posedness result that we will need.

Theorem 3.1.3 (Local well-posedness) *Let (d, s_c) satisfy (1.2), (1.3), (1.4), or (1.5). For any $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$, there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1). Moreover, this solution satisfies the following:*

- *(Local existence) I is an open neighborhood of 0.*

- *(Blowup criterion) If $\sup I$ is finite, then u blows up forward in time. If $\inf I$ is finite, then u blows up backward in time.*

- *(Scattering) If $\sup I = +\infty$ and u does not blow up forward in time, then u scatters forward in time; that is, there exists a unique $u_+ \in \dot{H}_x^{s_c}(\mathbb{R}^d)$ such that*

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} = 0. \tag{3.2}$$

Conversely, for any $u_+ \in \dot{H}_x^{s_c}(\mathbb{R}^d)$, there is a unique solution u to (1.1) so that (3.2) holds. The analogous statements hold backward in time.

- *(Small-data global existence) There exists $\eta_0 = \eta_0(d, p)$ such that if*

$$\|u_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} < \eta_0,$$

then u is global and scatters, with $S_{\mathbb{R}}(u) \lesssim \|u_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)}^{p(d+2)/2}$.

We will establish this theorem as a corollary of a local well-posedness result of Cazenave–Weissler [8] (Theorem 3.1.4) and a stability result (Theorem 3.2.2 or Theorem 3.3.5). The stability results will also play a key role in the reduction to almost periodic solutions in Chapter 4.

For the following local well-posedness result, one must assume that the initial data belongs to the inhomogeneous Sobolev space $H_x^{s_c}(\mathbb{R}^d)$. This assumption serves to simplify the proof (allowing for a contraction mapping argument in a norm without derivatives); we can remove it *a posteriori* by using the stability results we prove below.

Theorem 3.1.4 (Standard local well-posedness [8]) *Let $d \geq 1$, $0 < s_c < 1$, and $u_0 \in H_x^{s_c}(\mathbb{R}^d)$. If $I \ni 0$ is an interval such that*

$$\||\nabla|^{s_c} e^{it\Delta} u_0\|_{L_t^{p+2} L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \tag{3.3}$$

is sufficiently small, then there exists a unique solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1).

Remark 3.1.5 By Strichartz, we have

$$\||\nabla|^{s_c} e^{it\Delta} u_0\|_{L_t^{p+2} L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \lesssim \||\nabla|^{s_c} u_0\|_{L_x^2(\mathbb{R}^d)}.$$

Thus for small enough data, the quantity appearing in (3.3) will be sufficiently small with $I = \mathbb{R}$. One can also guarantee that the quantity appearing in (3.3) is sufficiently small by by taking $|I|$ small enough (cf. monotone convergence).

We now turn to the stability results. We will prove two versions of stability. The first, which will apply to the cases (1.3), (1.4), and (1.5), follows an approach that is fairly standard. The second, which will apply to (1.2), is a more refined result that only requires errors to be small in spaces that do not contain any derivatives. We model our presentation of these results after [38].

3.2 Stability for the cases (1.3), (1.4), and (1.5)

In this section we prove a stability result for (d, s_c) satisfying (1.3), (1.4), or (1.5). In all of these cases, we have $p > 1$. As we will see, this assumption allows for a very simple stability theory. On the other hand, when $p < 1$, developing a stability theory can become quite delicate. For a discussion in the energy-critical case, see [38, Section 3.4] and the references cited therein. See also [40] for a stability theory in the energy-supercritical regime, as well as [45] for a stability theory in the intercritical regime in high dimensions.

The results of this section appear originally in [49].

We begin with the following lemma.

Lemma 3.2.1 (Short-time perturbations) *Fix (d, s_c) satisfying (1.3), (1.4), or (1.5). Let I be a compact interval and $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}$ a solution to*

$$(i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^{s_c}(I\times\mathbb{R}^d)} \leq E. \tag{3.4}$$

Let $t_0 \in I$ and $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$. Then there exist ε_0, $\delta > 0$ (depending on E) such that for all $0 < \varepsilon < \varepsilon_0$, if

$$\||\nabla|^{s_c}\tilde{u}\|_{L_t^{\frac{p(d+2)}{2}} L_x^{\frac{2dp(d+2)}{d^2p+2dp-8}}(I\times\mathbb{R}^d)} \leq \delta, \tag{3.5}$$

$$\|u_0 - \tilde{u}(t_0)\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leq \varepsilon, \tag{3.6}$$

$$\||\nabla|^{s_c} e\|_{\dot{N}^0(I)} \leq \varepsilon, \tag{3.7}$$

then there exists $u : I \times \mathbb{R}^d \to \mathbb{C}$ solving $(i\partial_t + \Delta)u = F(u)$ with $u(t_0) = u_0$ satisfying

$$\||\nabla|^{s_c}(u - \tilde{u})\|_{\dot{S}^0(I)} \lesssim \varepsilon, \tag{3.8}$$

$$\||\nabla|^{s_c} u\|_{\dot{S}^0(I)} \lesssim E, \tag{3.9}$$

$$\||\nabla|^{s_c}(|u|^p u - |\tilde{u}|^p\tilde{u})\|_{\dot{N}^0(I)} \lesssim \varepsilon. \tag{3.10}$$

Proof. We prove the lemma under the additional hypothesis $u_0 \in L_x^2(\mathbb{R}^d)$; this allows us (by Theorem 3.1.4) to find a solution u, so that we are left to prove all of the estimates as *a priori* estimates. Once the lemma is proven, we can use approximation by $H_x^{s_c}(\mathbb{R}^d)$ functions (along with the lemma itself) to see that the lemma holds for $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$.

Define $w = u - \tilde{u}$, so that $(i\partial_t + \Delta)w = |u|^p u - |\tilde{u}|^p\tilde{u} - e$. Without loss of generality, assume $t_0 = \inf I$, and define

$$A(t) = \||\nabla|^{s_c}(|u|^p u - |\tilde{u}|^p\tilde{u})\|_{\dot{N}^0([t_0,t))}.$$

We first note that by Duhamel, Strichartz, (3.6), and (3.7), we get

$$\begin{aligned}
&\||\nabla|^{s_c} w\|_{\dot S^0([t_0,t))} \\
&\lesssim \||\nabla|^{s_c} w(t_0)\|_{L_x^2(\mathbb{R}^d)} + \||\nabla|^{s_c}(|u|^p u - |\tilde u|^p \tilde u)\|_{\dot N^0([t_0,t))} + \||\nabla|^{s_c} e\|_{\dot N^0(I)} \\
&\lesssim \varepsilon + A(t). \qquad (3.11)
\end{aligned}$$

Using this fact, together with Lemma 2.2.5, (3.5), and Sobolev embedding, we can estimate (with all spacetime norms over $[t_0,t)\times\mathbb{R}^d$)

$$\begin{aligned}
A(t) &\lesssim \||\nabla|^{s_c}(|\tilde u + w|^p(\tilde u + w) - |\tilde u|^p \tilde u)\|_{L_t^{\frac{p(d+2)}{2(p+1)}} L_x^{\frac{2dp(d+2)}{d^2p+6dp-8}}} \\
&\lesssim \||\nabla|^{s_c}\tilde u\|_{L_t^{\frac{p(d+2)}{2}} L_x^{\frac{2dp(d+2)}{d^2p+2dp-8}}} \|w\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}} \\
&\qquad + \||\nabla|^{s_c} w\|_{L_t^{\frac{p(d+2)}{2}} L_x^{\frac{2dp(d+2)}{d^2p+2dp-8}}} \|\tilde u + w\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}} \\
&\lesssim \delta[\varepsilon + A(t)]^p + [\varepsilon + A(t)][\delta^p + (\varepsilon + A(t))^p].
\end{aligned}$$

Thus, recalling $p>1$ and choosing δ,ε sufficiently small, we conclude $A(t)\lesssim\varepsilon$ for all $t\in I$, which gives (3.10). Combining (3.10) with (3.11), we also get (3.8). Finally, we can prove (3.9) as follows: by Strichartz, (3.8), (3.4), (3.7), (3.5), the fractional chain rule, and Sobolev embedding,

$$\begin{aligned}
\||\nabla|^{s_c} u\|_{\dot S^0(I)} &\lesssim \||\nabla|^{s_c}(u-\tilde u)\|_{\dot S^0(I)} + \||\nabla|^{s_c}\tilde u\|_{\dot S^0(I)} \\
&\lesssim \varepsilon + \||\nabla|^{s_c}\tilde u(t_0)\|_{L_x^2(\mathbb{R}^d)} + \||\nabla|^{s_c}(|\tilde u|^p\tilde u)\|_{\dot N^0(I)} + \||\nabla|^{s_c} e\|_{\dot N^0(I)} \\
&\lesssim \varepsilon + E + \||\nabla|^{s_c}\tilde u\|_{L_t^{\frac{p(d+2)}{2}} L_x^{\frac{2dp(d+2)}{d^2p+2dp-8}}(I\times\mathbb{R}^d)} \|\widetilde u\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}(I\times\mathbb{R}^d)} \\
&\lesssim E + \varepsilon + \delta^{p+1} \\
&\lesssim E
\end{aligned}$$

for ε and δ sufficiently small depending on E. ∎

With Lemma 3.2.1 established, we now turn to

Theorem 3.2.2 (Stability) *Fix (d, s_c) satisfying (1.3), (1.4), or (1.5). Let I be a compact time interval and $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}$ a solution to*

$$(i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^{s_c}(I\times\mathbb{R}^d)} \leq E, \tag{3.12}$$

$$S_I(\tilde{u}) \leq L. \tag{3.13}$$

Let $t_0 \in I$ and $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$. Then there exists $\varepsilon_1 = \varepsilon_1(E, L)$ such that if

$$\|u_0 - \tilde{u}(t_0)\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leq \varepsilon, \tag{3.14}$$

$$\||\nabla|^{s_c} e\|_{\dot{N}^0(I)} \leq \varepsilon \tag{3.15}$$

for some $0 < \varepsilon < \varepsilon_1$, then there exists a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to $(i\partial_t + \Delta)u = F(u)$ with $u(t_0) = u_0$ satisfying

$$\||\nabla|^{s_c}(u - \tilde{u})\|_{\dot{S}^0(I)} \leq C(E, L)\varepsilon, \tag{3.16}$$

$$\||\nabla|^{s_c} u\|_{\dot{S}^0(I)} \leq C(E, L). \tag{3.17}$$

Proof. Once again, we may assume $t_0 = \inf I$. To begin, we let $\eta > 0$ be a small parameter to be determined shortly. By (3.13), we may subdivide I into (finitely many, depending on η and L) intervals $J_k = [t_k, t_{k+1})$ so that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} \sim \eta$$

for each k. Then by Strichartz, (3.12), (3.15), and the fractional chain rule, we have

$$\begin{aligned}
\||\nabla|^{s_c}\tilde{u}\|_{\dot{S}^0(J_k)} &\lesssim \||\nabla|^{s_c}\tilde{u}(t_k)\|_{L_x^2(\mathbb{R}^d)} + \||\nabla|^{s_c}(|\tilde{u}|^p\tilde{u})\|_{\dot{N}^0(J_k)} + \||\nabla|^{s_c} e\|_{\dot{N}^0(J_k)} \\
&\lesssim E + \||\nabla|^{s_c}\tilde{u}\|_{\dot{S}^0(I)}\|\tilde{u}\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} + \varepsilon \\
&\lesssim E + \varepsilon + \eta^p\||\nabla|^{s_c}\tilde{u}\|_{\dot{S}^0(I)}.
\end{aligned}$$

Thus for $\varepsilon \le E$ and η sufficiently small, we find

$$\| |\nabla|^{s_c} \tilde{u} \|_{\dot{S}^0(J_k)} \lesssim E.$$

Adding these bounds, we find

$$\| |\nabla|^{s_c} \tilde{u} \|_{\dot{S}^0(I)} \le C(E, L). \tag{3.18}$$

Now, we take $\delta > 0$ as in Lemma 3.2.1 and subdivide I into finitely many, say $J_0 = J_0(C(E, L), \delta)$ intervals $I_j = [t_j, t_{j+1})$ so that

$$\| |\nabla|^{s_c} \tilde{u} \|_{L_t^{\frac{p(d+2)}{2}} L_x^{\frac{2dp(d+2)}{d^2p+2dp-8}}(I_j \times \mathbb{R}^d)} \le \delta$$

for each j. We now wish to proceed inductively. We may apply Lemma 3.2.1 on each I_j, provided we can guarantee

$$\| u(t_j) - \tilde{u}(t_j) \|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \le \varepsilon \tag{3.19}$$

for some $0 < \varepsilon < \varepsilon_0$ and each j (where ε_0 is as in Lemma 3.2.1). In the event that (3.19) holds for some j, applying Lemma 3.2.1 on $I_j = [t_j, t_{j+1})$ gives

$$\| |\nabla|^{s_c} (u - \tilde{u}) \|_{\dot{S}^0(I_j)} \le C(j)\varepsilon, \tag{3.20}$$

$$\| |\nabla|^{s_c} u \|_{\dot{S}^0(I_j)} \le C(j)E, \tag{3.21}$$

$$\| |\nabla|^{s_c} (|u|^p u - |\tilde{u}|^p \tilde{u}) \|_{\dot{N}^0(I_j)} \le C(j)\varepsilon. \tag{3.22}$$

Now, we first note that (3.19) holds for $j = 0$, provided we take $\varepsilon_1 < \varepsilon_0$. Next, assuming that (3.19) holds for $0 \le k \le j - 1$, we can use Strichartz, (3.14), (3.15), and the inductive hypothesis (3.22) to estimate

$$\begin{aligned}
&\| u(t_j) - \tilde{u}(t_j) \|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \\
&\quad \lesssim \| u(t_0) - \tilde{u}(t_0) \|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} + \| |\nabla|^{s_c} (|u|^p u - |\tilde{u}|^p \tilde{u}) \|_{\dot{N}^0([t_0, t_j))} + \| |\nabla|^{s_c} e \|_{\dot{N}^0([t_0, t_j))} \\
&\quad \lesssim \varepsilon + \sum_{k=0}^{j-1} C(k)\varepsilon + \varepsilon \\
&\quad < \varepsilon_0,
\end{aligned}$$

provided $\varepsilon_1 = \varepsilon_1(\varepsilon_0, J_0)$ is taken sufficiently small. Thus, by induction, we get (3.20) and (3.21) on each I_j. Adding these bounds over the I_j yields (3.16) and (3.17). ∎

Remark 3.2.3 Using arguments from [8, 9], one can establish Theorem 3.1.3 for cases (1.3), (1.4), and (1.5) for data in the imhomogeneous Sobolev space $H_x^{s_c}$. Using Theorem 3.3.5, one can then remove the assumption $u_0 \in L_x^2$ *a posteriori* (by approximating $u_0 \in \dot{H}_x^{s_c}$ by $H_x^{s_c}$- functions). We omit the standard details.

3.3 Stability for the case (1.2)

In this section, we develop a stability theory for (d, s_c) satisfying (1.2). That is, we take $d \geq 4$ and $s_c = 1/2$. Note that in this case we have $p = 4/(d-1)$. Compared to the results of the previous section, we will prove a 'refined' stability result for the case (1.2), in the sense that the results will not require errors to be small in spaces with derivatives.

The results of this section appear originally in [50].

We will make use of function spaces that are critical with respect to scaling, but do not involve any derivatives. In particular, for a time interval I, we define the following norms:

$$\|u\|_{X(I)} := \|u\|_{L_t^{\frac{4(d+1)}{d-1}} L_x^{\frac{2(d+1)}{d-1}} (I\times\mathbb{R}^d)}, \quad \|F\|_{Y(I)} := \|F\|_{L_t^{\frac{4(d+1)}{d+3}} L_x^{\frac{2(d+1)}{d+3}} (I\times\mathbb{R}^d)}.$$

We first relate the X-norm to the usual Strichartz norms. By Sobolev embedding, we get $\|u\|_{X(I)} \lesssim \|u\|_{\dot{S}^{1/2}(I)}$, while Hölder and Sobolev embedding together imply

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-1}} (I\times\mathbb{R}^d)} \lesssim \|u\|_{X(I)}^c \|u\|_{\dot{S}^{1/2}(I)}^{1-c} \quad \text{for some} \quad 0 < c(d) < 1. \tag{3.23}$$

Next, we record a Strichartz estimate, which one can prove via the standard approach (namely, by applying the dispersive estimate and Hardy–Littlewood–Sobolev).

Lemma 3.3.1 *Let I be a compact time interval and $t_0 \in I$. Then for all $t \in I$,*

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{X(I)} \lesssim \|F\|_{Y(I)}. \tag{3.24}$$

Finally, we collect some estimates that will allow us to control the nonlinearity.

Lemma 3.3.2 *Let (d, s_c) satisfy (1.2). Then, with spacetime norms over $I \times \mathbb{R}^d$, we have*

$$\|F(u)\|_{Y(I)} \lesssim \|u\|_{X(I)}^{\frac{d+3}{d-1}} \tag{3.25}$$

$$\|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \big\{\|u\|_{X(I)}^{\frac{4}{d-1}} + \|\tilde{u}\|_{X(I)}^{\frac{4}{d-1}}\big\}\|u - \tilde{u}\|_{X(I)} \tag{3.26}$$

$$\||\nabla|^{s_c} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|u\|_{X(I)}^{\frac{4}{d-1}} \|u\|_{\dot{S}^{1/2}(I)} \tag{3.27}$$

$$\||\nabla|^{s_c}[F(u) - F(\tilde{u})]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|u - \tilde{u}\|_{X(I)}^{\frac{4}{d-1}} \|\tilde{u}\|_{\dot{S}^{1/2}(I)} + \|u\|_{X(I)}^{\frac{4}{d-1}} \|u - \tilde{u}\|_{\dot{S}^{1/2}(I)}. \tag{3.28}$$

Proof. We first note that (3.25) follows from Hölder, while (3.26) follows from the fundamental theorem of calculus followed by Hölder.

Next, we see that (3.27) follows from Hölder and the fractional chain rule. Indeed,

$$\||\nabla|^{s_c} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|u\|_{L_t^{\frac{4(d+1)}{d-1}} L_x^{\frac{2(d+1)}{d-1}}}^{\frac{4}{d-1}} \||\nabla|^{s_c} u\|_{L_t^{\frac{2(d+1)}{d-1}} L_x^{\frac{2d(d+1)}{d^2-d+2}}}.$$

Using these same exponents with Lemma 2.2.5, we deduce (3.28). ∎

We may now state our first stability result.

Lemma 3.3.3 (Short-time perturbations) *Let (d, s_c) satisfy (1.2) and let I be a compact time interval, with $t_0 \in I$. Let $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to $(i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + e$ with $\tilde{u}(t_0) = \tilde{u}_0 \in \dot{H}_x^{1/2}$. Suppose*

$$\|\tilde{u}\|_{\dot{S}^{1/2}(I)} \leq E \quad \text{and} \quad \||\nabla|^{s_c} e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)} \leq E \tag{3.29}$$

for some $E > 0$. Let $u_0 \in \dot{H}_x^{1/2}(\mathbb{R}^d)$ satisfy

$$\|u_0 - \tilde{u}_0\|_{\dot{H}_x^{1/2}} \leq E, \tag{3.30}$$

and suppose that we have the smallness conditions

$$\|\tilde{u}\|_{X(I)} \leq \delta, \tag{3.31}$$

$$\|e^{i(t-t_0)\Delta}(u_0 - \tilde{u}_0)\|_{X(I)} + \|e\|_{Y(I)} \leq \varepsilon, \tag{3.32}$$

for some small $0 < \delta = \delta(E)$ *and* $0 < \varepsilon < \varepsilon_0(E)$*. Then there exists* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *solving* (1.1) *with* $u(t_0) = u_0$ *such that*

$$\|u - \tilde{u}\|_{X(I)} + \|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \varepsilon, \tag{3.33}$$

$$\|u - \tilde{u}\|_{\dot{S}^{1/2}(I)} + \||\nabla|^{s_c}[F(u) - F(\tilde{u})]\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \lesssim_E 1. \tag{3.34}$$

Proof. We first suppose $u_0 \in L_x^2$, so that Theorem 3.1.4 provides the solution u. We will then prove (3.33) and (3.34) as *a priori* estimates. After the lemma is proven for $u_0 \in H_x^{1/2}$, we can use approximation by $H_x^{1/2}$-functions to see that the lemma holds for $u_0 \in \dot{H}_x^{1/2}$.

Throughout the proof, spacetime norms will be over $I \times \mathbb{R}^d$.

We will first show

$$\|u\|_{X(I)} \lesssim \delta. \tag{3.35}$$

By the triangle inequality, (3.24), (3.25), (3.31), and (3.32), we get

$$\|e^{i(t-t_0)\Delta}\tilde{u}_0\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|F(\tilde{u})\|_{Y(I)} + \|e\|_{Y(I)} \lesssim \delta + \delta^{\frac{d+3}{d-1}} + \varepsilon.$$

Combining this estimate with (3.32) and using the triangle inequality then gives

$$\|e^{i(t-t_0)\Delta}u_0\|_{X(I)} \lesssim \delta$$

for δ and $\varepsilon \lesssim \delta$ sufficiently small. Thus, by (3.24) and (3.25), we get

$$\|u\|_{X(I)} \lesssim \delta + \|F(u)\|_{Y(I)} \lesssim \delta + \|u\|_{X(I)}^{\frac{d+3}{d-1}},$$

which (taking δ sufficiently small) implies (3.35).

We now turn to proving the desired estimates for $w := u - \tilde{u}$. Note first that w is a solution to $(i\partial_t + \Delta)w = F(u) - F(\tilde{u}) - e$, with $w(t_0) = u_0 - \tilde{u}_0$; thus, we can use (3.24), (3.26), (3.31), (3.32), and (3.35) to see

$$\begin{aligned}\|w\|_{X(I)} &\lesssim \|e^{i(t-t_0)\Delta}(u_0 - \tilde{u}_0)\|_{X(I)} + \|e\|_{Y(I)} + \|F(u) - F(\tilde{u})\|_{Y(I)} \\ &\lesssim \varepsilon + \big\{\|u\|_{X(I)}^{\frac{4}{d-1}} + \|\tilde{u}\|_{X(I)}^{\frac{4}{d-1}}\big\}\|w\|_{X(I)} \\ &\lesssim \varepsilon + \delta^{\frac{4}{d-1}}\|w\|_{X(I)}.\end{aligned}$$

Taking δ sufficiently small, we see that the first estimate in (3.33) holds. Using the first estimate in (3.33), along with (3.26), (3.31), and (3.35), we see that the remaining estimate in (3.33) holds, as well.

Next, by Strichartz, (3.28), (3.29), (3.30), (3.31), (3.33), and (3.35), we get

$$\begin{aligned}
\|w\|_{\dot{S}^{1/2}(I)} &\lesssim \|u_0 - \tilde{u}_0\|_{\dot{H}_x^{1/2}} + \||\nabla|^{s_c} e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \||\nabla|^{s_c}[F(u) - F(\tilde{u})]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\lesssim_E 1 + \|\tilde{u}\|_{\dot{S}^{1/2}(I)} \|w\|_{X(I)}^{\frac{4}{d-1}} + \|w\|_{\dot{S}^{1/2}(I)} \|u\|_{X(I)}^{\frac{4}{d-1}} \\
&\lesssim_E 1 + \delta^{\frac{4}{d-1}} \|w\|_{\dot{S}^{1/2}(I)}.
\end{aligned}$$

Taking $\delta = \delta(E)$ sufficiently small then gives the first estimate in (3.34). We get the remaining estimate in (3.34) by using (3.28) with (3.29), (3.33), (3.35), and the first estimate in (3.34). ∎

Remark 3.3.4 As mentioned above, the error e is only required to be small in a space without derivatives; it merely needs to be *bounded* in a space with derivatives. This will also be the case in Theorem 3.3.5 below. We will see the benefit of this refinement when we carry out the proof of Theorem 1.1.2 for the case (1.2) in Chapter 4.

We continue to the main result of this section.

Theorem 3.3.5 (Stability) *Let* (d, s_c) *satisfy* (1.2)*, and let* I *be a compact time interval, with* $t_0 \in I$*. Suppose* $\tilde{u}$ *is a solution to* $(i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + e$*, with* $\tilde{u}(t_0) = \tilde{u}_0$*. Suppose*

$$\|\tilde{u}\|_{\dot{S}^{1/2}(I)} \leq E \quad \textit{and} \quad \||\nabla|^{s_c} e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)} \leq E \tag{3.36}$$

for some $E > 0$*. Let* $u_0 \in \dot{H}_x^{1/2}(\mathbb{R}^d)$*, and suppose we have the smallness conditions*

$$\|u_0 - \tilde{u}_0\|_{\dot{H}_x^{1/2}(\mathbb{R}^d)} + \|e\|_{Y(I)} \leq \varepsilon \tag{3.37}$$

for some small $0 < \varepsilon < \varepsilon_1(E)$*. Then, there exists* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *solving* (1.1) *with* $u(t_0) = u_0$*, and there exists* $0 < c(d) < 1$ *such that*

$$\|u - \tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-1}}(I \times \mathbb{R}^d)} \lesssim_E \varepsilon^c. \tag{3.38}$$

One can derive Theorem 3.3.5 from Lemma 3.3.3 as in the previous section, namely, by applying Lemma 3.3.3 inductively. We omit these details, but pause to point out the following: this induction will actually yield the bounds

$$\|u - \tilde{u}\|_{X(I)} \lesssim \varepsilon \quad \text{and} \quad \|u - \tilde{u}\|_{\dot{S}^{1/2}(I)} \lesssim_E 1.$$

With these bounds in hand, we then use (3.23) to see that (3.38) holds.

Remark 3.3.6 The smallness condition on $u_0 - \tilde{u}_0$ appearing in (3.37) may actually be relaxed to the condition appearing in (3.32). In our setting, it will not be difficult to prove the stronger condition (see Lemma 4.4.2).

Remark 3.3.7 As mentioned at the end of the last section, we can deduce Theorem 3.1.3 for the cases (1.2) using the standard arguments of [8, 9] together with Theorem 3.3.5. We omit the standard details.

CHAPTER 4

Almost periodic solutions

In this chapter, we discuss almost periodic solutions. After giving some definitions and collecting some results, we carry out the proof of the reduction to almost periodic solutions, Theorem 1.1.2.

For a more extensive treatment of almost periodic solutions, one can refer to [38].

4.1 Definitions and basic results

In this section we define almost periodic solutions and collect some useful consequences of almost periodicity.

Definition 4.1.1 (Almost periodic solutions) *Let $s_c > 0$. A solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to* (1.1) *is* almost periodic (modulo symmetries) *if*

$$u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^d) \tag{4.1}$$

and there exist functions $x : I \to \mathbb{R}^d$, $N : I \to \mathbb{R}^+$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ so that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)|>\frac{C(\eta)}{N(t)}} \left||\nabla|^{s_c} u(t,x)\right|^2 dx + \int_{|\xi|>C(\eta)N(t)} |\xi|^{2s_c} |\widehat{u(t,x)}|^2 \, d\xi < \eta.$$

We call $N(t)$ the frequency scale function*, $x(t)$ the* spatial center*, and $C(\eta)$ the* compactness modulus function*.*

By the Arzelá–Ascoli theorem, a family of functions is precompact in $\dot{H}_x^{s_c}(\mathbb{R}^d)$ if and only if it is bounded and there exists a compactness modulus function C so that

$$\int_{|x|\geq C(\eta)} \big||\nabla|^{s_c} f(x)\big|^2\,dx + \int_{|\xi|>C(\eta)} |\xi|^{2s_c}|\widehat{f}(\xi)|^2\,d\xi \leq \eta$$

uniformly for all f in the family. Thus an equivalent formulation of Definition 4.1.1 is the following: u is almost periodic (modulo symmetries) if and only if

$$\{u(t) : t \in I\} \subset \{\lambda^{\frac{2}{p}} f(\lambda(x+x_0)) : \lambda \in (0,\infty),\ x_0 \in \mathbb{R}^d,\ f \in K\}$$

for some compact $K \subset \dot{H}_x^{s_c}(\mathbb{R}^d)$.

Remark 4.1.2 If u is a *radial* almost periodic solution, it can only concentrate near the spatial origin. Thus for a radial almost periodic solution, we may take $x(t) \equiv 0$.

The frequency scale function of an almost periodic solution obeys the following local constancy property (see [38, Lemma 5.18]).

Lemma 4.1.3 (Local constancy) *If $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan almost periodic solution to* (1.1)*, then there exists $\delta = \delta(u) > 0$ so that for all $t_0 \in I$,*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I.$$

Moreover

$$N(t) \sim_u N(t_0), \quad |x(t) - x(t_0)| \lesssim_u N(t_0)^{-1} \quad \textit{for} \quad |t - t_0| \leq \delta N(t_0)^{-2}.$$

We may use Lemma 4.1.3 to divide the lifespan I into *characteristic subintervals* J_k on which we can set $N(t) \equiv N_k$ for some N_k, with $|J_k| \sim_u N_k^{-2}$. This requires us to modify the compactness modulus function by a time-independent multiplicative factor.

Lemma 4.1.3 also provides information about the behavior of the frequency scale function at the blowup time (see [38, Corollary 5.19]).

Corollary 4.1.4 ($N(t)$ at blowup) *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to* (1.1). *If T is a finite endpoint of I, then $N(t) \gtrsim_u |T - t|^{-1/2}$. If I is infinite or semi-infinite, then for any $t_0 \in I$ we have $N(t) \gtrsim \langle t - t_0 \rangle^{-1/2}$.*

We may also relate the frequency scale function of an almost periodic solution to its Strichartz norms.

Lemma 4.1.5 (Spacetime bounds) *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be an almost periodic solution to* (1.1). *Then*

$$\int_I N(t)^2 \, dt \lesssim_u \| |\nabla|^{s_c} u \|^2_{L_t^2 L_x^{\frac{2d}{d-2}}(I \times \mathbb{R}^d)} \lesssim_u 1 + \int_I N(t)^2 \, dt.$$

To prove this lemma, one can adapt the proof of [38, Lemma 5.21]. The key is to note that $\int_I N(t)^2 \, dt$ counts the number of characteristic intervals J_k inside I, and that for each such subinterval we have

$$\| |\nabla|^{s_c} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbb{R}^d)} \sim_u 1.$$

We next record a 'reduced' Duhamel formula that holds for almost periodic solutions (see [38, Proposition 5.2]).

Proposition 4.1.6 (Reduced Duhamel formula) *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to* (1.1). *Then for all $t \in I$, we have*

$$u(t) = \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-s)\Delta} [F(u(s))] \, ds$$

as a weak limit in $\dot{H}_x^{s_c}(\mathbb{R}^d)$.

To conclude this section, we establish a corollary to the bilinear Strichartz estimate Lemma 2.3.4 for the case of an almost periodic solution on a characteristic subinterval.

Corollary 4.1.7 (Bilinear Strichartz) *Let $u : I \times \mathbb{R}^d$ be an almost periodic solution to* (1.1) *with $0 < s_c < \frac{d-1}{2}$. Suppose $u \in L_t^\infty \dot{H}_x^s$ for some $s > 0$ and let J_k be a characteristic subinterval. Then*

$$\| u_{\leq M} u_{>N} \|_{L_{t,x}^2(J_k \times \mathbb{R}^d)} \lesssim_u M^{\frac{d-1}{2} - s_c} N^{-\frac{1}{2} - s}.$$

Proof. This result will follow from Lemma 2.3.4, provided we can show

$$\||\nabla|^{s_c} u\|_{S^*(J_k)} \lesssim_u 1 \tag{4.2}$$

and

$$\|u_{>N}\|_{S^*(J_k)} \lesssim_u N^{-s}. \tag{4.3}$$

For (4.2), we first note that interpolating between (4.1) and Lemma 4.1.5 gives

$$\|u\|_{\dot{S}^{s_c}(J_k)} \lesssim_u 1.$$

Thus, using the fractional chain rule and Sobolev embedding, we find

$$\begin{aligned}\||\nabla|^{s_c} u\|_{S^*(J_k)} &\lesssim \|u\|_{L_t^\infty \dot{H}_x^{s_c}(J_k\times\mathbb{R}^d)} + \|u\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbb{R}^d)} \\ &\lesssim_u 1 + \|u\|^{p+1}_{\dot{S}^{s_c}(J_k)} \lesssim_u 1.\end{aligned}$$

For (4.3), we first apply Strichartz and the fractional chain rule to estimate

$$\|u\|_{\dot{S}^s(J_k)} \lesssim \|u\|_{L_t^\infty \dot{H}_x^s} + \|u\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} \|u\|_{\dot{S}^s(J_k)}.$$

As $\|u\|_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} \lesssim_u 1$, a standard bootstrap argument gives $\|u\|_{\dot{S}^s(J_k)} \lesssim_u 1$. Thus, using Bernstein, we find

$$\begin{aligned}\|u_{>N}\|_{S^*(J_k)} &\lesssim N^{-s}\|u\|_{L_t^\infty \dot{H}_x^s(J_k\times\mathbb{R}^3)} + N^{-s}\||\nabla|^s F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k\times\mathbb{R}^d)} \\ &\lesssim_u N^{-s} + N^{-s}\|u\|^p_{L_{t,x}^{\frac{p(d+2)}{2}}(J_k\times\mathbb{R}^d)} \|u\|_{\dot{S}^s(J_k)} \lesssim_u N^{-s}.\end{aligned}$$

This completes the proof of Corollary 4.1.7. ∎

4.2 Lower bounds

In this section we show that for almost periodic solutions, we can prove lower bounds for the quantities appearing in Morawetz estimates. In particular, these bounds will be given in terms of the frequency scale function $N(t)$. These estimates will play a key role in the preclusion of quasi-soliton solutions in Chapter 9.

We begin with the following lemma.

Lemma 4.2.1 *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a nonzero almost periodic solution to* (1.1)*, with I is a contiguous union of characteristic intervals J_k. Suppose (d, s_c) satisfies* (1.3)*,* (1.4)*, or* (1.5)*.*

If $\inf_{t\in I} N(t) \geq 1$, then there exists $C(u) > 0$ and $N_1 > 0$ so that for $N < N_1$,

$$\inf_{t\in I} N(t)^{2s_c} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{>N}(t,x)|^2 \, dx \gtrsim_u 1. \tag{4.4}$$

If $\sup_{t\in I} N(t) \leq 1$, then there exists $C(u) > 0$ and $N_1 > 0$ so that for $N > N_1$,

$$\inf_{t\in I} N(t)^{2s_c} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq N}(t,x)|^2 \, dx \gtrsim_u 1. \tag{4.5}$$

Proof of Lemma 4.2.1. We will prove (4.4) only, as the proof of (4.5) is similar.

We first establish that for $C(u)$ sufficiently large, we have

$$\inf_{t\in I} N(t)^{2s_c} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^2 \, dx \gtrsim_u 1. \tag{4.6}$$

To this end, we let $\eta_0 > 0$ and use almost periodicity to find $C_0 := C_0(\eta_0)$ large enough that

$$\| |\nabla|^{s_c} u_{>C_0 N(t)} \|_{L_t^\infty L_x^2} < \eta_0. \tag{4.7}$$

Then using Hölder, Bernstein, and Sobolev embedding, we can estimate

$$\begin{aligned} \Big| \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} & |u(t,x)|^2 - |u_{\leq C_0 N(t)}(t,x)|^2 \, dx \Big| \\ &\lesssim_u N(t)^{-s_c} \|u_{>C_0N(t)}(t)\|_{L_x^2} \|u(t)\|_{L_x^{\frac{dp}{2}}} \\ &\lesssim_u \eta_0 N(t)^{-2s_c} \end{aligned}$$

for $t \in I$. Thus, if we can show that for $C(u)$ sufficiently large, we have

$$\inf_{t\in I} N(t)^{2s_c} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0 N(t)}(t,x)|^2 \, dx \gtrsim_u 1, \tag{4.8}$$

then we will have (4.6) by choosing $\eta_0 = \eta_0(u)$ sufficiently small.

To prove (4.8), we first choose use almost periodicity and Sobolev embedding to choose $C(u) > 0$ large enough that

$$\inf_{t\in I} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^{\frac{dp}{2}}\,dx \gtrsim_u 1.$$

We then use Hölder, Sobolev embedding, and (4.7) to see

$$\left| \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^{\frac{dp}{2}} - |u_{\leq C_0 N(t)}|^{\frac{dp}{2}}\,dx \right| \lesssim \|u_{>C_0N(t)}\|_{L_t^\infty L_x^{\frac{dp}{2}}} \|u\|_{L_t^\infty L_x^{\frac{dp}{2}}}^{\frac{dp}{2}-1} \lesssim_u \eta_0$$

for $t \in I$. Thus for $\eta_0 = \eta_0(u)$ sufficiently small, we find

$$\inf_{t\in I} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0N(t)}(t,x)|^{\frac{dp}{2}}\,dx \gtrsim_u 1. \tag{4.9}$$

Finally, using Hölder and Bernstein, we can get

$$\begin{aligned}
&\int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0N(t)}(t,x)|^{\frac{dp}{2}}\,dx \\
&\qquad \lesssim \|u_{\leq C_0N(t)}(t)\|_{L_x^\infty(\mathbb{R}^d)}^{\frac{dp}{2}-2} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0N(t)}(t,x)|^2\,dx \\
&\qquad \lesssim_u N(t)^{2s_c} \|u(t)\|_{L_x^{\frac{dp}{2}}(\mathbb{R}^d)}^{\frac{dp}{2}-2} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0N(t)}(t,x)|^2\,dx \\
&\qquad \lesssim_u N(t)^{2s_c} \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\leq C_0N(t)}(t,x)|^2\,dx.
\end{aligned}$$

Together with (4.9), this implies (4.8), which in turn implies (4.6).

With (4.6) in place, we are now in a position to establish (4.4). We let $\eta_1 > 0$ be a small parameter to be determined shortly. As $\inf_{t\in I} N(t) \geq 1$, we may find $N_1 = N_1(\eta_1)$ so that

$$\|u_{\leq N}\|_{L_t^\infty L_x^{\frac{dp}{2}}} < \eta_1 \quad \text{for } N \leq N_1.$$

We then use Hölder and Sobolev embedding to estimate

$$\begin{aligned}
\left| \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^2 - |u_{>N}(t,x)|^2\,dx \right| &\lesssim_u N(t)^{-2s_c} \|u_{\leq N}\|_{L_t^\infty L_x^{\frac{dp}{2}}} \|u\|_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim_u \eta_1 N(t)^{-2s_c}
\end{aligned}$$

for $t \in I$ and $N \leq N_1$. Thus, choosing $\eta_1 = \eta_1(u)$ sufficiently small, we may use (4.6) to deduce (4.4). This completes the proof of Lemma 4.2.1. ∎

With Lemma 4.2.1 in place, we can now turn to the main result of this section.

Proposition 4.2.2 (Lower bounds) *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a nonzero almost periodic solution to* (1.1)*, with I a contiguous union of characteristic subintervals J_k. Suppose (d, s_c) satisfies* (1.3)*,* (1.4)*, or* (1.5)*.*

If $\inf_{t\in I} N(t) \geq 1$, then there exists $N_1 > 0$ so that for $N < N_1$,

$$-\int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u_{>N}(t,x)|^2 \Delta(\tfrac{1}{|x-y|})|u_{>N}(t,y)|^2\, dx\, dy\, dt \gtrsim_u \int_I N(t)^{3-4s_c}\, dt. \tag{4.10}$$

If $\inf_{t\in I} N(t) \geq 1$ and $x(t) \equiv 0$, then there exists $N_1 > 0$ so that for $N < N_1$,

$$\iint_{I\times\mathbb{R}^d} \frac{|u_{>N}(t,x)|^{p+2}}{|x|}\, dx\, dt \gtrsim_u \int_I N(t)^{3-2s_c}\, dt. \tag{4.11}$$

If $\inf_{t\in I} N(t) \leq 1$ and $x(t) \equiv 0$, then there exists $N_1 > 0$ so that for $N > N_1$,

$$\iint_{I\times\mathbb{R}^d} \frac{|u_{\leq N}(t,x)|^{p+2}}{|x|}\, dx\, dt \gtrsim_u \int_I N(t)^{3-2s_c}\, dt. \tag{4.12}$$

Proof of Proposition 4.2.2. We first prove (4.10). We consider the cases $d = 3$ and $d \in \{4, 5\}$ separately.

If $d = 3$, we have $-\Delta(\frac{1}{|x|}) = 4\pi\delta$. Using Hölder and (4.4), we see that there exists $C(u)$ and $N_1 > 0$ so that for $N < N_1$ we have

$$\begin{aligned}\int_I \int_{|x-x(t)\leq \frac{C(u)}{N(t)}} |u_{>N}(t,x)|^4\, dx\, dt &\gtrsim_u \int_I \bigg(\int_{|x-x(t)|\leq\frac{C(u)}{N(t)}} |u_{>N}(t,x)|^2\, dx\bigg)^2 N(t)^3\, dt \\ &\gtrsim_u \int N(t)^{3-4s_c}\, dt,\end{aligned}$$

which implies (4.10) for $d = 3$.

If $d \in \{4, 5\}$, we have $-\Delta(\frac{1}{|x|}) = \frac{d-3}{|x|^3}$. Using (4.4), we see that there exists $C(u)$ and $N_1 > 0$ so that for $N < N_1$ we have

$$\begin{aligned}&\int_I \iint_{|x-y|\leq\frac{2C(u)}{N(t)}} \frac{|u_{>N}(t,x)|^2 |u_{>N}(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt \\ &\qquad\gtrsim \int_I \big[\tfrac{N(t)}{2C(u)}\big]^3 \bigg(\int_{|x-x(t)|\leq\frac{C(u)}{N(t)}} |u_{>N}(t,x)|^2\, dx\bigg)^2 dt \\ &\qquad\gtrsim_u \int_I N(t)^{3-4s_c}\, dt,\end{aligned}$$

which implies (4.10) for $d \in \{4,5\}$.

We now turn to (4.11). Using (4.4), Hölder, and the assumption $x(t) \equiv 0$, we see that there exists $C(u) > 0$ and $N_1 > 0$ so that for $N < N_1$ we have

$$\begin{aligned}
\int_I \int_{|x| \leq \frac{C(u)}{N(t)}} \frac{|u_{>N}(t,x)|^{p+2}}{|x|}\,dx\,dt &\gtrsim_u \int_I N(t)^{3+ps_c} \Big(\int_{|x|\leq \frac{C(u)}{N(t)}} |u_{>N}(t,x)|^2\,dx \Big)^{\frac{p+2}{2}}\,dt \\
&\gtrsim_u \int_I N(t)^{3-2s_c}\,dt,
\end{aligned}$$

which implies (4.11).

To prove (4.12), one proceeds as in the proof of (4.11), using (4.5) instead of (4.4). This completes the proof of Proposition 4.2.2. ∎

4.3 Concentration compactness

In this section we record a linear profile decomposition for $e^{it\Delta}$, which we will utilize in the reduction to almost periodic solutions. We begin with a definition.

Definition 4.3.1 (Symmetry group) *For any position $x_0 \in \mathbb{R}^d$ and scaling parameter $\lambda > 0$, we define a unitary transformation $g_{x_0,\lambda} : \dot{H}_x^{s_c}(\mathbb{R}^d) \to \dot{H}_x^{s_c}(\mathbb{R}^d)$ by*

$$[g_{x_0,\lambda} f](x) := \lambda^{-\frac{2}{p}} f\big(\lambda^{-1}(x - x_0)\big),$$

where $s_c := \frac{d}{2} - \frac{2}{p}$. We let G denote the collection of such transformations. For a function $u : I \times \mathbb{R}^d \to \mathbb{C}$ we define $T_{g_{x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^d \to \mathbb{C}$ by the formula

$$[T_{g_{x_0,\lambda}} u](t,x) := \lambda^{-\frac{2}{p}} u\big(\lambda^{-2}t, \lambda^{-1}(x - x_0)\big),$$

where $\lambda^2 I := \{\lambda^2 t : t \in I\}$. Note that if u is a solution to (1.1)*, then $T_g u$ is a solution to* (1.1) *with initial data gu_0.*

Remark 4.3.2 We remark here that G forms a group under composition. The map $u \mapsto T_g u$ takes solutions to (1.1) to solutions with the same scattering size. Furthermore, u is a maximal-lifespan solution if and only if $T_g u$ is a maximal-lifespan solution.

We now state the linear profile decomposition that we will use in the reduction to almost periodic solutions. The first profile decompositions established for $e^{it\Delta}$ were adapted to the mass- and energy-critical settings [1, 7, 34, 47]; the case of non-conserved critical regularity was addressed in [53].

For the cases we consider in this thesis, we will be able to import the profile decomposition that we need directly from [53].

Lemma 4.3.3 (Linear profile decomposition [53]) *Let $0 < s_c < 1$ and let $\{u_n\}$ be a bounded sequence in $\dot{H}_x^{s_c}(\mathbb{R}^d)$. After passing to a subsequence if necessary, there exist functions $\{\phi^j\} \subset \dot{H}_x^{s_c}(\mathbb{R}^d)$, group elements $g_n^j \in G$ (with parameters x_n^j and λ_n^j), and times $t_n^j \in \mathbb{R}$ such that for all $J \geq 1$, we have the decomposition*

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + w_n^J$$

with the following properties:

- *For each j, either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$ as $n \to \infty$.*
- *For all n and all $J \geq 1$, we have $w_n^J \in \dot{H}_x^{s_c}(\mathbb{R}^d)$, with*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{p(d+2)}{2}}(\mathbb{R}\times\mathbb{R}^d)} = 0. \tag{4.13}$$

- *For any $j \neq k$, we have the following asymptotic orthogonality of parameters:*

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \to \infty \quad \text{as } n \to \infty. \tag{4.14}$$

- *We have the decoupling properties: for any $J \geq 1$,*

$$\lim_{n\to\infty} \left[\||\nabla|^{s_c} u_n\|_2^2 - \sum_{j=1}^{J} \||\nabla|^{s_c} \phi^j\|_2^2 - \||\nabla|^{s_c} w_n^J\|_2^2 \right] = 0, \tag{4.15}$$

and for any $1 \leq j \leq J$,

$$e^{-it_n^j \Delta}[(g_n^j)^{-1} w_n^J] \rightharpoonup 0 \quad \text{weakly in } \dot{H}_x^{s_c} \quad \text{as } n \to \infty. \tag{4.16}$$

The author of [53] deduces Lemma 4.3.3 from a linear profile decomposition adapted to the mass-critical equation. We remark here that it is also possible to establish Lemma 4.3.3 'from scratch'. We refer the interested reader to the lecture notes [38, 71]. There one can find the proof of the linear profile decomposition adapted to the energy-critical setting. However, the ideas in [38, 71] carry over to the $\dot{H}_x^{s_c}$-critical case, as well.

4.4 The reduction to almost periodic solutions

The goal of this section is to prove Theorem 1.1.2. As described in the introduction, the key ideas come from [34, 35] and are well-known. Thus, we will only outline the main steps of the argument, providing full details only when significant new difficulties arise in our setting. We will model our presentation after [39, Section 3].

The material in this section appeared originally in [49, 50].

We suppose that Theorem 1.1.1 fails. We then define the function $L : [0, \infty) \to [0, \infty]$ by

$$L(E) := \sup\{S_I(u) : u : I \times \mathbb{R}^d \to \mathbb{C} \text{ solving (1.1) with } \sup_{t \in I} \|u(t)\|^2_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leq E\},$$

where $S_I(u)$ is defined as in (3.1). For the cases (1.4) and (1.5), we restrict the supremum to radial solutions.

We note that L is a non-decreasing function, and that Theorem 3.1.3 implies

$$L(E) \lesssim E^{\frac{p(d+2)}{4}} \quad \text{for} \quad E < \eta_0, \tag{4.17}$$

where η_0 is the small-data threshold. Thus, there exists a unique 'critical' threshold $E_c \in (0, \infty]$ such that $L(E) < \infty$ for $E < E_c$ and $L(E) = \infty$ for $E > E_c$. The failure of Theorem 1.1.1 implies that $0 < E_c < \infty$.

The key to the proof of Theorem 1.1.2 is the following convergence result. With this result in hand, establishing Theorem 1.1.2 is a straightforward exercise (see [39, Section 3.2]).

Proposition 4.4.1 (Palais–Smale condition modulo symmetries) *Let $u_n : I_n \times \mathbb{R}^d \to$*

$\mathbb{C}$ *be a sequence of solutions to* (1.1) *such that*

$$\limsup_{n\to\infty} \|u_n\|^2_{L_t^\infty \dot{H}_x^{s_c}(I_n\times\mathbb{R}^d)} = E_c,$$

and suppose $t_n \in I_n$ *are such that*

$$\lim_{n\to\infty} S_{[t_n,\sup I_n)}(u_n) = \lim_{n\to\infty} S_{(\inf I_n,t_n]}(u_n) = \infty. \tag{4.18}$$

Then $\{u_n(t_n)\}$ *converges along a subsequence in* $\dot{H}_x^{s_c}(\mathbb{R}^d)/G$. *(Here* G *is as in Definition 4.3.1.)*

Proof of Proposition 4.4.1. We first translate so that each $t_n = 0$ and apply Lemma 4.3.3 to write

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j\Delta}\phi^j + w_n^J \tag{4.19}$$

along some subsequence. To prove Proposition 4.4.1, we need to show that there is exactly one profile ϕ^1, with $t_n^1 \equiv 0$ and $\|w_n^1\|_{\dot{H}_x^{s_c}} \to 0$.

First, using Theorem 3.1.3, for each j we define $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ to be the maximal-lifespan solution to (1.1) such that

$$\begin{cases} v^j(0) = \phi^j & \text{if } t_n^j \equiv 0, \\ v^j \text{ scatters to } \phi^j \text{ as } t \to \pm\infty & \text{if } t_n^j \to \pm\infty. \end{cases}$$

Next, we define nonlinear profiles $v_n^j : I_n^j \times \mathbb{R}^d \to \mathbb{C}$ by

$$v_n^j(t) = g_n^j v^j\big((\lambda_n^j)^{-2}t + t_n^j\big), \quad \text{where} \quad I_n^j = \{t : (\lambda_n^j)^{-2}t + t_n^j \in I^j\}.$$

The proof of Proposition 4.4.1 relies on the following three claims:

(i) There is at least one 'bad' profile ϕ^j, in the sense that

$$\limsup_{n\to\infty} S_{[0,\sup I_n^j)}(v_n^j) = \infty. \tag{4.20}$$

(ii) There can then be at *most* one profile (which we label ϕ^1), and $\|w_n^1\|_{\dot{H}_x^{s_c}} \to 0$.

(iii) We have $t_n^1 \equiv 0$.

We will provide a proof of (i) below. The proofs of (ii) and (iii) require only small variations of the analysis given for (i), so we will merely outline the arguments here. For (ii), one can adapt the argument of [39, Lemma 3.3] to show that the decoupling (4.15) persists in time (this is not obvious, as the $\dot{H}_x^{s_c}$-norm is not a conserved quantity for (1.1)). The critical nature of E_c may then be used to preclude the possibility of multiple profiles (and to show $\|w_n^1\|_{\dot{H}_x^{s_c}} \to 0$). For (iii), we only need to rule out the cases $t_n^1 \to \pm\infty$. To do this, one can argue by contradiction: if $t_n^1 \to \pm\infty$, one can use the stability results Theorem 3.2.2 or Theorem 3.3.5 (comparing u_n to $e^{it\Delta}u_n(0)$) to contradict (4.18). See [39, p. 391] for more details.

We now turn to the proof of (i). We first note that the decoupling (4.15) implies that the v_n^j are global and scatter for j sufficiently large, say for $j \geq J_0$; indeed, for j sufficiently large, the $\dot{H}_x^{s_c}$-norm of ϕ^j must be below the small-data threshold given in Theorem 3.1.3. Thus, we need to show that there is at least one bad profile ϕ^j (in the sense of (4.20)) in the range $1 \leq j < J_0$.

Suppose toward a contradiction that there are no bad profiles. By the blowup criterion of Theorem 3.1.3, this immediately implies that $\sup I_n^j = \infty$ for all j and for all n sufficiently large. In fact, we claim that we have the following:

$$\limsup_{J\to\infty} \limsup_{n\to\infty} \sum_{j=1}^{J} \|v_n^j\|^2_{\dot{S}^{s_c}([0,\infty))} \lesssim_{E_c} 1. \tag{4.21}$$

Indeed, for $\eta > 0$, the decoupling (4.15) implies the existence of $J_1 = J_1(\eta)$ such that

$$\sum_{j>J_1} \|\phi^j\|^2_{\dot{H}_x^{s_c}} \lesssim \eta.$$

Thus, choosing η smaller than the small-data threshold, Strichartz and a standard bootstrap argument give

$$\sum_{j>J_1} \|v_n^j\|^2_{\dot{S}^{s_c}([0,\infty))} \lesssim \sum_{j>J_1} \|\phi^j\|^2_{\dot{H}_x^{s_c}} \lesssim \eta.$$

As the v_n^j satisfy $S_{[0,\infty)}(v_n^j) \lesssim 1$ for n large, we may use Strichartz and another bootstrap argument to see $\|v_n^j\|_{\dot{S}^{s_c}} \lesssim 1$ for $1 \leq j \leq J_1$ and n large. We conclude that (4.21) holds.

We now wish to use (4.21), the orthogonality condition (4.14), and the stability results of Chapter 3 to deduce a bound on the scattering size of the u_n, thus contradicting (4.18). To this end, we define the approximate solutions

$$u_n^J(t) := \sum_{j=1}^J v_n^j(t) + e^{it\Delta} w_n^J$$

and the corresponding errors

$$e_n^J := (i\partial_t + \Delta) u_n^J - F(u_n^J) = \sum_{j=1}^J F(v_n^j) - F(u_n^J).$$

Regarding the approximate solutions, we first have the following.

Lemma 4.4.2 *The approximate solutions u_n^J satisfy*

$$\limsup_{J\to\infty} \limsup_{n\to\infty} \|u_n(0) - u_n^J(0)\|_{\dot{H}_x^{s_c}} = 0, \tag{4.22}$$

$$\limsup_{J\to\infty} \limsup_{n\to\infty} S_{[0,\infty)}(u_n^J) \lesssim_{E_c} 1, \tag{4.23}$$

Proof. We first note that (4.22) follows from the construction of the v^j.

To see that (4.23) holds, first note that by (4.13) and (4.21), it suffices to show

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left| S_{[0,\infty)}\left(\sum_{j=1}^J v_n^j \right) - \sum_{j=1}^J S_{[0,\infty)}(v_n^j) \right| = 0. \tag{4.24}$$

To establish (4.24), we can first use the pointwise inequality

$$\left| \left| \sum_{j=1}^J v_n^j \right|^{\frac{p(d+2)}{2}} - \sum_{j=1}^J |v_n^j|^{\frac{p(d+2)}{2}} \right| \lesssim_J \sum_{j\neq k} |v_n^j|^{\frac{p(d+2)}{2}-1} |v_n^k|$$

along with Hölder's inequality to see

$$\text{LHS}(4.24) \lesssim_J \sum_{j\neq k} \|v_n^j\|_{L_{t,x}^{\frac{p(d+2)}{2}}([0,\infty)\times\mathbb{R}^d)}^{\frac{p(d+2)}{2}-2} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{p(d+2)}{4}}([0,\infty)\times\mathbb{R}^d)}. \tag{4.25}$$

Now we follow an argument of Keraani (cf. [34, Lemma 2.7]): for $j \neq k$, we can approximate v^j and v^k by compactly supported functions in $\mathbb{R}\times\mathbb{R}^d$ and use the asymptotic orthogonality of parameters (4.14) to show

$$\limsup_{n\to\infty} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{p(d+2)}{4}}([0,\infty)\times\mathbb{R}^d)} = 0. \tag{4.26}$$

Thus, continuing from (4.25), we get that (4.24) (and therefore (4.23)) holds. ∎

We next need to control the errors e_n^J in order to show that the u_n^J are good approximate solutions to (1.1). For then, in light of Lemma 4.4.2, the stability results of Chapter 3 will allow us to deduce good bounds on the u_n from those enjoyed by u_n^J, giving us the desired contradiction.

To proceed, we need separate into two cases depending on the stability result that we wish to apply. We will first treat the case (1.2), that is, $s_c = 1/2$ in dimensions $d \geq 4$. Note that $s_c = 1/2$ corresponds to $p = 4/(d-1)$, so that $p \to 0$ as $d \to \infty$. For the case (1.2), we have proven a refined stability result (Theorem 3.3.5) that does not require errors to be small in a space with derivatives. This will greatly simplify the analysis of the errors e_n^J.

On the other hand, for the cases (1.3), (1.4), and (1.5), the stability result that we proved (Theorem 3.2.2) requires errors to be small in a space with s_c derivatives. As we will see, combining fractional derivatives with non-polynomial nonlinearities will present nontrivial technical difficulties. However, relying on the fact that $p > 1$ in all of the cases under consideration, we will ultimately be able to overcome these difficulties.

We turn to the case (1.2). First, using the assumption that there are no bad profiles, together with the orthogonality condition (4.14), one can use the arguments of [34] to arrive at the following lemma.

Lemma 4.4.3 (Orthogonality) *Let* (d, s_c) *satisfy* (1.2). *For* $j \neq k$, *we have*

$$\begin{aligned}\Big[\|v_n^j v_n^k\|_{L_t^{\frac{4(d+1)}{d+3}} L_x^{\frac{2d(d+1)}{2d^2-d-5}}} &+ \|(|\nabla|^{s_c} v_n^j)(|\nabla|^{s_c} v_n^k)\|_{L_{t,x}^{\frac{d+2}{d}}} \\ &+ \|\big(|\nabla|^{s_c} F(v_n^j)\big)\big(|\nabla|^{s_c} F(v_n^k)\big)^{\frac{d}{d+4}}\|_{L^1_{t,x}} \Big] \to 0 \quad as \quad n \to \infty, \end{aligned} \tag{4.27}$$

where all spacetime norms are taken over $[0, \infty) \times \mathbb{R}^d$.

We can now turn to controlling the errors e_n^J in the case (1.2). As we will see, due to the fact that we are estimating errors in spaces without derivatives, pointwise estimates as in [34] will suffice to establish the bounds we need. In order to apply Theorem 3.3.5, we will also need to bound the u_n^J in $\dot{S}^{1/2}$.

Lemma 4.4.4 *Let (d, s_c) satisfy (1.2). Then we have the following:*

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \||\nabla|^{s_c} e_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([0,\infty)\times\mathbb{R}^d)} \lesssim_{E_c} 1, \tag{4.28}$$

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \|u_n^J\|_{\dot S^{1/2}([0,\infty))} \lesssim_{E_c} 1. \tag{4.29}$$

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \|e_n^J\|_{L_t^{\frac{4(d+1)}{d+3}} L_x^{\frac{2(d+1)}{d+3}}([0,\infty)\times\mathbb{R}^d)} = 0. \tag{4.30}$$

Proof. We begin with (4.28). We will first derive the bound

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \||\nabla|^{s_c} u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \lesssim_{E_c} 1. \tag{4.31}$$

As $w_n^J \in \dot H_x^{1/2}$, it will suffice to show

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \|\sum_{j=1}^J |\nabla|^{s_c} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \lesssim_{E_c} 1. \tag{4.32}$$

To this end, we first note that as $\frac{2(d+2)}{d} \geq 2$, we may use (4.21) to see

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \sum_{j=1}^J \||\nabla|^{s_c} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \lesssim_{E_c} 1. \tag{4.33}$$

On the other hand, for fixed J, we can use (4.21) and (4.27) to see

$$\begin{aligned}
&\left| \|\sum_{j=1}^J |\nabla|^{s_c} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} - \sum_{j=1}^J \||\nabla|^{s_c} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \right| \\
&\lesssim_J \sum_{j\neq k} \||\nabla|^{s_c} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d}} \|(|\nabla|^{s_c} v_n^j)(|\nabla|^{s_c} v_n^k)\|_{L_{t,x}^{\frac{d+2}{d}}} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Then (4.33) implies (4.32), which in turn gives (4.31).

Next, by the fractional chain rule, (4.23), and (4.31), we get

$$\||\nabla|^{s_c} F(u_n^J)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-1}}}^{\frac{4}{d-1}} \||\nabla|^{s_c} u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim_{E_c} 1 \tag{4.34}$$

as $n, J\to\infty$, which handles one of the terms appearing in (4.28).

To complete the proof of (4.28), it remains to show

$$\limsup_{J\to\infty}\limsup_{n\to\infty} \|\sum_{j=1}^J |\nabla|^{s_c} F(v_n^j)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{2(d+2)}{d+4}} \lesssim_{E_c} 1.$$

We claim it will suffice to establish

$$\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j=1}^{J}\||\nabla|^{s_c}F(v_n^j)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{2(d+2)}{d+4}}\lesssim_{E_c}1. \tag{4.35}$$

Indeed, for fixed J, we have by (4.27)

$$\begin{aligned}\Big|\|\sum_{j=1}^{J}|\nabla|^{s_c}F(v_n^j)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{2(d+2)}{d+4}}-\sum_{j=1}^{J}\||\nabla|^{s_c}F(v_n^j)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{2(d+2)}{d+4}}\Big|\\ \lesssim_J\sum_{j\neq k}\||\nabla|^{s_c}F(v_n^j)\,||\nabla|^{s_c}F(v_n^k)|^{\frac{d}{d+4}}\|_{L_{t,x}^1}\to 0\quad\text{as } n\to\infty.\end{aligned}$$

To establish (4.35) and thereby complete the proof of (4.28), we use the fractional chain rule and Sobolev embedding to see

$$\begin{aligned}\sum_{j=1}^{J}\||\nabla|^{s_c}F(v_n^j)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{2(d+2)}{d+4}}&\lesssim\sum_{j=1}^{J}\big(\|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-1}}}^{\frac{4}{d-1}}\||\nabla|^{s_c}v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}\big)^{\frac{2(d+2)}{d+4}}\\ &\lesssim\sum_{j=1}^{J}\|v_n^j\|_{\dot S^{1/2}}^{\frac{2(d+2)(d+3)}{(d+4)(d-1)}}.\end{aligned}$$

Then (4.35) follows from (4.21) and the fact that $\frac{2(d+2)(d+3)}{(d+4)(d-1)}\geq 2$.

Now (4.29) follows from an application of Strichartz, (4.28) and (4.34).

It remains to establish (4.30). Here we argue as in [34]. We begin by rewriting

$$e_n^J=\Big[\sum_{j=1}^{J}F(v_n^j)-F\Big(\sum_{j=1}^{J}v_n^j\Big)\Big]+\big[F(u_n^J-e^{it\Delta}w_n^J)-F(u_n^J)\big]=:(e_n^J)_1+(e_n^J)_2.$$

We first fix J and $d\geq 5$. By Hölder, Sobolev embedding, (4.21), and (4.27), we have

$$\begin{aligned}\|(e_n^J)_1\|_{L_t^{\frac{4(d+1)}{d+3}}L_x^{\frac{2(d+1)}{d+3}}}&\lesssim_J\sum_{j\neq k}\|\,|v_n^jv_n^k|^{\frac{4}{d-1}}|v_n^j|^{\frac{d-5}{d-1}}\,\|_{L_t^{\frac{4(d+1)}{d+3}}L_x^{\frac{2(d+1)}{d+3}}}\\ &\lesssim_J\sum_{j\neq k}\|v_n^j\|_{L_t^{\frac{4(d+1)}{d+3}}L_x^{\frac{2d(d+1)}{d^2-d-4}}}^{\frac{d-5}{d-1}}\|v_n^jv_n^k\|_{L_t^{\frac{4(d+1)}{d+3}}L_x^{\frac{2d(d+1)}{2d^2-d-5}}}^{\frac{4}{d-1}}\to 0\end{aligned}$$

as $n\to\infty$. When $d=4$, we modify this argument as follows:

$$\begin{aligned}\|(e_n^J)_1\|_{L_t^{20/7}L_x^{10/7}}&\lesssim_J\sum_{j\neq k}\||v_n^j|^{1/3}v_n^jv_n^k\|_{L_t^{20/7}L_x^{10/7}}\\ &\lesssim_J\sum_{j\neq k}\|v_n^j\|_{L_t^\infty L_x^{8/3}}^{1/3}\|v_n^jv_n^k\|_{L_t^{20/7}L_x^{40/23}}\to 0\end{aligned}$$

as $n \to \infty$.

Next, we note that we have the pointwise estimate

$$|(e_n^J)_2| \lesssim |e^{it\Delta} w_n^J| |f_n^J|^{\frac{4}{d-1}},$$

where $f_n^J := u_n^J + e^{it\Delta} w_n^J$ satisfies $\|f_n^J\|_{\dot{S}^{1/2}} \lesssim_{E_c} 1$ as $n, J \to \infty$ (cf. (4.29) and the fact that $w_n^J \in \dot{H}_x^{1/2}$). Thus, we can use Hölder, Strichartz, Sobolev embedding, $w_n^J \in \dot{H}_x^{1/2}$, and (4.13) to see

$$\begin{aligned} \|(e_n^J)_2\|_{L_t^{\frac{4(d+1)}{d+3}} L_x^{\frac{2(d+1)}{d+3}}} &\lesssim \|e^{it\Delta} w_n^J\|_{L_t^{\frac{4(d+1)}{d-1}} L_x^{\frac{2(d+1)}{d-1}}} \|f_n^J\|^{\frac{4}{d-1}}_{L_t^{\frac{4(d+1)}{d-1}} L_x^{\frac{2(d+1)}{d-1}}} \\ &\lesssim \|e^{it\Delta} w_n^J\|^{\frac{d+2}{2(d+1)}}_{L_{t,x}^{\frac{2(d+2)}{d-1}}} \|w_n^J\|^{\frac{d}{2(d+1)}}_{\dot{H}_x^{1/2}} \|f_n^J\|^{\frac{4}{d-1}}_{\dot{S}^{1/2}} \to 0 \quad \text{as } n, J \to \infty. \end{aligned}$$

Combining the estimates for $(e_n^J)_1$ and $(e_n^J)_2$, we conclude that (4.30) holds. This completes the proof of Lemma 4.4.4. ∎

We can now see that for the case (1.2) we may use Lemmas 4.4.2 and 4.4.4 together with Theorem 3.3.5 to deduce that $S_{[0,\infty)}(u_n) \lesssim_{E_c} 1$ for n large, thus contradicting (4.18). We conclude that there is at least one bad profile, that is, (4.20) holds. Thus claim (i) above holds, which completes the proof of Proposition 4.4.1 and Theorem 1.1.2 in the case (1.2).

For the cases (1.3), (1.4), and (1.5), we instead wish to apply the stability result Theorem 3.2.2. Breaking e_n^J into $(e_n^J)_1$ and $(e_n^J)_2$ as above and applying the triangle inequality, we see that in order to apply Theorem 3.2.2 we will need the following lemma.

Lemma 4.4.5 *Let (d, s_c) satisfy* (1.3), (1.4), *or* (1.5). *Then we have the following:*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| |\nabla|^{s_c} \left(F\Big(\sum_{j=1}^J v_n^j \Big) - \sum_{j=1}^J F(v_n^j) \right) \right\|_{\dot{N}^0([0,\infty))} = 0, \tag{4.36}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \| |\nabla|^{s_c} \left(F(u_n^J - e^{it\Delta} w_n^J) - F(u_n^J) \right) \|_{\dot{N}^0([0,\infty))} = 0. \tag{4.37}$$

Before we begin the proof, we pause to discuss some of the new technical difficulties associated to establishing this decoupling. As we saw above when treating the case (1.2), the estimate (4.36) would be much simplier in the absence of derivatives. Indeed, in this case one could use the pointwise estimate

$$\Big| F\big(\sum_{j=1}^{J} v_n^j\big) - \sum_{j=1}^{J} F(v_n^j)\Big| \lesssim_J \sum_{j\neq k} |v_n^j|\,|v_n^k|^p$$

and follow the arguments of Keraani [34] directly. This is the approach taken in the mass-critical case, for example.

In the energy-critical setting there is a replacement for this estimate, namely

$$\Big|\nabla\Big(F\big(\sum_{j=1}^{J} v_n^j\big) - \sum_{j=1}^{J} F(v_n^j)\Big)\Big| \lesssim_J \sum_{j\neq k} |\nabla v_n^j|\,|v_n^k|^p.$$

Thus it is still possible to employ a pointwise estimate to exhibit terms containing v_n^j paired against v_n^k for some $j \neq k$, so that the orthogonality (4.14) can be used.

However, pointwise estimates such as these do not apply directly in our setting due to the nonlocal nature of $|\nabla|^{s_c}$.

In the energy-supercritical case, the authors of [40] were able to establish analogous pointwise estimates for a square function of Strichartz that shares estimates with fractional differentiation operators (see [59]). With such pointwise estimates in place, the usual arguments then finish the argument. This approach does not work in our setting, however, as it relies fundamentally on the fact that $s_c > 1$.

The authors of [32] dealt with the case $s_c = 1/2$ in dimension $d = 3$. However, in that case one has the algebraic nonlinearity $|u|^2u$. Exploiting this fact and employing a paraproduct estimate, the authors were able to place themselves back into a situation where the usual arguments apply.

In our case, we must deal simultaneously with a non-algebraic nonlinearity and a fractional number of derivatives. We proceed by opening up the proof of the fractional chain rule (Lemma 2.2.4) as given in [61, §2.4]. In particular, we employ the Littlewood–Paley square function (cf. Lemma 2.2.2), which allows us to work at the level of individual frequencies. By making use of maximal function and vector maximal function estimates, we can ultimately find a way to adapt the standard arguments.

Proof of (4.36)**.** By induction, it will suffice to treat the case of two summands; to simplify notation, we write $f = v_n^j$ and $g = v_n^k$ for some $j \neq k$, and we are left to show

$$\| |\nabla|^{s_c} \big(|f+g|^p (f+g) - |f|^p f - |g|^p g \big) \|_{\dot N^0([0,\infty))} \to 0 \tag{4.38}$$

as $n \to \infty$.

As alluded to above, the key will be to perform a decomposition in such a way that all of the resulting terms we need to estimate have f paired against g inside of a single integrand; for such terms, we will be able to use the asymptotic orthogonality of parameters (4.14) to our advantage.

We first rewrite

$$\begin{aligned} &|f+g|^p(f+g) - |f|^p f - |g|^p g \\ &\qquad = \big(|f+g|^p - |f|^p\big) f + \big(|f+g|^p - |g|^p\big) g. \end{aligned}$$

By symmetry, it will suffice to treat the first term. We turn therefore to estimating

$$\| |\nabla|^{s_c} \big[(|f+g|^p - |f|^p) f \big] \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}.$$

By Lemma 2.2.2, it will suffice to consider

$$\Big\| \Big(\sum \big| N^{s_c} P_N \big[(|f+g|^p - |f|^p) f \big] \big|^2 \Big)^{1/2} \Big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}. \tag{4.39}$$

Thus, we restrict our attention to a single frequency $N \in 2^{\mathbb{Z}}$. We let $\delta_y f(x) := f(x-y) - f(x)$, and let $\check\psi$ denote the convolution kernel of the Littlewood–Paley projection P_1. As $\psi(0) = 0$, we have

$$\textstyle\int \check\psi(y)\, dy = 0,$$

so that exploiting cancellation, we can write

$$\begin{aligned} &P_N\big(\big[|f(x)+g(x)|^p - |f(x)|^p \big] f(x) \big) \\ &\quad = \textstyle\int N^d \check\psi(Ny) \delta_y \big(\big[|f(x)+g(x)|^p - |f(x)|^p \big] f(x) \big)\, dy. \end{aligned} \tag{4.40}$$

We now rewrite

$$
\begin{aligned}
&\delta_y\big(\big[|f(x)+g(x)|^p-|f(x)|^p\big]f(x)\big)\\
&\quad=\delta_y f(x)\big[|f(x-y)+g(x-y)|^p-|f(x-y)|^p\big] &(4.41)\\
&\quad\ +f(x)\big[|f(x)+g(x-y)|^p-|f(x)+g(x)|^p\big] &(4.42)\\
&\quad\ +f(x)\big[|f(x-y)+g(x-y)|^p-|f(x-y)|^p+|f(x)|^p-|f(x)+g(x-y)|^p\big]. &(4.43)
\end{aligned}
$$

We estimate each term individually. First, we have

$$|(4.41)|\lesssim|\delta_y f(x)|\,|g(x-y)|\left\{|f(x-y)|^{p-1}+|g(x-y)|^{p-1}\right\}.$$

Next, we see

$$|(4.42)|\lesssim|f(x)|\,|\delta_y g(x)|\left\{|f(x)|^{p-1}+|g(x)|^{p-1}+|g(x-y)|^{p-1}\right\}.$$

We now turn to (4.43). First, if $1<p\le 2$, a simple argument using the fundamental theorem of calculus implies

$$|(4.43)|\lesssim|f(x)|\,|\delta_y f(x)|\,|g(x-y)|^{p-1}$$

(see Lemma 2.1.1 for details). For $p>2$, one instead finds

$$|(4.43)|\lesssim|f(x)|\,|\delta_y f(x)|\,|g(x-y)|\left\{|f(x)|^{p-2}+|f(x-y)|^{p-2}+|g(x-y)|^{p-2}\right\}.$$

To ease the exposition, we will restrict our attention here and below to the more difficult case $1<p\le 2$; once we have dealt with this case, it should be clear how to proceed when $p>2$.

Collecting terms, we continue from (4.40) to see

$$
\begin{aligned}
&\big|P_N\big(\big[|f(x)+g(x)|^p-|f(x)|^p\big]f(x)\big)\big|\\
&\quad\lesssim\int N^d|\check\psi(Ny)|\,|\delta_y f(x)|\,|g(x-y)|\left\{|f(x-y)|^{p-1}+|g(x-y)|^{p-1}\right\}dy &(4.44)\\
&\quad\ +\int N^d|\check\psi(Ny)|\,|f(x)|\,|\delta_y g(x)|\left\{|f(x)|^{p-1}+|g(x)|^{p-1}+|g(x-y)|^{p-1}\right\}dy &(4.45)\\
&\quad\ +\int N^d|\check\psi(Ny)|\,|f(x)|\,|\delta_y f(x)||g(x-y)|^{p-1}\,dy. &(4.46)
\end{aligned}
$$

One can see that we are already faced with several terms to estimate; moreover, to estimate any single term will require further decomposition. However, in the end, the same set of tools will suffice to handle every term that appears. Thus, let us deal with only (4.44) in detail; once we have seen how to handle this term, it should be clear that the same techniques apply to handle (4.45) and (4.46).

Turning to (4.44), we first write

$$
\begin{aligned}
(4.44) = &\textstyle\int N^d|\check{\psi}(Ny)|\,|\delta_y f(x)|\,|g(x-y)|\,|f(x-y)|^{p-1}\,dy && (4.47)\\
&+ \textstyle\int N^d|\check{\psi}(Ny)|\,|\delta_y f(x)|\,|g(x-y)|^p\,dy. && (4.48)
\end{aligned}
$$

For both of these terms, we will need to make use of some auxiliary inequalities in the spirit of [61, §2.3], which we record in Lemma 2.2.8.

We turn to (4.47). If we first write

$$
|\delta_y f(x)| \lesssim |f_{>N}(x)| + |f_{>N}(x-y)| + \sum_{K\le N} |\delta_y f_K(x)|, \tag{4.49}
$$

then putting Lemma 2.2.8 to use, we arrive at

$$
\begin{aligned}
(4.47) \lesssim\ & |f_{>N}(x)|\,M(g\,|f|^{p-1})(x) && (4.50)\\
&+ M(f_{>N}\,g\,|f|^{p-1})(x) && (4.51)\\
&+ \sum_{K\le N} \tfrac{K}{N} M(f_K)(x) M(g\,|f|^{p-1})(x) && (4.52)\\
&+ \sum_{K\le N} \tfrac{K}{N} M(M(f_K)\,g\,|f|^{p-1})(x). && (4.53)
\end{aligned}
$$

Similarly, we can decompose

$$
\begin{aligned}
(4.48) \lesssim\ & |f_{>N}(x)|\,M(|g|^p)(x) && (4.54)\\
&+ M(f_{>N}|g|^p)(x) && (4.55)\\
&+ \sum_{K\le N} \tfrac{K}{N} M(f_K)(x) M(|g|^p)(x) && (4.56)\\
&+ \sum_{K\le N} \tfrac{K}{N} M(M(f_K)|g|^p)(x). && (4.57)
\end{aligned}
$$

Let us now consider the contribution of (4.50) to the left-hand side of (4.38). Comparing with (4.39), we see it will suffice to estimate

$$\|\Big(\sum_N \big|N^{s_c} f_{>N} M(g\,|f|^{p-1})\big|^2\Big)^{1/2}\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}.$$

Using Hölder's inequality and maximal function estimates, we can control this term by

$$\|\Big(\sum_N \big|N^{s_c} f_{>N}\big|^2\Big)^{1/2}\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \||g|\,|f|^{p-1}\|_{L_{t,x}^{\frac{d+2}{2}}}.$$

We now recall that $f = v_n^j$ and $g = v_n^k$ for some $j \neq k$. Then, the first term is controlled by $\||\nabla|^{s_c} v_n^j\|_{S^0}$ (cf. Lemma 2.2.2), which in turn is bounded (recall that by assumption, all of the v_n^j have scattering size $\lesssim E_c$). The second term can be handled in the standard way; that is, this term vanishes in the limit due to the asymptotic orthogonality of parameters (4.14) (cf. [34, Lemma 2.7]).

Thus, we see that (4.50) is under control. A similar approach (this time using the vector maximal inequality) handles (4.51).

To estimate the contribution of (4.52) to the left-hand side of (4.38), we need to estimate

$$\|\Big(\sum_N \big|N^{s_c} \sum_{K\leq N} \tfrac{K}{N} M(f_K) M(g\,|f|^{p-1})\big|^2\Big)^{1/2}\|_{\dot N^0([0,\infty))}. \tag{4.58}$$

For this term, we need to make use of the following basic inequality: for a nonnegative sequence $\{a_K\}_{K\in 2^{\mathbb{Z}}}$ and $0 < s < 1$, one has

$$\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\Big|\sum_{K\leq N} \tfrac{K}{N} a_K\Big|^2 \lesssim \sum_{K\in 2^{\mathbb{Z}}} K^{2s}|a_K|^2 \tag{4.59}$$

(cf. [61, Lemma 4.2]). Using this inequality, along with Hölder, we can estimate

$$\begin{aligned}(4.58) &\lesssim \|\Big(\sum_K \big|K^{s_c} M(f_K)\big|^2\Big)^{1/2} M(g\,|f|^{p-1})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}\\ &\lesssim \|\Big(\sum_K |K^{s_c} M(f_K)|^2\Big)^{1/2}\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \||g|\,|f|^{p-1}\|_{L_{t,x}^{\frac{d+2}{2}}} \to 0\end{aligned}$$

as $n \to \infty$, exactly as before. Thus, (4.52) is under control; the same approach handles (4.53) (after an application of the vector maximal inequality).

Let us now turn to (4.54). As before, we sum over $N \in 2^{\mathbb{Z}}$ and find that we need to estimate

$$\Big\| \Big(\sum \big| N^{s_c} f_{>N} \big|^2 \Big)^{1/2} M(|g|^p) \Big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}. \tag{4.60}$$

Recalling that $f = v_n^j$ and $g = v_n^k$ for some $j \neq k$, we see that we are once again in a position to use the argument from [34].

To begin, we may assume without loss of generality that both

$$\Phi_1 := \Big(\sum |N^{s_c} P_{>N} v^j|^2 \Big)^{1/2} \quad \text{and} \quad \Phi_2 := M(|v^k|^p)$$

belong to $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$; indeed, C_c^∞-functions are dense in both $L_{t,x}^{\frac{2(d+2)}{d}}$ and $L_{t,x}^{\frac{d+2}{2}}$. We now wish to use the asymptotic orthogonality of parameters, that is,

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \to \infty \quad \text{as } n \to \infty, \tag{4.61}$$

to show (4.60)$\to 0$.

Consider first the case $\frac{\lambda_n^j}{\lambda_n^k} \to c > 0$ (along a subsequence, say). If we unravel the definition of the nonlinear profiles and change variables to move the symmetries onto Φ_2, we arrive at

$$\begin{aligned} &(4.60)^{\frac{2(d+2)}{d+4}} \\ &= \Big(\tfrac{\lambda_n^j}{\lambda_n^k} \Big)^{\frac{4(d+2)}{d+4}} \iint \Big| \Phi_1(s,y) \Phi_2\big(t_n^k + \big(\tfrac{\lambda_n^j}{\lambda_n^k}\big)^2 (s - t_n^j), \big(\tfrac{\lambda_n^j}{\lambda_n^k}\big) y + \tfrac{x_n^j - x_n^k}{\lambda_n^k} \big) \Big|^{\frac{2(d+2)}{d+4}} \, dy \, ds. \end{aligned}$$

Then, recalling (4.61), we see that as $n \to \infty$, either the spatial or temporal argument of Φ_2 must escape the support of Φ_1. Thus, in this case, we get (4.60)$\to 0$.

If instead we have $\frac{\lambda_n^j}{\lambda_n^k} \to 0$, then continuing from above, we can estimate

$$(4.60) \lesssim \big(\tfrac{\lambda_n^j}{\lambda_n^k} \big)^2 \|\Phi_1\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \|\Phi_2\|_{L_{t,x}^\infty}.$$

As $\Phi_1, \Phi_2 \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, we see that (4.60)$\to 0$ in this case, as well.

Finally, we can treat the case $\frac{\lambda_n^j}{\lambda_n^k} \to \infty$ just like the previous case; the only difference is that we change variables to move the symmetries onto Φ_1, instead of Φ_2. Thus, we have that (4.60)$\to 0$ in this third and final case.

We have now shown that (4.54) is under control. The same ideas can be used to handle (4.55), (4.56), and (4.57).

As mentioned above, this same set of ideas suffices to deal with all the remaining terms stemming from (4.36). ∎

Proof of (4.37)**.** For this term, we will need to make use (4.13). As we will see, the terms in which $e^{it\Delta}w_n^J$ appears without derivatives will be relatively easy to handle, as (4.13) will apply directly. On the other hand, the terms that only contain $|\nabla|^{s_c}e^{it\Delta}w_n^J$ will require a more careful analysis; in particular, we will need to carry out a local smoothing argument before we can make effective use of (4.13).

Defining $g := \sum_{j=1}^J v_n^j$ and $h := e^{it\Delta}w_n^J$, we are left to show

$$\lim_{J\to\infty}\limsup_{n\to\infty} \| |\nabla|^{s_c}\left(|g+h|^p(g+h)-|g|^p g\right)\|_{\dot N^0([0,\infty))} = 0. \tag{4.62}$$

We write

$$\begin{aligned} |g+h|^p(g+h)-|g|^pg &= |g+h|^p h &(4.63)\\ &\quad + (|g+h|^p-|g|^p)g &(4.64)\end{aligned}$$

and first restrict our attention to (4.63). We proceed as before, working at a single frequency and exploiting cancellation to write

$$\begin{aligned} |P_N(|g+h|^ph)(x)| &= \left|\int N^d\check\psi(Ny)\delta_y\big[|g(x)+h(x)|^ph(x)\big]\,dy\right| \\ &\leq \int N^d|\check\psi(Ny)|\,|g(x-y)+h(x-y)|^p\,|\delta_y h(x)|\,dy &(4.65)\\ &\quad + \int N^d|\check\psi(Ny)|\,|\delta_y\big[|g(x)+h(x)|^p\big]\,|h(x)|\,dy. &(4.66)\end{aligned}$$

We will deal only with (4.65), which is the more difficult term. Indeed, in all of the terms that stem from (4.66), we will have a copy of $e^{it\Delta}w_n^J$ appearing without derivatives, so that (4.13) will suffice. (For completeness, we will later show how to handle such a term; cf. (4.78) below.)

Proceeding as in (4.49), we write

$$(4.65) \lesssim \int N^d|\check{\psi}(Ny)|\,|g(x-y)+h(x-y)|^p\,|h_{>N}(x)|\,dy \tag{4.67}$$

$$+\int N^d|\check{\psi}(Ny)|\,|g(x-y)+h(x-y)|^p\,|h_{>N}(x-y)|\,dy \tag{4.68}$$

$$+\sum_{K\leq N}\int N^d|\check{\psi}(Ny)|\,|g(x-y)+h(x-y)|^p\,|\delta_y h_K(x)|\,dy. \tag{4.69}$$

Let us now deal only with (4.69); in doing so, we will see all of the ideas necessary to handle (4.67) and (4.68), as well. We first write

$$(4.69) \lesssim \sum_{K\leq N}\int N^d|\check{\psi}(Ny)|\,|g(x-y)|^p|\delta_y h_K(x)|\,dy \tag{4.70}$$

$$+\sum_{K\leq N}\int N^d|\check{\psi}(Ny)|\,|h(x-y)|^p|\delta_y h_K(x)|\,dy. \tag{4.71}$$

We only consider (4.70), as the contribution of (4.71) is easier to estimate (again, due to the presence of $e^{it\Delta}w_n^J$ without derivatives). Employing the inequalities of Lemma 2.2.8, we find

$$(4.70) \lesssim \sum_{K\leq N}\tfrac{K}{N}M(|g|^p)(x)M(h_K)(x)+\sum_{K\leq N}\tfrac{K}{N}M(|g|^pM(h_K))(x).$$

Let us now concern ourselves only with the first term above, as the second is similar. As before, to estimate the contribution of this term to (4.62) (and thereby complete our treatment of (4.63)), we need to sum over $N\in 2^{\mathbb{Z}}$. Using (4.59) and recalling the definitions of g and h, we write

$$\begin{aligned}
&\|\big(\sum_N \big|N^{s_c}\sum_{K\leq N}\tfrac{K}{N}M(|g|^p)M(h_K)\big|^2\big)^{1/2}\|_{\dot N^0}\\
&\quad\lesssim \|\big(\sum_N \big|N^{s_c}M(h_N)\big|^2\big)^{1/2}M(|g|^p)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}\\
&\quad\lesssim \|\big(\sum_N \big|N^{s_c}M(P_Ne^{it\Delta}w_n^J)|^2\big)^{1/2}M\big(\big|\sum_{j=1}^J v_n^j\big|^p\big)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}.
\end{aligned}$$

Thus, to complete our treatment of (4.63), we are left to show

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\big(\sum_N \big|N^{s_c}M(P_Ne^{it\Delta}w_n^J)|^2\big)^{1/2}M\big(\big|\sum_{j=1}^J v_n^j\big|^p\big)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{d+2}{2}}=0. \tag{4.72}$$

To begin, we let $\eta > 0$; then using (4.21), we see that there exists some $J_1 = J_1(\eta)$ so that

$$\sum_{j \geq J_1} \|v_n^j\|_{L_{t,x}^{\frac{p(d+2)}{2}}}^{\frac{p(d+2)}{2}} < \eta.$$

Using Hölder's inequality, maximal function and vector maximal function estimates, and Lemma 2.2.2, we can argue as we did to obtain (4.24) to see

$$\begin{aligned}
&\limsup_{n\to\infty} \|\big(\sum_N \big|N^{s_c} M(P_N e^{it\Delta} w_n^J)|^2\big)^{1/2} M\big(\big|\sum_{j\geq J_1} v_n^j\big|^p\big)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{d+2}{2}} \\
&\qquad\lesssim \limsup_{n\to\infty} \||\nabla|^{s_c} e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d+2}{2}} \sum_{j\geq J_1} \|v_n^j\|_{L_{t,x}^{\frac{p(d+2)}{2}}}^{\frac{p(d+2)}{2}} \\
&\qquad\lesssim \eta.
\end{aligned}$$

As $\eta > 0$ was arbitrary, we see that to establish (4.72), it will suffice to show

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\big(\sum_N \big|N^{s_c} M(P_N e^{it\Delta} w_n^J)\big|^2\big)^{1/2} M(|v_n^j|^p)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = 0 \tag{4.73}$$

for $1 \leq j < J_1$.

Restricting our attention to a single j and recalling the definition of v_n^j, we change variables and find we need to estimate

$$\|\big(\sum_N \big|(\lambda_n^j)^{\frac{2}{p}} N^{s_c} M P_N \big[e^{i[(\lambda_n^j)^2(t-t_n^j)]\Delta} w_n^J(\lambda_n^j x + x_n^j)\big]\big|^2\big)^{1/2} M(|v^j|^p)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}.$$

We will now carry out some reductions, inspired by the proof of [34, Proposition 3.4]: as $M(|v^j|^p)$ shares bounds with $|v^j|^p$, and v^j obeys good bounds (it has scattering size $\lesssim E_c$), we may replace $M(|v^j|^p)$ with some function Φ in $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$. If we then use Hölder's inequality, we find it suffices to estimate the first term in $L_{t,x}^2(K)$, where K is the (compact) support of this function Φ. The next step will be to use a local smoothing estimate on this (fixed) set K. Now, the norms that will appear in these estimates will have critical scaling; that is, they will be invariant under the change of variables that eliminates the parameters λ_n^j, x_n^j, and t_n^j. Thus, without loss of generality, we will ignore them from the start.

To establish (4.73) and complete our treatment of (4.63), we are therefore left to show

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\big(\sum_N \big|M(N^{s_c} P_N e^{it\Delta} w_n^J)\big|^2\big)^{1/2}\|_{L_{t,x}^2(K)} = 0 \tag{4.74}$$

for a fixed compact set $K \subset \mathbb{R} \times \mathbb{R}^d$.

To establish (4.74), we will need to rely on the fact that we are working on a compact set, so that we can carry out a local smoothing argument. Indeed, the term appearing above is morally like $|\nabla|^{s_c} e^{it\Delta} w_n^J$, over which we do not have sufficient control (cf. (4.16)). However, we do have good control over $e^{it\Delta} w_n^J$, in the form of (4.13). Thus, to succeed, we need to find a way to estimate the term above using fewer than s_c derivatives; this is exactly the role of local smoothing.

For the proof of (4.74), we will use a standard local smoothing result for the free propagator (Lemma 2.3.3), along with a few results from [57, Chapter V]. In particular, we need the following: if we choose $\varepsilon > 0$ so that $-d < -1 - \varepsilon$, then $|x|^{-1-\varepsilon}$ is an A_2 weight, so that M is bounded on $L^2(|x|^{-1-\varepsilon}\,dx)$.

Proof of (4.74) We can write $K \subset [-T, T] \times \{|x| \leq R\}$ for some $T, R > 0$. We fix some $N_0 \in 2^{\mathbb{Z}}$ and break into low and high frequencies:

$$\begin{aligned}\iint_K \sum_N \left|N^{s_c} M(P_N e^{it\Delta} w_n^J)\right|^2 dx\,dt \lesssim &\sum_{N \leq N_0} \iint_K \left|M(N^{s_c} P_N e^{it\Delta} w_n^J)\right|^2 dx\,dt \\ &+ \sum_{N > N_0} \iint_K \left|M(N^{s_c} P_N e^{it\Delta} w_n^J)\right|^2 dx\,dt.\end{aligned}$$

For the low frequencies, we use Hölder and maximal function estimates to write

$$\begin{aligned}&\sum_{N \leq N_0} \iint_K \left|M(N^{s_c} P_N e^{it\Delta} w_n^J)\right|^2 dx\,dt \\ &\qquad \lesssim \sum_{N \leq N_0} T^{\frac{p(d+2)-4}{p(d+2)}} R^{\frac{d(p(d+2)-4)}{p(d+2)}} \|M(N^{s_c} P_N e^{it\Delta} w_n^J)\|^2_{L_{t,x}^{\frac{p(d+2)}{2}}} \\ &\qquad \lesssim_K \sum_{N \leq N_0} N^{2s_c} \|e^{it\Delta} w_n^J\|^2_{L_{t,x}^{\frac{p(d+2)}{2}}} \\ &\qquad \lesssim_K N_0^{2s_c} \|e^{it\Delta} w_n^J\|^2_{L_{t,x}^{\frac{p(d+2)}{2}}}.\end{aligned}$$

For the high frequencies, we choose $\varepsilon > 0$ so that $-d < -1 - \varepsilon$. Then, using Lemma 2.3.3, Bernstein, and the fact that $|x|^{-1-\varepsilon} \in A_2$, we can estimate

$$\begin{aligned}
\sum_{N>N_0} \iint_K & \big|M(N^{s_c} P_N e^{it\Delta} w_n^J)\big|^2 \, dx \, dt \\
&\lesssim R^{1+\varepsilon} \sum_{N>N_0} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \big|M(N^{s_c} P_N e^{it\Delta} w_n^J)|^2 \langle x\rangle^{-1-\varepsilon} \, dx \, dt \\
&\lesssim_K \sum_{N>N_0} N^{2s_c} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \big|P_N e^{it\Delta} w_n^J\big|^2 \langle x\rangle^{-1-\varepsilon} \, dx \, dt \\
&\lesssim_K \sum_{N>N_0} N^{2s_c} \| |\nabla|^{-\frac{1}{2}} P_N w_n^J \|_{L_x^2(\mathbb{R}^d)}^2 \\
&\lesssim_K \sum_{N>N_0} N^{-1} \| |\nabla|^{s_c} w_n^J \|_{L_x^2(\mathbb{R}^d)}^2 \\
&\lesssim_K N_0^{-1} \| |\nabla|^{s_c} w_n^J \|_{L_x^2(\mathbb{R}^d)}^2.
\end{aligned}$$

Optimizing in the choice of N_0 now yields

$$\| \big(\sum_N \big|M(N^{s_c} P_N e^{it\Delta} w_n^J)\big|^2 \big)^{1/2} \|_{L_{t,x}^2(K)} \lesssim_K \| e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{p(d+2)}{2}}}^{\frac{1}{2s_c+1}} \| w_n^J \|_{\dot H_x^{s_c}(\mathbb{R}^d)}^{\frac{2s_c}{2s_c+1}},$$

which, by (4.13), gives (4.74). ∎

We have now dealt with (4.63), and so we finally turn to (4.64). As usual, we first restrict our attention to a single frequency N. We have dealt with a term of this form before (cf. (4.40)); proceeding in exactly the same way, we arrive at

$$\begin{aligned}
&\big|P_N\big([|g(x)+h(x)|^p - |g(x)|^p] g(x)\big)\big| \\
&\quad \lesssim \int N^d |\check\psi(Ny)| \, |\delta_y g(x)| \, |h(x-y)| \big\{ |g(x-y)|^{p-1} + |h(x-y)|^{p-1} \big\} \, dy && (4.75)\\
&\quad + \int N^d |\check\psi(Ny)| \, |g(x)| \, |\delta_y g(x)| \, |h(x-y)|^{p-1} \, dy && (4.76)\\
&\quad + \int N^d |\check\psi(Ny)| \, |g(x)| \, |\delta_y h(x)| \, \big\{ |g(x)|^{p-1} + |h(x)|^{p-1} + |h(x-y)|^{p-1} \big\} \, dy, && (4.77)
\end{aligned}$$

at least in the case $p \le 2$ (as above, we will only consider this case).

Note that all of the terms above are similar to terms we have handled before. Thus, we proceed in the same way, decomposing terms exactly as before. Whenever a term includes a copy of $e^{it\Delta} w_n^J$ without derivatives, things will be relatively straightforward, as one can rely on (4.13) (see (4.78) below for details); for the one term stemming from (4.77) in which $e^{it\Delta}w_n^J$ only appears with derivatives, we have to go through the same local smoothing argument given above (cf. the proof of (4.74)).

To conclude the proof of (4.37), we will see how to estimate the contribution of the term

$$\int N^d|\check{\psi}(Ny)|\,|\delta_y g(x)|\,|h(x-y)|\,|g(x-y)|^{p-1}\,dy. \tag{4.78}$$

Estimating $|\delta_y g(x)|$ as before, we find we need to bound the terms

$$\begin{aligned}&M(h|g|^{p-1})g_{>N} + M(h|g|^{p-1}g_{>N})\\ &\qquad+\sum_{K\le N}\tfrac{K}{N}M(h|g|^{p-1})M(g_K)+\sum_{K\le N}M(h|g|^{p-1}M(g_K)).\end{aligned}$$

Let us now see how to handle the contribution of the first term only, as the other three are similar. We begin by summing over $N\in 2^{\mathbb{Z}}$ and recalling the definitions of g and h; then, using Hölder, maximal function estimates, and Lemma 2.2.2, we can argue as we did to obtain (4.24) to see

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\|\big(\sum_N\big|N^{s_c}g_{>N}\big|^2\big)^{1/2}M(h|g|^{p-1})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}^{\frac{p(d+2)}{2(p-1)}}\\ &\lesssim\lim_{J\to\infty}\limsup_{n\to\infty}\||\nabla|^{s_c}\big(\sum_{j=1}^J v_n^j\big)\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{p(d+2)}{2(p-1)}}\|e^{it\Delta}w_n^J\|_{L_{t,x}^{\frac{p(d+2)}{2}}}^{\frac{p(d+2)}{2(p-1)}}\sum_{j=1}^J\|v_n^j\|_{L_{t,x}^{\frac{p(d+2)}{2}}}^{\frac{p(d+2)}{2}}.\end{aligned} \tag{4.79}$$

We turn to estimating the first term above. We first write

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\||\nabla|^{s_c}\big(\sum_{j=1}^J v_n^j\big)\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^2\\ &\lesssim\lim_{J\to\infty}\limsup_{n\to\infty}\left(\sum_{j=1}^J\||\nabla|^{s_c}v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^2+\sum_{j\ne k}\||\nabla|^{s_c}v_n^j|\nabla|^{s_c}v_n^k\|_{L_{t,x}^{\frac{d+2}{d}}}\right).\end{aligned} \tag{4.80}$$

Arguing as we did to obtain (4.26), we immediately get that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j\ne k}\||\nabla|^{s_c}v_n^j|\nabla|^{s_c}v_n^k\|_{L_{t,x}^{\frac{d+2}{d}}}=0. \tag{4.81}$$

Next, we let $\eta>0$; then, using (4.15), we can find $J(\eta)>0$ so that

$$\sum_{j>J(\eta)}\||\nabla|^{s_c}\phi^j\|_{L_x^2}^2<\eta.$$

Taking η sufficiently small and applying a standard bootstrap argument, we find

$$\sum_{j>J(\eta)} \||\nabla|^{s_c} v_n^j\|^2_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \sum_{j>J(\eta)} \||\nabla|^{s_c}\phi^j\|^2_{L_x^2} \lesssim \eta. \tag{4.82}$$

On the other hand, the fact that each v_n^j has scattering size $\lesssim E_c$ implies

$$\sum_{j=1}^{J(\eta)} \||\nabla|^{s_c} v_n^j\|^2_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim_{E_c} 1. \tag{4.83}$$

Combining (4.81), (4.82), and (4.83), we can continue from (4.80) to see

$$\lim_{J\to\infty} \limsup_{n\to\infty} \||\nabla|^{s_c}\big(\sum_{j=1}^{J} v_n^j\big)\|^2_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim_{E_c} 1.$$

Thus, continuing from (4.79) and using (4.21) and (4.13), we find

$$\lim_{J\to\infty} \limsup_{n\to\infty} \||\nabla|^{s_c}\big(\sum_{j=1}^{J} v_n^j\big)\|^{\frac{p(d+2)}{2(p-1)}}_{L_{t,x}^{\frac{2(d+2)}{d}}} \|e^{it\Delta} w_n^J\|^{\frac{p(d+2)}{2(p-1)}}_{L_{t,x}^{\frac{p(d+2)}{2}}} \sum_{j=1}^{J} \|v_n^j\|^{\frac{p(d+2)}{2}}_{L_{t,x}^{\frac{p(d+2)}{2}}} = 0,$$

as needed. This completes the proof of (4.37). ∎

Having established (4.36) and (4.37), we are now done with the proof of Lemma 4.4.5.

Thus for the cases (1.3), (1.4), and (1.5), we may use Lemmas 4.4.2 and 4.4.5 together with Theorem 3.2.2 to deduce that $S_{[0,\infty)}(u_n) \lesssim_{E_c} 1$ for n large, thus contradicting (4.18). We conclude that there is at least one bad profile, that is, (4.20) holds. Thus claim (i) above holds, which completes the proof of Proposition 4.4.1 and Theorem 1.1.2 in the cases (1.3), (1.4), (1.5). ∎

4.5 Further reductions

In this section, we give some further reductions to the solutions given by Theorem 1.1.2.

First, a rescaling argument as in [37, 39, 65] allows us to restrict attention to almost periodic solutions that do not escape to arbitrarily low (or high) frequencies on half of their maximal lifespan, say $[0, T_{max})$. We will not include the details here, but see [65, Theorem 3.3], for example.

For cases (1.2), (1.3), and (1.4), we restrict to solutions that do not escape to arbitrarily low frequencies. For (1.5), we instead restrict to solutions that do not escape to arbitrarily high frequencies, which we note implies $T_{max} = \infty$ (cf. Corollary 4.1.4). The choice of whether to suppress low or high frequencies is motivated by whether the critical regularity is higher or lower than the scaling of the *a priori* estimates (e.g. Morawetz estimates) we plan to use.

We next recall that using Lemma 4.1.3, we can subdivide the lifespan into characteristic subintervals J_k on which $N(t) \equiv N_k$, with $|J_k| \sim_u N_k^{-2}$.

For the case (1.2), we translate so that $x(0) = 0$. Modifying $x(t)$ by $O(N(t)^{-1})$, we can make $x(t)$ piecewise linear on each J_k, with $|\dot{x}(t)| \sim_u N(t)$ for $t \in I_k^\circ$. Thus we get $|\dot{x}(t)| \sim_u N(t)$ for a.e. $t \in [0, T_{max})$, so that $|x(t)| \lesssim_u \int_0^t N(s)\, ds$ for $t \in [0, T_{max})$.

Finally, for the cases (1.4) and (1.5), we recall that radial almost periodic solutions have $x(t) \equiv 0$.

Putting all the pieces together, we arrive at the following theorems.

Theorem 4.5.1 *Suppose Theorem 1.1.1 fails and* (d, s_c) *satisfies* (1.2). *Then there exists an almost periodic solution* $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ *to* (1.1) *such that* $S_{[0,T_{max})}(u) = \infty$, $\inf_{t\in[0,T_{max})} N(t) \geq 1$, *and* $|x(t)| \lesssim_u \int_0^t N(s)\, ds$ *for all* $t \in [0, T_{max})$.

Theorem 4.5.2 *Suppose Theorem 1.1.1 fails and* (d, s_c) *satisfies* (1.3). *Then there exists an almost periodic solution* $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ *to* (1.1) *such that* $S_{[0,T_{max})} = \infty$ *and*

$$N(t) \equiv N_k \geq 1 \quad \textit{for } t \in J_k, \quad \textit{with} \quad [0, T_{max}) = \cup J_k \quad \textit{and} \quad |J_k| \sim_u N_k^{-2}.$$

Furthermore, one of the following holds:

$$\int_0^{T_{max}} N(t)^{3-4s_c}\, dt < \infty \qquad \text{(frequency-cascade)}$$

$$\int_0^{T_{max}} N(t)^{3-4s_c}\, dt = \infty \qquad \text{(quasi-soliton).}$$

Theorem 4.5.3 *Suppose Theorem 1.1.1 fails and* (d, s_c) *satisfies* (1.4). *Then there exists an almost periodic solution* $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ *to* (1.1) *such that* $S_{[0,T_{max})} = \infty$, $x(t) \equiv 0$, *and*

$$N(t) \equiv N_k \geq 1 \quad \textit{for } t \in J_k, \quad \textit{with} \quad [0, T_{max}) = \cup J_k \quad \textit{and} \quad |J_k| \sim_u N_k^{-2}.$$

Furthermore, one of the following holds:

$$\int_0^{T_{max}} N(t)^{3-2s_c}\, dt < \infty \qquad \text{(frequency-cascade)}$$
$$\int_0^{T_{max}} N(t)^{3-2s_c}\, dt = \infty \qquad \text{(quasi-soliton).}$$

Theorem 4.5.4 *Suppose Theorem 1.1.1 fails and* (d, s_c) *satisfies* (1.5). *Then there exists an almost periodic solution* $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ *to* (1.1) *such that* $S_{[0,\infty)} = \infty$, $x(t) \equiv 0$, *and*

$$N(t) \equiv N_k \leq 1 \quad \textit{for } t \in J_k, \quad \textit{with} \quad [0, T_{max}) = \cup J_k \quad \textit{and} \quad |J_k| \sim_u N_k^{-2}.$$

Furthermore, one of the following holds:

$$\int_0^{\infty} N(t)^{3-2s_c}\, dt < \infty \qquad \text{(frequency-cascade)}$$
$$\int_0^{\infty} N(t)^{3-2s_c}\, dt = \infty \qquad \text{(quasi-soliton).}$$

CHAPTER 5

The $\dot{H}^{1/2}$-critical case

In this chapter, we treat the case (1.2). In particular, we preclude the existence of almost periodic solutions as in Theorem 4.5.1. We break into two cases, namely $T_{max} < \infty$ and $T_{max} = \infty$. As we will see, the fact that the Lin–Strauss Morawetz has critical scaling for the $\dot{H}_x^{1/2}$-critical problem makes it a very effective tool in this setting.

The results in this chapter appeared originally in [50].

5.1 Finite-time blowup

In this section, we use Proposition 4.1.6, Strichartz estimates, and conservation of mass to preclude the existence of almost periodic solutions as in Theorem 4.5.1 with $T_{max} < \infty$.

Theorem 5.1.1 *There are no almost periodic solutions $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) as in Theorem 4.5.1 with $T_{max} < \infty$.*

Proof. Suppose that u were such a solution. Then for $t \in [0, T_{max})$ and $N > 0$, Proposition 4.1.6, Strichartz, Hölder, Bernstein, and Sobolev embedding give

$$\begin{aligned}
\|P_N u(t)\|_{L_x^2} &\lesssim \|P_N(|u|^{\frac{4}{d-1}} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([t,T_{max})\times\mathbb{R}^d)} \\
&\lesssim (T_{max} - t)^{1/2} N^{1/2} \||u|^{\frac{4}{d-1}} u\|_{L_t^\infty L_x^{\frac{2d}{d+3}}} \\
&\lesssim (T_{max} - t)^{1/2} N^{1/2} \|u\|_{L_t^\infty \dot{H}_x^{1/2}}^{\frac{d+3}{d-1}}.
\end{aligned}$$

As $u \in L_t^\infty \dot{H}_x^{1/2}$, we deduce

$$\|P_{\leq N} u(t)\|_{L_x^2} \lesssim_u (T_{max} - t)^{1/2} N^{1/2} \quad \text{for all } t \in I \text{ and } N > 0. \tag{5.1}$$

On the other hand, an application of Bernstein gives

$$\|P_{>N}u\|_{L_t^\infty L_x^2} \lesssim N^{-1/2}\|u\|_{L_t^\infty \dot{H}_x^{1/2}} \lesssim_u N^{-1/2} \quad \text{for all } N > 0. \tag{5.2}$$

We now let $\eta > 0$. We choose N large enough that $N^{-1/2} < \eta$, and subsequently choose t close enough to T_{max} that $(T_{max} - t)^{1/2}N^{1/2} < \eta$. Combining (5.1) and (5.2), we then get $\|u(t)\|_{L_x^2} \lesssim_u \eta$.

As η was arbitrary and mass is conserved, we conclude $\|u(t)\|_{L_x^2} = 0$ for all $t \in [0, T_{max})$. Thus $u \equiv 0$, which contradicts the fact that u blows up. ∎

5.2 The Lin–Strauss Morawetz inequality

In this section, we use the Lin–Strauss Morawetz inequality to preclude the existence of almost periodic solutions as in Theorem 4.5.1 such that $T_{max} = \infty$.

Proposition 5.2.1 (Lin–Strauss Morawetz inequality, [46]) *Let $d \geq 3$ and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to $(i\partial_t + \Delta)u = |u|^p u$. Then*

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^{p+2}}{|x|}\, dx\, dt \lesssim \|u\|^2_{L_t^\infty \dot{H}_x^{1/2}(I\times\mathbb{R}^d)}. \tag{5.3}$$

As in [32], we will use this estimate to establish the following

Theorem 5.2.2 *There are no almost periodic solutions $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) as in Theorem 4.5.1.*

Proof. Suppose u were such a solution. In particular, u is nonzero, so that by almost periodicity and Sobolev embedding we may find $C(u) > 0$ such that

$$\int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^{\frac{2d}{d-1}}\, dx \gtrsim_u 1 \quad \text{uniformly for } t \in [0,\infty).$$

Applying Hölder and rearranging, this implies

$$\int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u(t,x)|^{\frac{2(d+1)}{d-1}}\, dx \gtrsim_u N(t) \quad \text{uniformly for } t \in [0,\infty). \tag{5.4}$$

We now let $T > 1$ and use $u \in L_t^\infty \dot{H}_x^{1/2}$, (5.3), and (5.4) to see

$$1 \gtrsim_u \int_1^T \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} \frac{|u(t,x)|^{\frac{2(d+1)}{d-1}}}{|x|}\, dx\, dt \gtrsim_u \int_1^T \frac{N(t)}{|x(t)| + N(t)^{-1}}\, dt.$$

As $\inf_{t\in[1,\infty)} N(t) \geq 1$, to derive a contradiction it will suffice to show that

$$\lim_{T\to\infty} \int_1^T \frac{N(t)}{1 + |x(t)|}\, dt = \infty. \tag{5.5}$$

Recalling that $|x(t)| \lesssim_u \int_0^t N(s)\, ds$ for all $t \geq 0$, we get

$$\int_1^T \frac{N(t)}{1+|x(t)|}\, dt \gtrsim_u \int_1^T \frac{d}{dt} \log\left(1 + \int_0^t N(s)\, ds\right) dt \gtrsim_u \log\left(\frac{1 + \int_0^T N(s)\, ds}{1 + \int_0^1 N(s)\, ds}\right).$$

As $\inf_{t\in[1,\infty)} N(t) \geq 1$, we conclude that (5.5) holds, as needed. ∎

CHAPTER 6

Long-time Strichartz estimates

In this chapter, we develop long-time Strichartz estimates for almost periodic solutions to (1.1). Such estimates were first developed by Dodson [18] in the study of the mass-critical NLS. They have since appeared in the energy-critical setting [43, 70], the energy-supercritical setting [48], and the intercritical setting [49, 51].

The results in this chapter appeared originally in [49, 51].

6.1 Estimates adapted to the Lin–Strauss Morawetz, $s_c < 1/2$

In this section, we establish a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz inequality for almost periodic solutions as in Theorem 4.5.4. In particular, we assume (d, s_c) satisfies (1.5), that is, $d = 3$ and $0 < s_c < 1/2$. A key ingredient in the proof is the bilinear Strichartz estimate, Corollary 4.1.7.

We work under the assumption

$$u \in L_t^\infty \dot{H}_x^s([0,\infty) \times \mathbb{R}^3) \tag{6.1}$$

for some $s \geq s_c$. We know from (4.1) that (6.1) holds for $s = s_c$. In Chapter 7 we will show that in the case of (1.5), rapid frequency-cascade solutions actually satisfy (6.1) for $s > s_c$.

Throughout this section, we make use of the following notation for almost periodic solutions to (1.1) as in Theorem 4.5.4.

$$A_I(N) := \|u_{>N}\|_{L_t^2 L_x^6(I \times \mathbb{R}^d)}, \tag{6.2}$$

$$K_I := \int_I N(t)^{3-2s_c}\, dt \sim_u \sum_{J_k \subset I} N_k^{1-2s_c}. \tag{6.3}$$

The main result of this section is the following.

Proposition 6.1.1 (Long-time Strichartz estimate) *Let $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.4. Let $I \subset [0,\infty)$ be a compact time interval, which is a contiguous union of characteristic subintervals J_k. Suppose* (6.1) *holds for some $s_c \leq s < 3/2 + s_c$. Then for any $N > 0$, we have*

$$A_I(N) \lesssim_u N^{-s_c} + N^{-\sigma(s)} K_I^{1/2}, \tag{6.4}$$

where $\sigma(s) := 1/2 + s - s_c$.

In particular, using (4.1)*, we have*

$$A_I(N) \lesssim_u N^{-s_c} + N^{-1/2} K_I^{1/2}. \tag{6.5}$$

Moreover, for any $\varepsilon > 0$, there exists $N_0(\varepsilon) > 0$ so that for $N \geq N_0$,

$$A_I(N) \lesssim_u \varepsilon\big(N^{-s_c} + N^{-1/2} K_I^{1/2}\big). \tag{6.6}$$

We will prove Proposition 6.1.1 by induction. The inductive step will rely on the following lemma.

Lemma 6.1.2 *Let $\eta > 0$ and u, I, s, σ be as above. For any $N > 0$, we have*

$$\begin{aligned}\|P_{>N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}(I\times\mathbb{R}^d)} \lesssim_u C_\eta \sup_{J_k \subset I} \|u_{>\eta N}\|^{2s_c}_{L_t^\infty \dot{H}_x^s(J_k\times\mathbb{R}^3)} N^{-\sigma(s)} K_I^{1/2} \\ + \sum_{M \leq \eta N} (\tfrac{M}{N})^2 A_I(M),\end{aligned}$$

Proof of Lemma 6.1.2. Throughout the proof, all spacetime norms will be taken over $I \times \mathbb{R}^3$ unless stated otherwise.

We begin by writing $F(u) = F(u_{\leq \eta N}) + F(u) - F(u_{\leq \eta N})$. We use Bernstein, the chain rule, Sobolev embedding, and (4.1) estimate

$$\begin{aligned}\|P_{>N} F(u_{\leq \eta N})\|_{L_t^2 L_x^{6/5}} &\lesssim N^{-2} \|\Delta F(u_{\leq \eta N})\|_{L_t^2 L_x^{6/5}} \\ &\lesssim N^{-2} \||\nabla|^{s_c} u\|^p_{L_t^\infty L_x^2} \sum_{M \leq \eta N} \|\Delta u_M\|_{L_t^2 L_x^6} \\ &\lesssim_u \sum_{M \leq \eta N} (\tfrac{M}{N})^2 A_I(M).\end{aligned} \tag{6.7}$$

Next, we use almost periodicity to choose $C(\eta)$ large enough that

$$\| |\nabla|^{s_c} u_{>C(\eta)N(t)} \|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} < \eta^2. \tag{6.8}$$

By almost periodicity and the embedding $\dot{H}_x^{s_c} \hookrightarrow L_x^{3p/2}$ we may choose $C(\eta)$ possibly even larger to guarantee

$$\|(1-\chi_{\frac{C(\eta)}{N(t)}})u_{\leq C(\eta)N(t)}\|_{L_t^\infty L_x^{3p/2}(I\times\mathbb{R}^d)} < \eta^2, \tag{6.9}$$

where χ_R denotes the characteristic function of $\{|x| \leq R\}$.

We now write

$$F(u) - F(u_{\leq \eta N}) \lesssim u_{>\eta N} \O\big\{(u_{\leq C(\eta)N(t)})^p + (u_{>C(\eta)N(t)})^p\big\},$$

so that

$$\begin{aligned}
\|P_{>N}\big(F(u) - F(u_{\leq\eta N})\big)\|_{L_t^2 L_x^{6/5}} &\lesssim \|u_{>\eta N}(u_{>C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}} &(6.10)\\
&\quad + \|(1-\chi_{\frac{C(\eta)}{N(t)}})u_{>\eta N}(u_{\leq C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}} &(6.11)\\
&\quad + \|\chi_{\frac{C(\eta)}{N(t)}} u_{>\eta N}(u_{\leq C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}}. &(6.12)
\end{aligned}$$

Using Hölder, (4.1), and (6.8), we estimate the contribution of (6.10) as follows:

$$\|u_{>\eta N}(u_{>C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}} \lesssim \|u_{>\eta N}\|_{L_t^2 L_x^6} \|u_{>C(\eta)N(t)}\|_{L_t^\infty L_x^{3p/2}}^p \lesssim \eta^2 A_I(\eta N). \tag{6.13}$$

Similarly, we estimate the contribution of (6.11) as follows:

$$\begin{aligned}
&\|(1-\chi_{\frac{C(\eta)}{N(t)}})u_{>\eta N}(u_{\leq C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}}\\
&\quad\lesssim \|(1-\chi_{\frac{C(\eta)}{N(t)}})u_{\leq C(\eta)N(t)}\|_{L_t^\infty L_x^{3p/2}} \|u\|_{L_t^\infty L_x^{3p/2}}^{p-1} \|u_{>\eta N}\|_{L_t^2 L_x^6}\\
&\quad\lesssim_u \eta^2 A_I(\eta N).
\end{aligned} \tag{6.14}$$

Finally, we estimate the contribution of (6.12). We first restrict our attention to a single characteristic subinterval J_k. We define the following exponents:

$$q = \tfrac{2(p^2+2p-4)}{3p-4}, \quad r_0 = \tfrac{3p(p^2+2p-4)}{8p-p^2-8}, \quad r = \tfrac{6(p^2+2p-4)}{3p^2-4}.$$

Note that as $4/3 < p < 2$, we have $4 < q < \infty$, $2 < r_0 < 6$, and $2 < r < 3$. We also note that we have the embedding $\dot{H}_x^{s_c,r} \hookrightarrow L_x^{r_0}$ and that (q,r) is an admissible pair.

With all spacetime norms over $J_k \times \mathbb{R}^3$, we use Hölder, the bilinear Strichartz estimate (Corollary 4.1.7), Sobolev embedding, Lemma 4.1.5, (4.1), and (6.1) to estimate

$$\begin{aligned}
&\|\chi_{\frac{C(\eta)}{N_k}} u_{>\eta N}(u_{\leq C(\eta)N_k})^p\|_{L_t^2 L_x^{6/5}} \\
&\lesssim \|\chi_{\frac{C(\eta)}{N_k}}\|_{L_x^{\frac{6}{(1-2s_c)^2}}} \|u_{>\eta N} u_{\leq C(\eta)N_k}\|_{L_{t,x}^2}^{1-2s_c} \|u_{>\eta N}\|_{L_t^\infty L_x^2}^{2s_c} \|u_{\leq C(\eta)N_k}\|_{L_t^q L_x^{r_0}}^{p-1+2s_c} \\
&\lesssim_u C_\eta N_k^{-(1-2s_c)^2/2} \big[N_k^{1-s_c} N^{-1/2-s}\big]^{1-2s_c} N^{-2s\cdot s_c} \|u_{>\eta N}\|_{L_t^\infty \dot{H}_x^s}^{2s_c} \||\nabla|^{s_c} u\|_{L_t^q L_x^r}^{p-1+2s_c} \\
&\lesssim_u C_\eta N_k^{1/2-s_c} N^{-(1/2+s-s_c)} \|u_{>\eta N}\|_{L_t^\infty \dot{H}_x^s}^{2s_c}.
\end{aligned}$$

Summing over $J_k \subset I$ and using (6.3), we find

$$\|\chi_{\frac{C(\eta)}{N(t)}} u_{>\eta N}(u_{\leq C(\eta)N(t)})^p\|_{L_t^2 L_x^{6/5}} \lesssim_u C_\eta \sup_{J_k \subset I} \|u_{>\eta N}\|_{L_t^\infty \dot{H}_x^s(J_k \times \mathbb{R}^3)}^{2s_c} N^{-\sigma(s)} K_I^{1/2}. \tag{6.15}$$

We may now add the estimates (6.7), (6.13), (6.14), and (6.15) to complete the proof of Lemma 6.1.2. ∎

We turn to the proof of Proposition 6.1.1.

Proof of Proposition 6.1.1 We proceed by induction. For the base case, we let $N \leq \inf_{t\in I} N(t) \leq 1$, so that $N^{-2(s-s_c)}\big(\frac{N(t)}{N}\big)^{1-2s_c} \geq 1$ for $t \in I$. We use Bernstein and Lemma 4.1.5 to estimate

$$\begin{aligned}
A_I(N)^2 &\lesssim N^{-2s_c} \||\nabla|^{s_c} u_{>N}\|_{L_t^2 L_x^6(I\times\mathbb{R}^d)}^2 \lesssim_u N^{-2s_c} + N^{-2s_c} \int_I N(t)^2\,dt \\
&\lesssim_u N^{-2s_c} + N^{-1-2(s-s_c)} K_I.
\end{aligned}$$

Thus for $N \leq \inf_{t\in I} N(t)$, we have

$$A_I(N) \leq C_u\big[N^{-s_c} + N^{-\sigma(s)} K_I^{1/2}\big]. \tag{6.16}$$

This inequality remains true if we replace C_u by any larger constant.

We now suppose that (6.16) holds at frequencies $\leq N/2$; we will use Lemma 6.1.2 to show that it holds at frequency N.

Applying Strichartz, Bernstein, Lemma 6.1.2, (4.1), and (6.1), we find

$$\begin{aligned} A_I(N) &\leq \tilde{C}_u\big[N^{-s_c}\inf_{t\in I}\|u_{>N}(t)\|_{\dot{H}_x^{s_c}} + C_\eta \sup_{J_k\subset I}\|u_{>\eta N}\|^{2s_c}_{L_t^\infty \dot{H}_x^s(J_k\times\mathbb{R}^3)} N^{-\sigma(s)}K_I^{1/2} \\ &\qquad + \sum_{M\leq \eta N}(\tfrac{M}{N})^2 A_I(M)\big] \qquad (6.17) \\ &\leq \tilde{C}_u\big[N^{-s_c} + C_\eta N^{-\sigma(s)}K_I^{1/2} + \sum_{M\leq\eta N}(\tfrac{M}{N})^2 A_I(M)\big]. \end{aligned}$$

We now let $\eta < 1/2$ and note that $s < 3/2 + s_c$ gives $\sigma(s) < 2$. Thus, using the inductive hypothesis, we find

$$\begin{aligned} A_I(N) &\leq \tilde{C}_u\big[N^{-s_c} + C_\eta N^{-\sigma(s)}K_I^{1/2} + \sum_{M\leq\eta N}(\tfrac{M}{N})^2(C_u M^{-s_c} + C_u M^{-\sigma(s)}K_I^{1/2})\big] \\ &\leq \tilde{C}_u\big[N^{-s_c} + C_\eta N^{-\sigma(s)}K_I^{1/2}\big] + C_u\tilde{C}_u\big[\eta^{2-s_c}N^{-s_c} + \eta^{2-\sigma(s)}N^{-\sigma(s)}K_I^{1/2}\big]. \end{aligned}$$

If we now choose η sufficiently small depending on $\tilde{C}_u$, we get

$$A_I(N) \leq \tilde{C}_u(N^{-s_c} + C_\eta N^{-\sigma(s)}K_I^{1/2}) + \tfrac{1}{2}C_u(N^{-s_c} + N^{-\sigma(s)}K_I^{1/2}).$$

Finally, if we choose C_u possibly larger so that $C_u \geq 2(1+C_\eta)\tilde{C}_u$, then the above inequality implies

$$A_I(N) \leq C_u(N^{-s_c} + N^{-\sigma(s)}K_I^{1/2}),$$

as was needed to show. This completes the proof of (6.4).

The estimate (6.5) follows directly from (6.4) with $s = s_c$. With (6.5) in place, we can prove (6.6) by continuing from (6.17), choosing η sufficiently small, and noting that $\sup_{t\in I} N(t) \leq 1$ implies

$$\lim_{N\to\infty}\big[\inf_{t\in I}\|u_{>N}(t)\|_{\dot{H}_x^{s_c}} + \sup_{J_k\subset I}\|u_{>\eta N}\|^{2s_c}_{L_t^\infty \dot{H}_x^{s_c}(J_k\times\mathbb{R}^3)}\big] = 0.$$

for any $\eta > 0$. This completes the proof of Proposition 6.1.1. ∎

6.2 Estimates adapted to the Lin–Strauss Morawetz, $s_c > 1/2$

In this section, we prove a long-time Strichartz estimate adapted to the Lin–Strauss Morawetz inequality. We will work under the assumption

$$u \in L_t^\infty \dot{H}_x^s([0, T_{max}) \times \mathbb{R}^3) \tag{6.18}$$

for some $s \leq s_c$. We have from (4.1) that (6.18) holds for $s = s_c$. In Chapter 7, we will show that in the case of (1.4), rapid frequency-cascade solutions actually satisfy (6.18) for $s < s_c$.

Throughout this section, we use the following notation for almost periodic solutions to (1.1) as in Theorem 4.5.4:

$$A_I(N) := \| |\nabla|^{s_c} u_{\leq N} \|_{L_t^2 L_x^6 (I \times \mathbb{R}^d)}, \tag{6.19}$$

$$K_I := \int_I N(t)^{3-2s_c}\, dt \sim_u \sum_{J_k \subset I} N_k^{1-2s_c}. \tag{6.20}$$

The main result of this section is the following.

Proposition 6.2.1 (Long-time Strichartz estimate) *Let $u : [0, T_{max}) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.3. Let $I \subset [0, T_{max})$ be a compact time interval, which is a contiguous union of characteristic subintervals J_k. Suppose* (6.18) *holds for some $s_c - 1/2 < s \leq s_c$. Then for any $N > 0$, we have*

$$A_I(N) \lesssim_u 1 + N^{\sigma(s)} K_I^{1/2}, \tag{6.21}$$

where $\sigma(s) := 2s_c - s - 1/2$.

In particular, using (4.1)*, we have*

$$A_I(N) \lesssim_u 1 + N^{s_c - 1/2} K_I^{1/2}. \tag{6.22}$$

Moreover, for any $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon) > 0$ so that for any $N \leq N_0$,

$$A_I(N) \lesssim_u \varepsilon(1 + N^{s_c - 1/2} K_I^{1/2}). \tag{6.23}$$

We prove Proposition 6.2.1 by induction. The inductive step relies on the following.

Lemma 6.2.2 *Let $\eta > 0$ and u, I, s, σ be as above. For any $N > 0$, we have*

$$\||\nabla|^{s_c} P_{\leq N}\big(F(u)\big)\|_{L_t^2 L_x^6(I\times\mathbb{R}^d)} \lesssim_u C_\eta \sup_{J_k\subset I} \|u_{\leq N/\eta}\|_{L_t^\infty \dot{H}_x^s(J_k\times\mathbb{R}^3)} N^{\sigma(s)} K_I^{-1/2} + \sum_{M\geq N/\eta} \big(\tfrac{M}{N}\big)^{s_c} A_I(M).$$

Proof. Throughout the proof, all spacetime norms are taken over $I\times\mathbb{R}^3$ unless indicated otherwise.

We fix $0 < \eta < 1$. Using almost periodicity, we may choose $c(\eta)$ sufficiently small so that

$$\||\nabla|^{s_c} u_{\leq c(\eta)N(t)}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} < \eta. \tag{6.24}$$

We decompose the nonlinearity as follows:

$$F(u) = F(u_{\leq N/\eta}) + [F(u) - F(u_{\leq N/\eta})].$$

We first restrict our attention to an individual characteristic subinterval J_k. Using the fractional chain rule, Hölder, the triangle inequality, and Sobolev embedding, we estimate

$$\begin{aligned}
&\||\nabla|^{s_c} P_{\leq N} F(u_{\leq N/\eta})\|_{L_t^2 L_x^{6/5}(J_k\times\mathbb{R}^d)} \\
&\quad\lesssim \|u_{\leq N/\eta}\|^p_{L_t^\infty L_x^{3p/2}(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)} \\
&\quad\lesssim \||\nabla|^{s_c} P_{\leq c(\eta)N_k} u_{\leq N/\eta}\|^p_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)} \\
&\qquad + \||\nabla|^{s_c} P_{> c(\eta)N_k} u_{\leq N/\eta}\|^p_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)}.
\end{aligned}$$

For the first term, we use (6.24) to get

$$\begin{aligned}
&\||\nabla|^{s_c} P_{\leq c(\eta)N_k} u_{\leq N/\eta}\|^p_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)} \\
&\quad\lesssim \eta^{s_c} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)}.
\end{aligned} \tag{6.25}$$

For the next term, we note that we only need to consider the case $c(\eta)N_k < N/\eta$, in which case we have $1 \lesssim C_\eta(\frac{N}{N_k})^{s_c-1/2}$. Using Bernstein, Lemma 4.1.5, and (4.1), we estimate

$$\begin{aligned}
\||\nabla|^{s_c} & P_{>c(\eta)N_k} u_{\leq N/\eta}\|^p_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^2 L_x^6(J_k\times\mathbb{R}^d)} \\
&\lesssim_u C_\eta(\tfrac{N}{N_k})^{s_c-1/2} \||\nabla|^{s_c} u_{\leq N/\eta}\|_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \\
&\lesssim_u C_\eta(\tfrac{N}{N_k})^{s_c-1/2} N^{s_c-s} \|u_{\leq N/\eta}\|_{L_t^\infty \dot H_x^s(J_k\times\mathbb{R}^3)}. \qquad (6.26)
\end{aligned}$$

Summing (6.25) and (6.26) over $J_k \subset I$ and using (6.20), we find

$$\begin{aligned}
\||\nabla|^{s_c} & P_{\leq N} F(u_{\leq N/\eta})\|_{L_t^2 L_x^{6/5}} \\
&\lesssim_u \eta^{s_c} A_I(N/\eta) + C_\eta \sup_{J_k\subset I} \|u_{\leq N/\eta}\|_{L_t^\infty \dot H_x^s(J_k\times\mathbb{R}^3)} N^{\sigma(s)} K_I^{1/2}.
\end{aligned}$$

Next we use Bernstein, Hölder, Sobolev embedding, and (4.1) to estimate

$$\begin{aligned}
\||\nabla|^{s_c} P_{\leq N}\big(F(u) - F(u_{\leq N/\eta})\big)\|_{L_t^2 L_x^{6/5}} &\lesssim N^{s_c} \|u\|^p_{L_t^\infty L_x^{3p/2}} \sum_{M > N/\eta} \|u_M\|_{L_t^2 L_x^6} \\
&\lesssim_u \sum_{M>N/\eta} \big(\tfrac{N}{M}\big)^{s_c} A_I(M).
\end{aligned}$$

Collecting the estimates, we complete the proof of Lemma 6.2.2. ∎

We turn to the proof of Proposition 6.2.1.

Proof of Proposition 6.2.1 We proceed by induction. For the base case, we take $N > \sup_{t\in I} N(t) \geq 1$, so that $N^{2(s_c-s)}\big(\frac{N}{N(t)}\big)^{2s_c-1} \geq 1$ for $t \in I$. Lemma 4.1.5 gives

$$A_I(N)^2 \lesssim_u 1 + \textstyle\int_I N(t)^2\,dt \lesssim_u 1 + N^{2(s_c-s)} N^{2s_c-1} K_I.$$

Thus for $N > \sup_{t\in I} N(t)$, we have

$$A_I(N) \leq C_u\big[1 + N^{\sigma(s)} K_I^{1/2}\big]. \qquad (6.27)$$

This inequality clearly remains true if we replace C_u by any larger constant.

We now suppose that (6.27) holds at frequencies $\geq 2N$; we will use Lemma 6.2.2 to show that it holds at frequency N.

Applying Strichartz, Lemma 6.2.2, (4.1), and (7.12) gives

$$\begin{aligned} A_I(N) &\le \tilde{C}_u\big[\inf_{t\in I}\|u_{\le N}(t)\|_{\dot{H}^{s_c}_x} + C_\eta \sup_{J_k\subset I}\|u_{\le N/\eta}\|_{L^\infty_t \dot{H}^s_x(J_k\times\mathbb{R}^3)} N^{\sigma(s)} K_I^{1/2} \\ &\qquad + \sum_{M\ge N/\eta}\big(\tfrac{N}{M}\big)^{s_c} A_I(M)\big] \qquad (6.28) \\ &\le \tilde{C}_u\big[1 + C_\eta N^{-\sigma(s)}K_I^{1/2} + \sum_{M\ge N/\eta}\big(\tfrac{N}{M}\big)^{s_c} A_I(M)\big]. \end{aligned}$$

We let $\eta < 1/2$ and notice that $s > s_c - 1/2$ guarantees $\sigma(s) < s_c$. Thus, using the inductive hypothesis, we find

$$\begin{aligned} A_I(N) &\le \tilde{C}_u\big[1 + C_\eta N^{\sigma(s)}K_I^{1/2} + \sum_{M\ge N/\eta}\big(\tfrac{N}{M}\big)^{s_c}(C_u + C_u M^{\sigma(s)}K_I^{1/2})\big] \\ &\le \tilde{C}_u\big[1 + C_\eta N^{\sigma(s)}K_I^{1/2}\big] + C_u\tilde{C}_u\big[\eta^{s_c} + \eta^{s_c-\sigma(s)}N^{\sigma(s)}K_I^{1/2}\big] \end{aligned}$$

Choosing η sufficiently small depending on $\tilde{C}_u$, we find

$$A_I(N) \le \tilde{C}_u(1 + C_\eta N^{\sigma(s)}K_I^{1/2}) + \tfrac{1}{2}C_u(1 + N^{\sigma(s)}K_I^{1/2}).$$

Finally, choosing C_u possibly even larger to guarantee $C_u \ge 2(1+C_\eta)\tilde{C}_u$, we deduce from the above inequality that

$$A_I(N) \le C_u(1 + N^{\sigma(s)}K_I^{1/2}),$$

as was needed to show. This completes the proof of (6.21).

The estimate (6.22) follows directly from (6.21) with $s = s_c$. With (6.22) in place, we can prove (6.23) by continuing from (6.28), choosing η sufficiently small, and noting that $\inf_{t\in I} N(t) \ge 1$ implies

$$\lim_{N\to 0}\big[\inf_{t\in I}\|u_{\le N}(t)\|_{\dot{H}^{s_c}_x} + \sup_{J_k\subset I}\|u_{\le N/\eta}\|_{L^\infty_t \dot{H}^{s_c}_x(J_k\times\mathbb{R}^3)}\big] = 0$$

for any $\eta > 0$. This completes the proof of Proposition 6.2.1. $\blacksquare$

6.3 Estimates adapted to the interaction Morawetz

In this section, we prove a long-time Strichartz estimate adapted to the interaction Morawetz inequality for almost periodic solutions as in Theorem 4.5.2. In particular, we assume (d, s_c)

satisfies (1.3). Key ingredients in the proof will be a paraproduct estimate, Lemma 2.2.6, as well as a bilinear Strichartz estimate, Corollary 4.1.7.

Throughout this section, we make use of the following notation for almost periodic solutions to (1.1) as in Theorem 4.5.2.

$$A_I(N) := \||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)},$$

$$K_I := \int_I N(t)^{3-4s_c}\, dt \sim_u \sum_{J_k\subset I} N_k^{1-4s_c}. \tag{6.29}$$

The main result of this section is the following.

Proposition 6.3.1 (Long-time Strichartz estimate) *Let $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.2. Let $I \subset [0, T_{max})$ be a compact time interval, which is a contionguous union of characteristic subintervals J_k. Then for any $N > 0$, we have*

$$A_I(N) \lesssim_u 1 + N^{2s_c-1/2} K_I^{1/2}, \tag{6.30}$$

Moreover, for any $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon)$ such that for all $N \leq N_0$,

$$A_I(N) \lesssim_u \varepsilon(1 + N^{2s_c-1/2} K_I^{1/2}). \tag{6.31}$$

We prove Proposition 6.3.1 by induction. The inductive step relies on the following.

Lemma 6.3.2 *Let $\eta, \eta_0 > 0$. Let u, I as above. There exists $\nu > 0$ so that for any $N > 0$, we have*

$$\begin{aligned}\||\nabla|^{s_c} P_{\leq N}\big(F(u)\big)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \lesssim_u C_{\eta,\eta_0} N^{2s_c-1/2} K_I^{1/2} + \eta^\nu A_I(N/\eta_0)\\ + \sum_{M>N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M).\end{aligned} \tag{6.32}$$

Proof of Lemma 6.3.2. Let $0 < \eta, \eta_0 < 1$. Using almost periodicity, we may choose $c(\eta)$ sufficiently small so that

$$\||\nabla|^{s_c} u_{\leq cN(t)}\|_{L_t^\infty L_x^2} \leq \eta. \tag{6.33}$$

We next decompose the nonlinearity $|u|^p u$ and estimate the resulting pieces. The particular decomposition we choose depends on the ambient dimension.

Case 1. When $d = 3$, we have $2 < p < 4$, and we decompose as follows:

$$\begin{aligned} |u|^p u = {} & (|u|^p + |u|^{p-2}\bar{u}u_{\leq N/\eta_0})u_{>N/\eta_0} \\ & + |u|^{p-2}\bar{u}(P_{>cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0} \\ & + |u|^{p-2}\bar{u}(P_{\leq cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0}. \end{aligned} \tag{6.34}$$

To estimate the contribution of the first term on the right-hand side of (6.34), we let

$$G := |u|^p + |u|^{p-2}\bar{u}u_{\leq N/\eta_0}$$

and use Bernstein, Lemma 2.2.6, and Hölder to estimate

$$\begin{aligned} \||\nabla|^{s_c} P_{\leq N}(Gu_{>N/\eta_0})\|_{L_t^2 L_x^{6/5}} & \lesssim N^{\frac{3}{2}s_c} \||\nabla|^{-\frac{1}{2}s_c}(Gu_{>N/\eta_0})\|_{L_t^2 L_x^{6/5}} \\ & \lesssim N^{\frac{3}{2}s_c} \||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \||\nabla|^{-\frac{1}{2}s_c} u_{>N/\eta_0}\|_{L_t^2 L_x^6} \\ & \lesssim \||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \sum_{M>N/\eta_0} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} A_I(M). \end{aligned} \tag{6.35}$$

To estimate the contribution of the first term above, we first use the fractional chain rule and Sobolev embedding to see

$$\||\nabla|^{\frac{1}{2}s_c}|u|^p\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \lesssim \|u\|_{L_t^\infty L_x^{\frac{3p}{2}}}^{p-1} \||\nabla|^{\frac{1}{2}s_c} u\|_{L_t^\infty L_x^{\frac{12p}{3p+4}}} \lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^p \lesssim_u 1,$$

while by the fractional product rule, the fractional chain rule, and Sobolev embedding we get

$$\begin{aligned} & \||\nabla|^{\frac{1}{2}s_c}(|u|^{p-2}\bar{u}u_{\leq N/\eta_0})\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \\ & \quad \lesssim \|u\|_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{\frac{1}{2}s_c}(|u|^{p-2}\bar{u})\|_{L_t^\infty L_x^{\frac{12p}{11p-12}}} + \|u\|_{L_t^\infty L_x^{\frac{3p}{2}}}^{p-1} \||\nabla|^{\frac{1}{2}s_c} u\|_{L_t^\infty L_x^{\frac{12p}{3p+4}}} \\ & \quad \lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^{\frac{3p}{2}}}^{p-2} \||\nabla|^{\frac{1}{2}s_c} u\|_{L_t^\infty L_x^{\frac{12p}{3p+4}}} + \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^p \\ & \quad \lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^p \lesssim_u 1. \end{aligned}$$

Thus, continuing from (6.35), we see

$$\begin{aligned}\big\||\nabla|^{s_c} P_{\leq N}\big((|u|^p + |u|^{p-2}\bar{u}u_{\leq N/\eta_0})u_{>N/\eta_0}\big)\big\|_{L_t^2 L_x^{6/5}} \\ \lesssim_u \sum_{M>N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M).\end{aligned}\tag{6.36}$$

Next, we turn to estimating the contribution of the second term in (6.34). We begin by restricting our attention to an individual $J_k \times \mathbb{R}^d$. Note that we only need to consider the case $cN_k \leq N/\eta_0$, in which case we have

$$(\tfrac{N}{cN_k})^{s_c} \leq c^{-(2s_c-1/2)}\eta_0^{-(s_c-1/2)}(\tfrac{N}{N_k})^{2s_c-1/2} \leq C_{\eta,\eta_0}(\tfrac{N}{N_k})^{2s_c-1/2}$$

for some positive constant C_{η,η_0}. Hence we can use Bernstein, Hölder, Sobolev embedding, and Lemma 4.1.5 to estimate

$$\begin{aligned}&\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}\|_{L_t^2 L_x^{6/5}} \\ &\qquad\lesssim N^{s_c}\||u|^{p-2}\bar{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}\|_{L_t^2 L_x^{6/5}} \\ &\qquad\lesssim N^{s_c}\|u\|^{p-1}_{L_t^\infty L_x^{\frac{3p}{2}}}\|P_{>cN_k}u_{\leq N/\eta_0}\|_{L_t^4 L_x^3}\|u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} \\ &\qquad\lesssim_u N^{s_c}(cN_k)^{-s_c}\||\nabla|^{s_c}u_{\leq N/\eta_0}\|^2_{L_t^4 L_x^3} \\ &\qquad\lesssim_u C_{\eta,\eta_0}(\tfrac{N}{N_k})^{2s_c-1/2}.\end{aligned}\tag{6.37}$$

Summing the estimates (6.37) over the characteristic subintervals $J_k \subset I$ then gives

$$\begin{aligned}\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}(P_{>cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0})\|_{L_t^2 L_x^{6/5}} \\ \lesssim_u C_{\eta,\eta_0} N^{2s_c-1/2}K_I^{1/2}.\end{aligned}\tag{6.38}$$

Before proceeding to the next term in (6.34), we note that in obtaining estimate (6.37), we could have held onto the term $\||\nabla|^{s_c}u_{\leq N/\eta_0}\|_{L_t^4 L_x^3}$, which (by interpolation) we can estimate by

$$\begin{aligned}\||\nabla|^{s_c}u_{\leq N/\eta_0}\|_{L_t^4 L_x^3} &\lesssim \||\nabla|^{s_c}u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^\infty L_x^2}\||\nabla|^{s_c}u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^2 L_x^6} \\ &\lesssim_u \||\nabla|^{s_c}u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^\infty L_x^2}.\end{aligned}$$

In this case, summing the estimates yields

$$\begin{aligned}\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}(P_{>cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0})\|_{L_t^2 L_x^{6/5}}&\\ \lesssim_u \sup_{J_k\subset I} \||\nabla|^{s_c} u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} C_{\eta,\eta_0} N^{2s_c-\frac{1}{2}} K_I^{1/2}.&\qquad (6.39)\end{aligned}$$

This variant of (6.38) will be important when we eventually need to exhibit smallness in (6.31).

To estimate the contribution of the final term in (6.34), we begin with an application of the fractional product rule and Hölder to see

$$\begin{aligned}&\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}(P_{\leq cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0})\|_{L_t^2 L_x^{6/5}}\\ &\lesssim \||\nabla|^{s_c}(|u|^{p-2}\bar{u})\|_{L_t^\infty L_x^{\frac{6p}{7p-8}}} \|P_{\leq cN(t)}u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} \|u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} &(6.40)\\ &\quad + \|u\|^{p-1}_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c} P_{\leq cN(t)}u_{\leq N/\eta_0}\|_{L_t^4 L_x^3} \|u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} &(6.41)\\ &\quad + \|u\|^{p-1}_{L_t^\infty L_x^{\frac{3p}{2}}} \|P_{\leq cN(t)}u_{\leq N/\eta_0}\|_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c} u_{\leq N/\eta_0}\|_{L_t^2 L_x^6}. &(6.42)\end{aligned}$$

We first note that by the fractional chain rule and Sobolev embedding, we get

$$\||\nabla|^{s_c}(|u|^{p-2}\bar{u})\|_{L_t^\infty L_x^{\frac{6p}{7p-8}}} \lesssim \|u\|^{p-2}_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \lesssim_u 1.$$

Using Sobolev embedding, interpolation, and (6.33), we also see

$$\begin{aligned}\|P_{\leq cN(t)}u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} &\lesssim \||\nabla|^{s_c} P_{\leq cN(t)}u_{\leq N/\eta_0}\|_{L_t^4 L_x^3}\\ &\lesssim \||\nabla|^{s_c} P_{\leq cN(t)}u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^\infty L_x^2} \||\nabla|^{s_c} P_{\leq cN(t)}u_{\leq N/\eta_0}\|^{\frac{1}{2}}_{L_t^2 L_x^6}\\ &\lesssim \eta^{1/2} A_I(N/\eta_0)^{1/2}.\end{aligned}$$

Estimating similarly gives

$$\|u_{\leq N/\eta_0}\|_{L_t^4 L_x^{\frac{6p}{4-p}}} \lesssim_u A_I(N/\eta_0)^{1/2}.$$

Plugging these last three estimates into (6.40), (6.41), and (6.42) and employing a few more instances of Sobolev embedding and (6.33) finally gives

$$\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}(P_{\leq cN(t)}u_{\leq N/\eta_0})u_{\leq N/\eta_0})\|_{L_t^2 L_x^{6/5}} \lesssim_u \eta^{1/2} A_I(N/\eta_0). \qquad (6.43)$$

Collecting the estimates (6.36), (6.38), and (6.43), we see that in the case $d = 3$, we have the estimate

$$\|P_{\leq N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}(I\times\mathbb{R}^d)} \lesssim_u C_{\eta,\eta_0} N^{2s_c-\frac{1}{2}} K_I^{1/2} + \eta^{1/2} A_I(N/\eta_0) + \sum_{M>N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M). \tag{6.44}$$

Comparing (6.44) to (6.32), we see that Lemma 6.3.2 holds for $d = 3$.

Case 2. When $d \in \{4,5\}$, we have $\frac{4}{d-1} < p < \frac{4}{d-2}$. In particular, we have $1 < p < 2$.

Again, we wish to decompose the nonlinearity and estimate each piece. This time, we decompose as follows:

$$\begin{aligned} |u|^p u = |u|^p u_{>N/\eta_0} & \\ + |u_{>cN(t)}|^p P_{\leq cN(t)} u_{\leq N/\eta_0} & \\ + |u_{>cN(t)}|^p P_{>cN(t)} u_{\leq N/\eta_0} & \\ + (|u|^p - |u_{>cN(t)}|^p) u_{\leq N/\eta_0}. & \end{aligned} \tag{6.45}$$

We estimate the contribution of the first term on the right-hand side of (6.45) similarly to the case $d = 3$; in particular, by Bernstein, Hölder, and Lemma 2.2.6, we have

$$\begin{aligned} \||\nabla|^{s_c} & P_{\leq N}(|u|^p u_{>N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ & \lesssim N^{\frac{3}{2}s_c} \||\nabla|^{-\frac{1}{2}s_c}(|u|^p u_{>N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ & \lesssim N^{\frac{3}{2}s_c} \||\nabla|^{\frac{1}{2}s_c}|u|^p\|_{L_t^\infty L_x^{\frac{4dp}{p(d+8)-4}}} \||\nabla|^{-\frac{1}{2}s_c} u_{>N/\eta_0}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\ & \lesssim \||\nabla|^{\frac{1}{2}s_c}|u|^p\|_{L_t^\infty L_x^{\frac{4dp}{p(d+8)-4}}} \sum_{M>N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M). \end{aligned} \tag{6.46}$$

As we can use the fractional chain rule and Sobolev embedding to estimate

$$\||\nabla|^{\frac{1}{2}s_c}|u|^p\|_{L_t^\infty L_x^{\frac{4dp}{p(d+8)-4}}} \lesssim \|u\|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} \||\nabla|^{\frac{1}{2}s_c} u\|_{L_t^\infty L_x^{\frac{4dp}{dp+4}}} \lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^p \lesssim_u 1,$$

we can continue from (6.46) to get

$$\||\nabla|^{s_c} P_{\leq N}(|u|^p u_{>N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim_u \sum_{M>N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M). \tag{6.47}$$

Next, we turn to estimating the second term in (6.45). Restricting our attention to an individual characteristic subinterval J_k, we first apply Bernstein, Hölder, and the fractional product rule to see

$$\begin{aligned}
&\big\||\nabla|^{s_c} P_{\leq N}\big(|u_{>cN_k}|^p P_{\leq cN_k} u_{\leq N/\eta_0}\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad\lesssim N^{s_c-\frac14}\big\||\nabla|^{\frac14}\big(|u_{>cN_k}|^p P_{\leq cN_k} u_{\leq N/\eta_0}\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad\lesssim N^{s_c-\frac14}\big\||\nabla|^{\frac14}|u_{>cN_k}|^p\big\|_{L_t^4 L_x^{\frac{2dp}{p(d+3)-4}}}\big\|P_{\leq cN_k} u_{\leq N/\eta_0}\big\|_{L_t^4 L_x^{\frac{2dp}{4-p}}} \qquad (6.48)\\
&\qquad+N^{s_c-\frac14}\|u_{>cN_k}\|^p_{L_t^{4p} L_x^{\frac{4dp^2}{p(2d+5)-8}}}\big\||\nabla|^{\frac14} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|_{L_t^4 L_x^{\frac{4dp}{8-p}}}. \qquad (6.49)
\end{aligned}$$

Using Hölder, the fractional chain rule, Sobolev embedding, Bernstein, interpolation, (6.33), and Young's inequality, we can estimate

$$\begin{aligned}
(6.48) &\lesssim N^{s_c-\frac14}\|u_{>cN_k}\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}}\big\||\nabla|^{\frac14} u_{>cN_k}\big\|_{L_t^4 L_x^{\frac{2d}{d-1}}}\big\||\nabla|^{s_c} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|_{L_t^4 L_x^{\frac{2d}{d-1}}} \\
&\lesssim_u N^{s_c-\frac14}(cN_k)^{\frac14-s_c}\big\||\nabla|^{s_c} u_{>cN_k}\big\|_{L_t^4 L_x^{\frac{2d}{d-1}}} \\
&\quad\times\big\||\nabla|^{s_c} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|^{\frac12}_{L_t^\infty L_x^2}\big\||\nabla|^{s_c} u_{\leq N/\eta_0}\big\|^{\frac12}_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\lesssim_u C_\eta\left(\tfrac{N}{N_k}\right)^{s_c-\frac14}\eta^{\frac12} A_{J_k}\big(\tfrac{N}{\eta_0}\big)^{\frac12} \\
&\lesssim_u C_\eta\left(\tfrac{N}{N_k}\right)^{2s_c-1/2}+\eta A_{J_k}\big(\tfrac{N}{\eta_0}\big),
\end{aligned}$$

for some positive constant C_η. Using Lemma 4.1.5 as well, we can estimate similarly

$$\begin{aligned}
(6.49) &\lesssim N^{s_c-\frac14}(cN_k)^{\frac14-s_c}\big\||\nabla|^{(s_c-\frac14)/p} u_{>cN_k}\big\|^p_{L_t^{4p} L_x^{\frac{4dp^2}{p(2d+5)-8}}} \\
&\quad\times\big\||\nabla|^{s_c} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|_{L_t^4 L_x^{\frac{2d}{d-1}}} \\
&\lesssim C_\eta\left(\tfrac{N}{N_k}\right)^{s_c-\frac14}\big\||\nabla|^{s_c} u_{>cN_k}\big\|^p_{L_t^{4p} L_x^{\frac{2dp}{dp-1}}} \\
&\quad\times\big\||\nabla|^{s_c} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|^{\frac12}_{L_t^\infty L_x^2}\big\||\nabla|^{s_c} u_{\leq N/\eta_0}\big\|^{\frac12}_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\lesssim_u C_\eta\left(\tfrac{N}{N_k}\right)^{2s_c-1/2}+\eta A_{J_k}\big(\tfrac{N}{\eta_0}\big).
\end{aligned}$$

Collecting the estimates for (6.48) and (6.49) and summing over the intervals $J_k \subset I$, we arrive at

$$\begin{aligned}
&\big\| |\nabla|^{s_c} P_{\leq N}\big(|u_{>cN(t)}|^p P_{\leq cN(t)} u_{\leq N/\eta_0}\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\qquad \lesssim_u C_\eta N^{2s_c-\frac{1}{2}} K_I^{1/2} + \eta A_I(N/\eta_0). \qquad (6.50)
\end{aligned}$$

Before proceeding, we note that for both (6.48) and (6.49), we could have instead estimated

$$\begin{aligned}
&\big\| |\nabla|^{s_c} P_{\leq cN_k} u_{\leq N/\eta_0}\big\|^{\frac{1}{2}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \\
&\quad \lesssim \big\| |\nabla|^{s_c} u_{\leq cN_k}\big\|^{\frac{1}{4}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \big\| |\nabla|^{s_c} u_{\leq N/\eta_0}\big\|^{\frac{1}{4}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \\
&\quad \lesssim \eta^{\frac{1}{4}} \big\| |\nabla|^{s_c} u_{\leq N/\eta_0}\big\|^{\frac{1}{4}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)}.
\end{aligned}$$

If we had done this, upon summing we could have ended up with the alternate estimate

$$\begin{aligned}
&\big\| |\nabla|^{s_c} P_{\leq N}\big(|u_{>cN(t)}|^p P_{\leq cN(t)} u_{\leq N/\eta_0}\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad \lesssim \sup_{J_k\subset I} \big\| |\nabla|^{s_c} u_{\leq N/\eta_0}\big\|^{\frac{1}{2}}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} C_\eta N^{2s_c-\frac{1}{2}} K_I^{1/2} + \eta^{1/2} A_I(N/\eta_0). \qquad (6.51)
\end{aligned}$$

This variant of (6.50) will be important when we need to exhibit smallness in (6.31).

To estimate the contribution of the third term in (6.45), we first define the following:

$$\begin{cases}
\theta := \frac{dp-4-p}{4-p} \in [0,1), & \sigma := \frac{p^2(d^2+2d-2)-4p(4d+1)+48}{4p(dp-8)} \in (0,s_c), \\
r_1 := \frac{4dp(dp-8)}{p^2(d^2-2d-2)+p(28-8d)-16}, & r_2 := \frac{4dp(dp-8)}{p^2(d^2+2d-2)-4p(2d+1)-16}.
\end{cases}$$

With this choice of parameters, we have

$$\begin{cases}
s_c + \theta(\frac{d-1}{2} - s_c) = 2s_c - \frac{1}{2}, \\
-\theta(s_c + \frac{1}{2}) - 2\sigma(1-\theta) = -(2s_c - \frac{1}{2})
\end{cases}$$

and (by Sobolev embedding)

$$\dot{H}^{s_c,\frac{2d}{d-2}} \hookrightarrow \dot{H}^{\sigma,r_1}, \quad \dot{H}^{s_c,2} \hookrightarrow \dot{H}^{\sigma,r_2}.$$

Then restricting our attention to an individual J_k, we can use Bernstein, Hölder, the bilinear

Strichartz estimate (Corollary 4.1.7), and Sobolev embedding to estimate

$$\begin{aligned}
&\| |\nabla|^{s_c} P_{\leq N}(|u_{>cN_k}|^p P_{>cN_k} u_{\leq N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad \lesssim N^{s_c} \|u_{>cN_k}\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \|u_{>cN_k} P_{>cN_k} u_{\leq N/\eta_0}\|^{\theta}_{L^2_{t,x}} \\
&\qquad \times \|u_{>cN_k}\|^{1-\theta}_{L_t^2 L_x^{r_1}} \|P_{>cN_k} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^{r_2}} \\
&\quad \lesssim_u N^{s_c} \left(\tfrac{N}{\eta_0}\right)^{\theta(\frac{d-1}{2}-s_c)} (cN_k)^{-\theta(s_c+\frac{1}{2})} \\
&\qquad \times \|u_{>cN_k}\|^{1-\theta}_{L_t^2 L_x^{r_1}} \|P_{>cN_k} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^{r_2}} \\
&\quad \lesssim_u B(\eta_0) N^{2s_c-\frac{1}{2}} (cN_k)^{-\theta(s_c+\frac{1}{2})-2\sigma(1-\theta)} \\
&\qquad \times \| |\nabla|^{\sigma} u_{>cN_k}\|^{1-\theta}_{L_t^2 L_x^{r_1}} \| |\nabla|^{\sigma} P_{>cN_k} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^{r_2}} \\
&\quad \lesssim_u C_{\eta,\eta_0} \left(\tfrac{N}{N_k}\right)^{2s_c-1/2} \| |\nabla|^{s_c} u_{>cN_k}\|^{1-\theta}_{L_t^2 L_x^{\frac{2d}{d-2}}} \| |\nabla|^{s_c} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^2} \\
&\quad \lesssim_u C_{\eta,\eta_0} \left(\tfrac{N}{N_k}\right)^{2s_c-1/2}
\end{aligned} \tag{6.52}$$

for some positive constant C_{η,η_0}. If we sum the estimates (6.52) over the intervals $J_k \subset I$, we arrive at

$$\| |\nabla|^{s_c} P_{\leq N}(|u_{>cN(t)}|^p P_{>cN(t)} u_{\leq N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim_u C_{\eta,\eta_0} N^{2s_c-\frac{1}{2}} K_I^{1/2}. \tag{6.53}$$

Before moving on to the fourth (and final) term in (6.45), we note that if we had held on to the term $\| |\nabla|^{s_c} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^2}$ when deriving (6.52), then upon summing we would get

$$\begin{aligned}
&\| |\nabla|^{s_c} P_{\leq N}(|u_{>cN(t)}|^p P_{>cN(t)} u_{\leq N/\eta_0})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\qquad \lesssim_u \sup_{J_k \subset I} \| |\nabla|^{s_c} u_{\leq N/\eta_0}\|^{1-\theta}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} C_{\eta,\eta_0} N^{2s_c-\frac{1}{2}} K_I^{1/2}.
\end{aligned} \tag{6.54}$$

This variant of (6.53) will be important when we eventually need to exhibit smallness in (6.31).

We now turn to the final term in (6.45), beginning with an application of the fractional product rule and Hölder:

$$
\begin{aligned}
&\big\| |\nabla|^{s_c} P_{\leq N}\big((|u|^p - |u_{>cN(t)}|^p) u_{\leq N/\eta_0}\big) \big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\qquad \lesssim \big\| |\nabla|^{s_c} (|u|^p - |u_{>cN(t)}|^p) \big\|_{L_t^\infty L_x^{\frac{2dp}{p(d+4)-4}}} \big\| u_{\leq N/\eta_0} \big\|_{L_t^2 L_x^{\frac{dp}{2-p}}} && (6.55)\\
&\qquad\quad + \big\| |u|^p - |u_{>cN(t)}|^p \big\|_{L_t^\infty L_x^{\frac{d}{2}}} \big\| |\nabla|^{s_c} u_{\leq N/\eta_0} \big\|_{L_t^2 L_x^{\frac{2d}{d-2}}}. && (6.56)
\end{aligned}
$$

By Lemma 2.2.5, Sobolev embedding, and (6.33), we first estimate

$$
\begin{aligned}
(6.55) &\lesssim \big\| |\nabla|^{s_c} u_{>cN(t)} \big\|_{L_t^\infty L_x^2} \| u_{\leq cN(t)} \|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} A_I(N/\eta_0) \\
&\quad + \big\| |\nabla|^{s_c} u_{\leq cN(t)} \big\|_{L_t^\infty L_x^2} \| u \|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} A_I(N/\eta_0) \\
&\lesssim_u (\eta^{p-1} + \eta) A_I(N/\eta_0).
\end{aligned}
$$

On the other hand, by Sobolev embedding, Hölder, and (6.33), we get

$$
(6.56) \lesssim \big(\|u\|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} + \|u_{>cN(t)}\|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} \big) \|u_{\leq cN(t)}\|_{L_t^\infty L_x^{\frac{dp}{2}}} A_I(N/\eta_0) \lesssim_u \eta A_I(N/\eta_0).
$$

Thus we can estimate the contribution of the final term in (6.45) by

$$
\big\| |\nabla|^{s_c} P_{\leq N}\big((|u|^p - |u_{>cN(t)}|^p) u_{\leq N/\eta_0}\big) \big\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim_u \eta^{p-1} A_I(N/\eta_0). \tag{6.57}
$$

Collecting the estimates (6.47), (6.50), (6.53), and (6.57), we see that in Case 2, we have the estimate

$$
\begin{aligned}
\big\| |\nabla|^{s_c} P_{\leq N}\big(F(u)\big) \big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} &\lesssim_u C_{\eta,\eta_0} N^{2s_c - \frac{1}{2}} K_I^{1/2} + \eta^{\min\{\frac{1}{2}, p-1\}} A_I(N/\eta_0) \\
&\quad + \sum_{M > N/\eta_0} \big(\tfrac{N}{M}\big)^{\frac{3}{2}s_c} A_I(M). && (6.58)
\end{aligned}
$$

Comparing (6.58) to (6.32), we see that Lemma 6.3.2 holds for $d \in \{4, 5\}$. ∎

We turn to the proof of Proposition 6.3.1.

Proof of Proposition 6.3.1. We proceed by induction. For the base case, we let $N \geq$

$\sup_{t\in I} N(t)$, so that $\left(\frac{N}{N(t)}\right)^{4s_c-1} \geq 1$ for $t \in I$. Thus, using Lemma 4.1.5, we estimate

$$\begin{aligned} A_I(N)^2 &\lesssim_u 1 + \int_I N(t)^2\,dt \\ &\lesssim_u 1 + \int_I N(t)^{3-4s_c} N^{4s_c-1}\,dt \\ &\lesssim_u 1 + N^{4s_c-1} K_I. \end{aligned}$$

Thus for $N \geq \sup_{t\in I} N(t)$, we have

$$A_I(N) \leq C_u \left[1 + N^{2s_c-\frac{1}{2}} K_I^{1/2}\right] \tag{6.59}$$

for $N \geq \sup_{J_k\subset I} N_k$. Of course, this inequality remains true if we replace C_u by any larger constant.

We now suppose (6.59) holds at frequency N and use the recurrence relation (6.32) to show it holds at frequency $N/2$. First, applying Strichartz and (6.32), we find

$$A_I(N) \leq \widetilde{C}_u\big[1 + C_{\eta,\eta_0} N^{2s_c-\frac{1}{2}} K_I^{1/2} + \eta^\nu A_I(\tfrac{N}{\eta_0}) + \sum_{M>N/\eta_0} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} A_I(M)\big]. \tag{6.60}$$

To simplify notation, we will let $\alpha := 2s_c - \frac{1}{2}$. Then, if we take $\eta_0 < \frac{1}{2}$ and use the inductive hypothesis, (6.60) becomes

$$\begin{aligned} A_I(\tfrac{N}{2}) &\leq \widetilde{C}_u\big[1 + C_{\eta,\eta_0}(\tfrac{N}{2})^\alpha K_I^{1/2} + \eta^\nu C_u(1 + \eta_0^{-\alpha}(\tfrac{N}{2})^\alpha K_I^{1/2}) \\ &\quad + C_u \sum_{M>N/2\eta_0} \left(\tfrac{N}{2M}\right)^{\frac{3}{2}s_c} (1 + M^\alpha K_I^{1/2})\big] \\ &\leq \widetilde{C}_u\big[1 + C_{\eta,\eta_0}(\tfrac{N}{2})^\alpha K_I^{1/2} + \eta^\nu C_u(1 + \eta_0^{-\alpha}(\tfrac{N}{2})^\alpha K_I^{1/2}) \\ &\quad + C_u \eta_0^{\frac{3}{2}s_c} + C_u \eta_0^{\frac{1}{2}(1-s_c)}(\tfrac{N}{2})^\alpha K_I^{1/2}\big] \\ &= \widetilde{C}_u\left[1 + C_{\eta,\eta_0}(\tfrac{N}{2})^\alpha K_I^{1/2}\right] + C_u\big[(\eta^\nu + \eta_0^{\frac{3}{2}s_c})\widetilde{C}_u \\ &\quad + \big(\eta_0^{-\alpha}\eta^\nu + \eta_0^{\frac{1}{2}(1-s_c)}\big)\widetilde{C}_u(\tfrac{N}{2})^\alpha K_I^{1/2}\big]. \end{aligned} \tag{6.61}$$

Notice that we had convergence of the sum above precisely because $s_c < 1$. If we choose η_0 possibly even smaller depending on $\widetilde{C}_u$, and η sufficiently small depending on $\widetilde{C}_u$ and η_0, we can guarantee

$$(6.61) \leq \widetilde{C}_u\left[1 + C_{\eta,\eta_0}(\tfrac{N}{2})^\alpha K_I^{1/2}\right] + \tfrac{1}{2}C_u\left[1 + (\tfrac{N}{2})^\alpha K_I^{1/2}\right].$$

If we now choose C_u possibly larger so that $C_u \geq 2(1+C_{\eta,\eta_0})\widetilde{C}_u$, then this inequality implies that (6.59) holds at $N/2$, as was needed to show. This completes the proof of (6.30).

It remains to establish (6.31). To begin, fix $\varepsilon > 0$. To exhibit the smallness in (6.31), we need to revisit the proof of the recurrence relation for $A_I(N)$, paying closer attention to the terms that gave rise to the expression $N^{2s_c-\frac{1}{2}}K_I^{1/2}$. More precisely, we use (6.39) instead of (6.38); (6.51) instead of (6.50); and (6.54) instead of (6.53). In this case, after an application of Strichartz we arrive at the estimate

$$A_I(N) \lesssim_u f(N) + f(N)N^{2s_c-\frac{1}{2}}K_I^{1/2} + \eta^\nu A_I(\tfrac{N}{\eta_0}) + \sum_{M>N/\eta_0} (\tfrac{N}{M})^{\frac{3}{2}s_c} A_I(M), \tag{6.62}$$

where $f(N)$ has the form

$$\begin{aligned} f(N) = \| |\nabla|^{s_c} u_{\leq N} \|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \\ + C_{\eta,\eta_0} \sum_{i=1}^4 \sup_{J_k\subset I} \| |\nabla|^{s_c} u_{\leq N/\eta_0} \|^{\theta_i}_{L_t^\infty L_x^2(J_k\times\mathbb{R}^d)} \end{aligned} \tag{6.63}$$

for some $\theta_i \in (0,1]$. Here the particular values of the θ_i are not important; we will only need the fact that each $\theta_i > 0$. Combining the updated recurrence relation (6.62) with the newly proven estimate (6.30) and once again simplifying notation via $\alpha = 2s_c - \frac{1}{2}$, we see

$$\begin{aligned} A_I(N) &\lesssim_u f(N) + f(N)N^\alpha K_I^{1/2} + \eta^\nu(1+\eta_0^{-\alpha}N^\alpha K_I^{1/2}) + \eta_0^{\frac{3}{2}s_c}(1+\eta_0^{-\alpha}N^\alpha K_I^{1/2}) \\ &\lesssim_u f(N) + \eta^\nu + \eta_0^{\frac{3}{2}s_c} + \Big[f(N) + \eta^\nu\eta_0^{-\alpha} + \eta_0^{\frac{1}{2}(1-s_c)}\Big] N^\alpha K_I^{1/2}. \end{aligned} \tag{6.64}$$

To complete the argument, we will need the fact that for fixed $\eta, \eta_0 > 0$, we have

$$\lim_{N\to 0} f(N) = 0, \tag{6.65}$$

which is a consequence of almost periodicity and the fact that $\inf_{t\in[0,T_{max})} N(t) \geq 1$.

Then, continuing from (6.64), we choose η_0 small enough that $\eta_0^{\frac{3}{2}s_c} + \eta_0^{\frac{1}{2}(1-s_c)} < \varepsilon$, and choose η sufficiently small depending on η_0 so that $\eta^\nu + \eta_0^{-\alpha}\eta^\nu < \varepsilon$. Finally, using (6.65), we choose $N_0 = N_0(\varepsilon)$ so that $f(N) < \varepsilon$ for $N \leq N_0$. With this choice of parameters, (6.64) becomes

$$A_I(N) \lesssim_u \varepsilon(1 + N^{2s_c-\frac{1}{2}}K_I^{1/2})$$

for $N \leq N_0$, which completes the proof of (6.31). ∎

CHAPTER 7

Frequency-cascades

In this chapter, we employ the long-time Strichartz estimates proved in the previous chapter to preclude the existence of frequency-cascade solutions to (1.1). We will see that for frequency-cascades, the long-time Strichartz estimates are strong enough to prove either additional decay or additional regularity. Combining this additional information with conservation of mass or energy, we can rule out the possibility of frequency-cascades.

The results in this chapter appeared originally in [49, 51].

7.1 The radial setting, $s_c < 1/2$

In this section, we preclude the existence of almost periodic solutions u as in Theorem 4.5.4 for which

$$K_{[0,\infty)} = \int_0^\infty N(t)^{3-2s_c}\,dt < \infty. \tag{7.1}$$

We show that (7.1) and Proposition 6.1.1 imply that such a solution would possess additional regularity. We then use the additional regularity and the conservation of energy to derive a contradiction.

We note here that (7.1) implies

$$\lim_{t\to\infty} N(t) = 0. \tag{7.2}$$

We begin with the following lemma.

Lemma 7.1.1 (Improved regularity) *Let $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.4. Suppose*

$$u \in L_t^\infty \dot{H}_x^s([0,\infty) \times \mathbb{R}^3) \quad \textit{for some} \quad s_c \leq s < 3/2 + s_c. \tag{7.3}$$

If (7.1) *holds, then*

$$u \in L_t^\infty \dot{H}_x^\sigma([0,\infty) \times \mathbb{R}^3) \quad \textit{for all} \quad s_c \leq \sigma < \sigma(s), \tag{7.4}$$

where $\sigma(s) := 1/2 + s - s_c$.

Proof. Throughout the proof, we take all spacetime norms over $[0,\infty) \times \mathbb{R}^3$.

We will first use Proposition 6.1.1 and (7.1) to establish

$$A_{[0,\infty)}(N) \lesssim_u N^{-\sigma(s)}. \tag{7.5}$$

Let $I_n \subset [0,\infty)$ be a nested sequence of compact subintervals, each of which is a contiguous union of characteristic subintervals J_k. We let $\eta > 0$ and apply Bernstein, Strichartz, Lemma 6.1.2, and (7.3) to estimate

$$A_{I_n}(N) \lesssim_u N^{-s_c} \inf_{t \in I_n} \|u_{>N}(t)\|_{\dot{H}_x^{s_c}} + C_\eta N^{-\sigma(s)} K_{I_n}^{1/2} + \sum_{M \leq \eta N} (\tfrac{M}{N})^2 A_{I_n}(M).$$

As (6.1.1) gives $A_{I_n}(N) \lesssim_u N^{-s_c} + N^{-\sigma(s)} K_{I_n}^{1/2}$, we may choose η sufficiently small and continue from above to get

$$A_{I_n}(N) \lesssim_u N^{-s_c} \inf_{t \in I_n} \|u_{>N}(t)\|_{\dot{H}_x^{s_c}} + N^{-\sigma(s)} K_{I_n}^{1/2}. \tag{7.6}$$

Using (7.2), we see that for any $N > 0$ we have

$$\lim_{t \to \infty} \|u_{>N}(t)\|_{\dot{H}_x^{s_c}} = 0.$$

Hence sending $n \to \infty$, continuing from (7.6), and using (7.1), we get

$$A_{[0,\infty)}(N) \lesssim_u N^{-\sigma(s)}.$$

We now show that (7.5) implies

$$\|u_{>N}\|_{L_t^\infty L_x^2([0,\infty)\times\mathbb{R}^d)} \lesssim_u N^{-\sigma(s)}. \tag{7.7}$$

We first use Proposition 4.1.6 and Strichartz to estimate

$$\|u_{>N}\|_{L_t^\infty L_x^2} \lesssim \|P_{>N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}}.$$

We write $F(u) = F(u_{\leq N}) + F(u) - F(u_{\leq N})$. Noting that $s < 3/2 + s_c$ implies $\sigma(s) < 2$, we use Bernstein, the chain rule, (4.1), and (7.5) to estimate

$$\begin{aligned}\|P_{>N}\big(F(u_{\leq N})\big)\|_{L_t^2 L_x^{6/5}} &\lesssim N^{-2} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^p \sum_{M\leq N} \|\Delta u_M\|_{L_t^2 L_x^6} \\ &\lesssim_u \sum_{M\leq N} (\tfrac{M}{N})^2 M^{-\sigma(s)} \lesssim_u N^{-\sigma(s)}.\end{aligned}$$

We next use Hölder, Sobolev embedding, (4.1), and (7.5) to estimate

$$\|P_{>N}\big(F(u) - F(u_{\leq N})\big)\|_{L_t^2 L_x^{6/5}} \lesssim \|u\|_{L_t^\infty L_x^{3p/2}}^p \|u_{>N}\|_{L_t^2 L_x^6} \lesssim_u N^{-\sigma(s)}.$$

Adding the last two estimates gives (7.7).

Finally, we use (7.7) to prove (7.4). We fix $s_c \leq \sigma < \sigma(s)$ and use Bernstein, (4.1), and (7.7) to estimate

$$\begin{aligned}\||\nabla|^\sigma u\|_{L_t^\infty L_x^2} &\lesssim \||\nabla|^{s_c} u_{\leq 1}\|_{L_t^\infty L_x^2} + \sum_{M>1} M^\sigma \|u_M\|_{L_t^\infty L_x^2} \\ &\lesssim_u 1 + \sum_{M>1} M^{\sigma - \sigma(s)} \lesssim_u 1.\end{aligned}$$

This completes the proof of Lemma 7.1.1. ∎

We now iterate Lemma 7.1.1 to establish additional regularity.

Proposition 7.1.2 (Additional regularity) *Let $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.4. If (7.1) holds, then $u \in L_t^\infty \dot{H}_x^{1+\varepsilon}$ for some $\varepsilon > 0$.*

Proof. As $0 < s_c < 1/2$, we may choose c such that

$$2s_c < \tfrac{2}{3-2s_c} < c < \tfrac{3+2s_c}{4} < 1.$$

We define $s_0 = s_c$, and for $n \geq 0$ we define $s_{n+1} = c \cdot \sigma(s_n)$, where as above $\sigma(s) := 1/2 + s - s_c$. The constraint $c > 2s_c$ guarantees that the sequence s_n is increasing and bounded above by $\ell := \frac{c(1-2s_c)}{2(1-c)}$. In fact, elementary arguments show that the sequence s_n converges to ℓ, and the constraint $c > \frac{2}{3-2s_c}$ guarantees that $\ell > 1$.

We have that $s_n \geq s_c$ for all $n \geq 0$, while the constraint $c < \frac{3+2s_c}{4}$ guarantees $s_n < 3/2+s_c$ for all $n \geq 0$. Thus, noting that $s_n \leq s_{n+1} < \sigma(s_n)$ for each $n \geq 0$, we deduce from Lemma 7.1.1 that

$$u \in L_t^\infty \dot{H}_x^{s_n} \implies u \in L_t^\infty \dot{H}_x^{s_{n+1}} \quad \text{for all} \quad n \geq 0.$$

As (4.1) gives $u \in L_t^\infty \dot{H}_x^{s_0}$, we get by induction that $u \in L_t^\infty \dot{H}_x^{s_n}$ for all $n \geq 0$. As $s_n \to \ell > 1$, we conclude that $u \in L_t^\infty \dot{H}_x^{1+\varepsilon}$ for some $\varepsilon > 0$. ∎

Combining Proposition 7.1.2 with almost periodicity and the conservation of energy, we preclude the existence of rapid frequency cascades.

Theorem 7.1.3 (No frequency-cascades) *There are no almost periodic solutions u as in Theorem 4.5.4 such that* (7.1) *holds.*

Proof. Suppose u were such a solution and let $\eta > 0$. By almost periodicity, we may find $C(\eta)$ large enough that $\||\nabla|^{s_c} u_{>C(\eta)N(t)}\|_{L_t^\infty L_x^2} < \eta$. Thus, by interpolation and Proposition 7.1.2, we have

$$\|\nabla u_{>C(\eta)N(t)}\|_{L_t^\infty L_x^2} \lesssim \||\nabla|^{s_c} u_{>C(\eta)N(t)}\|_{L_t^\infty L_x^2}^{\frac{\varepsilon}{1+\varepsilon-s_c}} \||\nabla|^{1+\varepsilon} u\|_{L_t^\infty L_x^2}^{\frac{1-s_c}{1+\varepsilon-s_c}} \lesssim_u \eta^{\frac{\varepsilon}{1+\varepsilon-s_c}}$$

for some $\varepsilon > 0$.

On the other hand, by Bernstein and (4.1) we have

$$\|\nabla u_{\leq C(\eta)N(t)}(t)\|_{L_x^2} \lesssim_u [C(\eta)N(t)]^{1-s_c} \quad \text{for any} \quad t \in [0, \infty).$$

Thus we find

$$\|\nabla u(t)\|_{L_x^2} \lesssim_u \eta^{\frac{\varepsilon}{1+\varepsilon-s_c}} + [C(\eta)N(t)]^{1-s_c} \quad \text{for any} \quad t \in [0,\infty).$$

Using (7.2) and the fact that $\eta > 0$ was arbitrary, we deduce that

$$\|\nabla u(t)\|_{L_x^2} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.8}$$

We next use Hölder and Sobolev embedding to estimate

$$\|u(t)\|_{L_x^{p+2}} \lesssim \|u(t)\|_{L_x^{3p/2}}^{\frac{p}{p+2}} \|u(t)\|_{L_x^6}^{\frac{2}{p+2}} \lesssim \||\nabla|^{s_c} u(t)\|_{L_x^2}^{\frac{p}{p+2}} \|\nabla u(t)\|_{L_x^2}^{\frac{2}{p+2}},$$

so that (4.1) and (7.8) imply

$$\|u(t)\|_{L_x^{p+2}} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.9}$$

Adding (7.8) and (7.9) implies that $E[u(t)] \to 0$ as $t \to \infty$. By the conservation of energy, we conclude $E[u(t)] \equiv 0$. Thus we must have $u \equiv 0$, which contradicts the fact that u blows up. This completes the proof of Theorem 7.1.3. ∎

7.2 The radial setting, $s_c > 1/2$

In this section we preclude the existence of almost periodic solutions as in Theorem 4.5.3 for which

$$K_{[0,T_{max})} = \int_0^{T_{max}} N(t)^{3-2s_c}\, dt < \infty. \tag{7.10}$$

We show that (7.10) and Proposition 6.2.1 imply that such solutions would possess additional decay. We then use the conservation of mass to derive a contradiction.

Note that we have

$$\lim_{t\to T_{max}} N(t) = \infty, \tag{7.11}$$

whether T_{max} is finite or infinite. Indeed, in the case $T_{max} < \infty$ this follows from Corollary 4.1.4, while in the case $T_{max} = \infty$ this follows from (7.10) and (6.20).

We begin with the following lemma.

Lemma 7.2.1 (Improved decay) *Let $u : [0, T_{max}) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.3. Suppose*

$$u \in L_t^\infty \dot{H}_x^s([0, T_{max}) \times \mathbb{R}^3) \quad \text{for some} \quad s_c - 1/2 < s \leq s_c. \tag{7.12}$$

If (7.10) *holds, then*

$$u \in L_t^\infty \dot{H}_x^\sigma([0, T_{max}) \times \mathbb{R}^3) \quad \text{for all} \quad s - \sigma(s) < \sigma \leq s_c, \tag{7.13}$$

where $\sigma(s) := 2s_c - s - 1/2$.

Proof. Throughout the proof, we take all spacetime norms over $[0, T_{max}) \times \mathbb{R}^3$.

We first use Proposition 6.2.1 and (7.10) to show

$$A_{[0,T_{max})}(N) \lesssim_u N^{\sigma(s)}. \tag{7.14}$$

Let $I_n \subset [0, T_{max})$ be a nested sequence of compact time intervals, each of which is a contiguous union of chracteristic subintervals. We let $\eta > 0$ and apply Strichartz, Lemma 6.2.2, and (7.12) to estimate

$$A_{I_n}(N) \lesssim_u \inf_{t \in I_n} \|u_{\leq N}(t)\|_{\dot{H}_x^{s_c}} + C_\eta N^{\sigma(s)} K_{I_n}^{1/2} + \sum_{M \geq N/\eta} \left(\tfrac{N}{M}\right)^{s_c} A_{I_n}(M).$$

As (6.21) gives $A_{I_n}(N) \lesssim_u 1 + N^{\sigma(s)} K_{I_n}^{1/2}$, we may choose η sufficiently small and continue from above to get

$$A_{I_n}(N) \lesssim_u \inf_{t \in I_n} \|u_{\leq N}\|_{\dot{H}_x^{s_c}} + N^{\sigma(s)} K_{I_n}^{1/2}. \tag{7.15}$$

Using (7.11), we get that for any $N > 0$ we have $\lim_{t \to T_{max}} \|u_{\leq N}\|_{\dot{H}_x^{s_c}} = 0$. Thus sending $n \to \infty$, continuing from (7.15), and using (7.10), we deduce that (7.14) holds.

We next show that (7.14) and (7.12) imply

$$\||\nabla|^s u_{\leq N}\|_{L_t^\infty L_x^2} \lesssim_u N^{\sigma(s)}. \tag{7.16}$$

We first use Proposition 4.1.6 and Strichartz to estimate

$$\||\nabla|^s u_{\leq N}\|_{L_t^\infty L_x^2} \lesssim_u \||\nabla|^s P_{\leq N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}}.$$

We decompose the nonlinearity as $F(u) = F(u_{\leq N}) + [F(u) - F(u_{\leq N})]$. Noting that $s_c - 1/2 < s \leq s_c$ implies $6 \leq \frac{3p}{2+ps-p} < \infty$, we can first use Hölder, the fractional chain rule, Sobolev embedding, (4.1), and (7.12) to estimate

$$\begin{aligned}\||\nabla|^s F(u_{\leq N})\|_{L_t^2 L_x^{6/5}} &\lesssim \|u\|_{L_t^\infty L_x^{3p/2}}^{p-1} \|u\|_{L_t^\infty L_x^{\frac{6}{3-2s}}} \||\nabla|^s u_{\leq N}\|_{L_t^2 L_x^{\frac{3p}{2+ps-p}}} \\ &\lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^{3p/2}}^{p-1} \||\nabla|^s u\|_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^6} \\ &\lesssim_u N^{\sigma(s)}.\end{aligned}$$

Next, we note that $F(u) - F(u_{\leq N}) = \emptyset(u_{>N} u^p)$ and that $s > s_c - 1/2$ implies $\sigma(s) < s_c$. Thus we can use Bernstein, Lemma 2.2.6, the fractional chain rule, Sobolev embedding, (4.1), (7.12), and (7.14) to estimate

$$\begin{aligned}&\||\nabla|^s P_{\leq N}\big(F(u) - F(u_{\leq N})\big)\|_{L_t^2 L_x^{6/5}} \\ &\lesssim N^{s_c} \||\nabla|^{-(s_c - s)}\big(u^p u_{>N}\big)\|_{L_t^2 L_x^{6/5}} \\ &\lesssim N^{s_c} \||\nabla|^{s_c - s}(u^p)\|_{L_t^\infty L_x^{\frac{3p}{5p-4-2ps}}} \sum_{M>N} \||\nabla|^{-(s_c-s)} u_M\|_{L_t^2 L_x^{\frac{3p}{2+ps-p}}} \\ &\lesssim \|u\|_{L_t^\infty L_x^{3p/2}}^{p-2} \|u\|_{L_t^\infty L_x^{\frac{6}{3-2s}}} \||\nabla|^{s_c - s} u\|_{L_t^\infty L_x^{\frac{6}{3-2s}}} \sum_{M>N} \big(\tfrac{N}{M}\big)^{s_c} \||\nabla|^s u_M\|_{L_t^2 L_x^{\frac{3p}{2+ps-p}}} \\ &\lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^{p-1} \||\nabla|^s u\|_{L_t^\infty L_x^2} \sum_{M>N} \big(\tfrac{N}{M}\big)^{s_c} \||\nabla|^{s_c} u_M\|_{L_t^2 L_x^6} \\ &\lesssim_u \sum_{M>N} \big(\tfrac{N}{M}\big)^{s_c} M^{\sigma(s)} \lesssim_u N^{\sigma(s)}.\end{aligned}$$

Note that in the case $s = s_c$, we would simply use Hölder instead of Lemma 2.2.6 and the fractional chain rule.

The last two estimates together imply (7.16).

Finally, we use (7.16) to prove (7.13). We fix $s - \sigma(s) < \sigma \leq s_c$ and use Bernstein, (4.1), and (7.16) to estimate

$$\begin{aligned}\||\nabla|^\sigma u\|_{L_t^\infty L_x^2} &\lesssim \||\nabla|^{s_c} u_{\geq 1}\|_{L_t^\infty L_x^2} + \sum_{M \leq 1} M^{\sigma - s} \||\nabla|^s u_M\|_{L_t^\infty L_x^2} \\ &\lesssim_u 1 + \sum_{M \leq 1} M^{\sigma - s + \sigma(s)} \lesssim_u 1.\end{aligned}$$

This completes the proof of Lemma 7.2.1. ∎

We now iterate Lemma 7.2.1 to establish additional decay.

Proposition 7.2.2 (Additional decay) *Let $u : [0, T_{max}) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.3. If* (7.10) *holds, then $u \in L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $\varepsilon > 0$.*

Proof. Let $0 < \delta < 1/4 < s_c - 1/2$ and for each $n \geq 0$ define $s_n := s_c - n\delta$. We have from Lemma 7.2.1 that

$$u \in L_t^\infty \dot{H}_x^{s_n} \implies u \in L_t^\infty \dot{H}_x^{\sigma} \quad \text{for all} \quad 0 \leq n < \tfrac{1}{2\delta} \quad \text{and} \quad s_n - \sigma(s_n) < \sigma \leq s_c.$$

The restriction $n < \frac{1}{2\delta}$ guarantees $s_n > s_c - 1/2$. As above, $\sigma(s) := 2s_c - s - 1/2$.

As (4.1) gives $u \in L_t^\infty \dot{H}_x^{s_0}$ and the constraint $0 < \delta < s_c - 1/2$ guarantees $s_n - \sigma(s_n) < s_{n+1} \leq s_c$ for all $n \geq 0$, we get by induction that

$$u \in L_t^\infty \dot{H}_x^{\sigma} \quad \text{for all} \quad 0 \leq n < \tfrac{1}{2\delta} \quad \text{and} \quad s_n - \sigma(s_n) < \sigma \leq s_c. \tag{7.17}$$

As $\delta < 1/4$, we may find n^* so that $\frac{1}{4\delta} < n^* < \frac{1}{2\delta}$. As the constraint $n^* > \frac{1}{4\delta}$ implies $s_{n^*} - \sigma(s_{n^*}) < 0$, we deduce from (7.17) that $u \in L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $\varepsilon > 0$. This completes the proof of Proposition 7.2.2. ∎

Finally, we turn to the following.

Theorem 7.2.3 (No frequency-cascades) *There are no almost periodic solutions as in Theorem 4.5.3 such that* (7.10) *holds.*

Proof. Suppose u were such a solution and let $\eta > 0$. By almost periodicity, we may find $c(\eta)$ small enough that $\| |\nabla|^{s_c} u_{\leq c(\eta)N(t)} \|_{L_t^\infty L_x^2} < \eta$. Thus, by interpolation and Proposition 7.2.2, we have

$$\|u_{\leq c(\eta)N(t)}\|_{L_t^\infty L_x^2} \lesssim \| |\nabla|^{s_c} u_{\leq c(\eta)N(t)} \|_{L_t^\infty L_x^2}^{\frac{\varepsilon}{s_c+\varepsilon}} \| |\nabla|^{-\varepsilon} u \|_{L_t^\infty L_x^2}^{\frac{s_c}{s_c+\varepsilon}} \lesssim_u \eta^{\frac{\varepsilon}{s_c+\varepsilon}}$$

for some $\varepsilon > 0$.

On the other hand, using Bernstein and (4.1) we get

$$\|u_{>c(\eta)N(t)}(t)\|_{L_x^2} \lesssim_u [c(\eta)N(t)]^{-s_c} \quad \text{for any} \quad t \in [0, T_{max}).$$

Thus

$$\|u(t)\|_{L_x^2} \lesssim_u \eta^{\frac{\varepsilon}{s_c+\varepsilon}} + [c(\eta)N(t)]^{-s_c} \quad \text{for any} \quad t \in [0, T_{max}).$$

Using (7.11) and the fact that $\eta > 0$ was arbitrary, we deduce

$$\|u(t)\|_{L_x^2} \to 0 \quad \text{as} \quad t \to T_{max}.$$

By the conservation of mass, we conclude that $\mathcal{M}[u(t)] \equiv 0$. Thus we must have that $u \equiv 0$, which contradicts that u blows up. This completes the proof of Theorem 7.2.3. ∎

7.3 The non-radial setting

In this section, we preclude the existence of almost periodic solutions as in Theorem 4.5.2 for which

$$K_{[0,T_{max})} = \int_0^{T_{max}} N(t)^{3-4s_c}\, dt < \infty. \tag{7.18}$$

We will show that (7.18) and Proposition 6.3.1 imply that such a solution would possess additinoal decay. We then use the conservation of mass to derive a contradiction.

Theorem 7.3.1 (No frequency-cascades) *There are no almost periodic solutions as in Theorem 4.5.2 such that* (7.18) *holds.*

Proof. We argue by contradiction. Suppose u were such a solution. By Corollary 4.1.4, we have

$$\lim_{t \to T_{max}} N(t) = \infty,$$

whether T_{max} is finite or infinite (cf. (6.29)). Thus

$$\lim_{t \to T_{max}} \||\nabla|^{s_c} u_{\leq N}(t)\|_{L_x^2(\mathbb{R}^d)} = 0 \qquad \text{for any } N > 0. \tag{7.19}$$

We now let I_n be a nested sequence of compact subintervals of $[0, T_{max})$, each of which is a contiguous union of characteristics intervals J_k. On each I_n, we apply Proposition 6.3.1; specifically, for fixed $\eta, \eta_0 > 0$, we use the recurrence relation (6.32), the estimate (6.30), and the hypothesis (7.18) to see

$$\begin{aligned} A_{I_n}(N) &\lesssim_u \inf_{t \in I_n} \| |\nabla|^{s_c} u_{\leq N}(t) \|_{L_x^2(\mathbb{R}^d)} + C_{\eta,\eta_0} N^{2s_c - 1/2} K_{I_n}^{1/2} + \sum_{M > N/\eta_0} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} A_{I_n}(M) \\ &\lesssim_u \inf_{t \in I_n} \| |\nabla|^{s_c} u_{\leq N}(t) \|_{L_x^2(\mathbb{R}^d)} + C_{\eta,\eta_0} N^{2s_c - 1/2} + \sum_{M > N/\eta_0} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} A_{I_n}(M). \end{aligned}$$

Arguing as we did to obtain (6.30), we conclude

$$A_{I_n}(N) \lesssim_u \inf_{t \in I_n} \| |\nabla|^{s_c} u_{\leq N}(t) \|_{L_x^2(\mathbb{R}^d)} + N^{2s_c - 1/2}.$$

Letting $n \to \infty$ and using (7.19) then gives

$$A_{[0,T_{max})}(N) \lesssim_u N^{2s_c - 1/2} \qquad \text{for all } N > 0. \tag{7.20}$$

We now claim that (7.20) implies

Lemma 7.3.2

$$\| |\nabla|^{s_c} u_{\leq N} \|_{L_t^\infty L_x^2([0,T_{max}) \times \mathbb{R}^d)} \lesssim_u N^{2s_c - 1/2} \quad \textit{for all} \quad N > 0. \tag{7.21}$$

Proof of Lemma 7.3.2. Let $N > 0$. We first use Proposition 4.1.6 and Strichartz to estimate

$$\| |\nabla|^{s_c} u_{\leq N} \|_{L_t^\infty L_x^2([0,T_{max}) \times \mathbb{R}^d)} \lesssim \| |\nabla|^{s_c} P_{\leq N}(|u|^p u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T_{max}) \times \mathbb{R}^d)}. \tag{7.22}$$

To proceed, we decompose the nonlinearity and estimate the individual pieces; as before, the particular decomposition we use depends on the ambient dimension. In the estimates that follow, spacetime norms will be taken over $[0, T_{max}) \times \mathbb{R}^d$.

Case 1. When $d = 3$, we decompose

$$|u|^p u = |u|^{p-2} \bar{u} u_{\leq N}^2 + (|u|^{p-2} \bar{u} u_{>N} + 2|u|^{p-2} \bar{u} u_{\leq N}) u_{>N}.$$

We can use Hölder, the fractional product rule, fractional chain rule, Sobolev embedding, interpolation, and (7.20) to estimate the contribution of the first piece as follows:

$$
\begin{aligned}
&\||\nabla|^{s_c} P_{\leq N}(|u|^{p-2}\bar{u}u_{\leq N}^2)\|_{L_t^2 L_x^{6/5}} \\
&\lesssim \||\nabla|^{s_c}(|u|^{p-2}\bar{u})\|_{L_t^\infty L_x^{\frac{6p}{7p-8}}} \|u_{\leq N}\|^2_{L_t^4 L_x^{\frac{6p}{4-p}}} + \|u\|^{p-1}_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c}(u_{\leq N}^2)\|_{L_t^2 L_x^{\frac{6p}{4+p}}} \\
&\lesssim \|u\|^{p-2}_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|^2_{L_t^4 L_x^3} + \||\nabla|^{s_c} u\|^{p-1}_{L_t^\infty L_x^2} \|u_{\leq N}\|_{L_t^\infty L_x^{\frac{3p}{2}}} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^6} \\
&\lesssim_u \||\nabla|^{s_c} u\|^{p-1}_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^6} + N^{2s_c - 1/2} \\
&\lesssim_u N^{2s_c - 1/2}.
\end{aligned}
$$

To estimate the contribution of the second piece, we denote

$$G = |u|^{p-2}\bar{u}u_{>N} + 2|u|^{p-2}\bar{u}u_{\leq N}$$

and use Bernstein, Hölder, Lemma 2.2.6, and (7.20) to see

$$
\begin{aligned}
\||\nabla|^{s_c} P_{\leq N}(G u_{>N})\|_{L_t^2 L_x^{6/5}} &\lesssim N^{\frac{3}{2}s_c} \||\nabla|^{-\frac{1}{2}s_c}(G u_{>N})\|_{L_t^2 L_x^{6/5}} \\
&\lesssim N^{\frac{3}{2}s_c} \||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \||\nabla|^{-\frac{1}{2}s_c} u_{>N}\|_{L_t^2 L_x^6} \\
&\lesssim \||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \sum_{M>N} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} \||\nabla|^{s_c} u_M\|_{L_t^2 L_x^6} \\
&\lesssim_u \||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} N^{2s_c - 1/2}. \qquad (7.23)
\end{aligned}
$$

A few applications of the fractional product rule, fractional chain rule, and Sobolev embedding give

$$\||\nabla|^{\frac{1}{2}s_c} G\|_{L_t^\infty L_x^{\frac{12p}{11p-4}}} \lesssim \||\nabla|^{s_c} u\|^p_{L_t^\infty L_x^2} \lesssim_u 1,$$

so that continuing from (7.23), we get

$$\||\nabla|^{s_c} P_{\leq N}\big((|u|^{p-2}\bar{u}u_{>N} + 2|u|^{p-2}\bar{u}u_{\leq N})u_{>N}\big)\|_{L_t^2 L_x^{6/5}} \lesssim_u N^{2s_c - 1/2}.$$

Thus we see that the claim holds in this first case.

Case 2. When $d \in \{4, 5\}$, we decompose

$$|u|^p u = |u|^p u_{\leq N} + |u|^p u_{>N}.$$

We employ Hölder, the fractional product rule, the fractional chain rule, Sobolev embedding, and (7.20) to estimate the contribution of the first piece as follows:

$$\begin{aligned}
&\| |\nabla|^{s_c} P_{\leq N}(|u|^p u_{\leq N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad \lesssim \| |\nabla|^{s_c} |u|^p \|_{L_t^\infty L_x^{\frac{2dp}{p(d+4)-4}}} \|u_{\leq N}\|_{L_t^2 L_x^{\frac{dp}{2-p}}} + \|u\|^p_{L_t^\infty L_x^{\frac{dp}{2}}} \| |\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\quad \lesssim_u \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \| |\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \| |\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} + N^{2s_c-1/2} \\
&\quad \lesssim_u N^{2s_c-1/2}.
\end{aligned}$$

For the second piece, we use Hölder, Bernstein, Lemma 2.2.6, the fractional chain rule, and Sobolev embedding to see

$$\begin{aligned}
&\| |\nabla|^{s_c} P_{\leq N}(|u|^p u_{>N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad \lesssim N^{\frac{3}{2}s_c} \| |\nabla|^{-\frac{1}{2}s_c} (|u|^p u_{>N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad \lesssim N^{\frac{3}{2}s_c} \| |\nabla|^{\frac{1}{2}s_c} |u|^p \|_{L_t^\infty L_x^{\frac{4dp}{p(d+8)-4}}} \| |\nabla|^{-\frac{1}{2}s_c} u_{>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\quad \lesssim \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \| |\nabla|^{\frac{1}{2}s_c} u\|_{L_t^\infty L_x^{\frac{4dp}{dp+4}}} \sum_{M>N} \left(\tfrac{N}{M}\right)^{\frac{3}{2}s_c} \| |\nabla|^{s_c} u_M\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\quad \lesssim_u \| |\nabla|^{s_c} u\|^p_{L_t^\infty L_x^2} N^{2s_c-1/2} \\
&\quad \lesssim_u N^{2s_c-1/2}.
\end{aligned}$$

Thus we see that the claim holds in this second case.

This completes the proof of Lemma 7.3.2. ∎

We now wish to use (7.21) to prove the following lemma.

Lemma 7.3.3 (Additional decay)

$$u \in L_t^\infty \dot{H}_x^{-\varepsilon}([0, T_{max}) \times \mathbb{R}^d) \qquad \textit{for some } \varepsilon > 0.$$

Proof of Lemma 7.3.3 Recalling that $s_c > \frac{1}{2}$, we may choose $\varepsilon > 0$ such that $s_c - \frac{1}{2} - \varepsilon > 0$

and use Bernstein and (7.21) to see

$$
\begin{aligned}
\| |\nabla|^{-\varepsilon} u \|_{L_t^\infty L_x^2} &\lesssim \sum_{N \leq 1} N^{-s_c-\varepsilon} \| |\nabla|^{s_c} u_N \|_{L_t^\infty L_x^2} + \sum_{N>1} N^{-s_c-\varepsilon} \| |\nabla|^{s_c} u_N \|_{L_t^\infty L_x^2} \\
&\lesssim_u \sum_{N \leq 1} N^{-s_c-\varepsilon} N^{2s_c-1/2} + 1 \lesssim_u 1.
\end{aligned}
$$

This completes the proof of Lemma 7.3.3. ∎

With Lemma 7.3.3 at hand, we are ready to complete the proof of Theorem 7.3.1. Fix $t \in [0, T_{max})$ and $\eta > 0$. Using almost periodicity, we may find $c(\eta) > 0$ so that

$$
\int_{|\xi| \leq c(\eta) N(t)} |\xi|^{2s_c} |\widehat{u}(t,\xi)|^2 \, d\xi \leq \eta.
$$

Interpolating with $u \in L_t^\infty \dot{H}_x^{-\varepsilon}$, we get

$$
\int_{|\xi| \leq c(\eta) N(t)} |\widehat{u}(t,\xi)|^2 \; d\xi \lesssim_u \eta^{\frac{\varepsilon}{s_c+\varepsilon}}.
$$

On the other hand, we have

$$
\int_{|\xi| \geq c(\eta) N(t)} |\widehat{u}(\xi,t)|^2 \, d\xi \leq (c(\eta) N(t))^{-2s_c} \int |\xi|^{2s_c} |\widehat{u}(t,\xi)|^2 \, d\xi \lesssim_u (c(\eta) N(t))^{-2s_c}.
$$

Adding these last estimates and using Plancherel, we conclude that for all $t \in [0, T_{max})$, we have

$$
0 \leq \mathcal{M}(u(t)) := \int |u(t,x)|^2 \, dx \lesssim_u \eta^{\frac{\varepsilon}{s_c+\varepsilon}} + (c(\eta) N(t))^{-2s_c}.
$$

As $\lim_{t \to T_{max}} N(t) = \infty$ and η was arbitrary, we conclude that $\mathcal{M}[u(t)] \to 0$ as $t \to T_{max}$. By the conservation of mass, we conclude that $\mathcal{M}[u(t)] \equiv 0$. Thus we must have that $u \equiv 0$, which contradicts that u blows up. This completes the proof of Theorem 7.3.1 ∎

CHAPTER 8

Frequency-localized Morawetz inequalities

In this chapter, we use the long-time Strichartz estimates of Chapter 6 to prove frequency-localized Morawetz estimates, which we will then use to rule out the existence of quasi-solitons in Chapter 9.

The results in this chapter appeared originally in [49, 51].

8.1 Frequency-localized Lin–Strauss Morawetz, $s_c < 1/2$

In this section, we use Proposition 6.1.1 to prove a frequency-localized Lin–Strauss Morawetz inequality. As $s_c < 1/2$, we prove an estimate that is localized to low frequencies.

The main result of this section is the following.

Proposition 8.1.1 (Frequency-localized Morawetz) *Let $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.4. Let $I \subset [0,\infty)$ be a compact time interval, which is a contiguous union of characteristic subintervals J_k. Then for any $\eta > 0$, there exists $N_0 = N_0(\eta)$ such that for $N > N_0$, we have*

$$\iint_{I\times\mathbb{R}^3} \frac{|u_{\leq N}(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim_u \eta(N^{1-2s_c} + K_I), \tag{8.1}$$

where K_I is as in (6.3).

To prove Proposition 8.1.1, we begin as in the proof of the standard Lin–Strauss Morawetz inequality (1.7). We truncate the high frequencies of the solution and work with $u_{\leq N}$ for some $N > 0$. As $u_{\leq N}$ is not a true solution to (1.1), we need to control error terms arising from this frequency projection. To do this, we choose N large enough to capture 'most' of the solution and use the estimates proved in Chapter 6. We make these notions precise in the following lemma.

Lemma 8.1.2 (Low and high frequency control) *Let u, I, K_I be as above. With all spacetime norms over $I \times \mathbb{R}^3$, we have the following.*

For any $N > 0$ and $s > 1/2$,

$$\||\nabla|^s u_{\leq N}\|_{L_t^2 L_x^6} \lesssim_u N^{s-s_c}(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.2}$$

For any $\eta > 0$ and $s > s_c$, there exists $N_1 = N_1(s,\eta)$ such that for $N > N_1$,

$$\||\nabla|^s u_{\leq N}\|_{L_t^\infty L_x^2} \lesssim_u \eta N^{s-s_c}. \tag{8.3}$$

For any $\eta > 0$, there exists $N_2 = N_2(\eta) > 0$ such that for $N > N_2$, we have

$$\|u_{>N}\|_{L_t^2 L_x^6} \lesssim_u \eta N^{-s_c}(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.4}$$

Proof. For (8.2), we let $s > 1/2$ and use (6.5) to estimate

$$\begin{aligned}
\||\nabla|^s u_{\leq N}\|_{L_t^2 L_x^6} &\lesssim \sum_{M \leq N} M^s \|u_M\|_{L_t^2 L_x^6} \\
&\lesssim_u \sum_{M \leq N} M^{s-s_c}(1 + M^{2s_c-1}K_I)^{1/2} \\
&\lesssim_u N^{s-s_c}(1 + N^{2s_c-1}K_I)^{1/2}.
\end{aligned}$$

For (8.3), we first let $\eta > 0$. Using almost periodicity and the fact that $\sup N(t) \leq 1$, we may find $C(\eta) > 0$ so that $\||\nabla|^{s_c} u_{>C(\eta)}\|_{L_t^\infty L_x^2} < \eta$. Thus we can use Bernstein to see

$$\begin{aligned}
&\||\nabla|^s u_{\leq N}\|_{L_t^\infty L_x^2} \\
&\quad\lesssim C(\eta)^{s-s_c}\||\nabla|^{s_c} u_{\leq C(\eta)}\|_{L_t^\infty L_x^2} + N^{s-s_c}\||\nabla|^{s_c} u_{C(\eta)<\cdot\leq N}\|_{L_t^\infty L_x^2} \\
&\quad\lesssim_u C(\eta)^{s-s_c} + \eta N^{s-s_c}.
\end{aligned}$$

Choosing $N_1 \gg \eta^{-1/(s-s_c)}C(\eta)$, we recover (8.3).

Finally, we note that (8.4) is just a restatement of (6.6). ∎

We turn to the proof of Proposition 8.1.1.

Proof of Proposition 8.1.1. We take all spacetime norms over $I \times \mathbb{R}^3$.

We let $0 < \eta \ll 1$ and choose

$$N > \max\{N_1(\tfrac{1}{2}, \eta), N_1(\tfrac{1+s_c}{2}, \eta^2), N_1(1, \eta), \tfrac{1}{\eta^2} N_2(\eta^2)\},$$

where N_1 and N_2 are as in Lemma 8.1.2. In particular, interpolating (8.2) and (8.3) with $s = (1 + s_c)/2$, we get

$$\| |\nabla|^{(1+s_c)/2} u_{\leq N} \|_{L_t^4 L_x^3} \lesssim_u \eta N^{(1-s_c)/2} (1 + N^{2s_c - 1} K_I)^{1/4}. \tag{8.5}$$

Moreover, as $\eta^2 N > N_2$, we can apply (8.4) to $u_{>\eta^2 N}$ to get

$$\|u_{>\eta^2 N}\|_{L_t^2 L_x^6} \lesssim \eta N^{-s_c} (1 + N^{2s_c - 1} K_I)^{1/2}. \tag{8.6}$$

We define the Morawetz action

$$\mathrm{Mor}(t) := 2 \,\mathrm{Im} \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot \nabla u_{\leq N}(t,x)\, \bar{u}_{\leq N}(t,x)\, dx.$$

A standard computation using $(i\partial_t + \Delta) u_{\leq N} = P_{\leq N}\big(F(u)\big)$ gives

$$\partial_t \mathrm{Mor}(t) \gtrsim \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot \{P_{\leq N}\big(F(u)\big), u_{\leq N}\}_P \, dx, \tag{8.7}$$

where the *momentum bracket* $\{\cdot, \cdot\}_P$ is defined by $\{f, g\}_P := \mathrm{Re}(f \nabla \bar{g} - g \nabla \bar{f})$.

Noting that $\{F(u), u\}_P = -\frac{p}{p+2} \nabla(|u|^{p+2})$, we integrate by parts in (8.7) to get

$$\partial_t \mathrm{Mor}(t) \gtrsim \int \frac{|u_{\leq N}(t,x)|^{p+2}}{|x|}\, dx + \int \frac{x}{|x|} \cdot \{P_{\leq N}\big(F(u)\big) - F(u_{\leq N}), u_{\leq N}\}_P \, dx.$$

Thus, by the fundamental theorem of calculus we have

$$\begin{aligned} &\iint_{I \times \mathbb{R}^3} \frac{|u_{\leq N}(t,x)|^{p+2}}{|x|}\, dx\, dt \\ &\qquad \lesssim \|\mathrm{Mor}\|_{L_t^\infty(I)} + \left| \iint_{I \times \mathbb{R}^3} \frac{x}{|x|} \cdot \{P_{\leq N}\big(F(u)\big) - F(u_{\leq N}), u_{\leq N}\}_P \, dx\, dt \right|. \end{aligned}$$

To complete the proof of Proposition 8.1.1, it therefore suffices to show

$$\|\mathrm{Mor}\|_{L_t^\infty(I)} \lesssim_u \eta N^{1-2s_c}, \tag{8.8}$$

$$\left| \iint_{I\times\mathbb{R}^3} \frac{x}{|x|} \cdot \{P_{\leq N}\big(F(u)\big) - F(u_{\leq N}), u_{\leq N}\}_P \, dx\, dt \right| \lesssim_u \eta(N^{1-2s_c} + K_I). \tag{8.9}$$

To prove (8.8), we use Bernstein, (2.4), and (8.3) to estimate

$$\begin{aligned} \|\mathrm{Mor}\|_{L_t^\infty(I)} &\lesssim \||\nabla|^{-1/2}\nabla u_{\leq N}\|_{L_t^\infty L_x^2} \||\nabla|^{1/2}(\tfrac{x}{|x|}u_{\leq N})\|_{L_t^\infty L_x^2} \\ &\lesssim \||\nabla|^{1/2} u_{\leq N}\|_{L_t^\infty L_x^2}^2 \lesssim_u \eta N^{1-2s_c}. \end{aligned}$$

We now turn to (8.9). We begin by rewriting

$$\begin{aligned} &\{P_{\leq N}\big(F(u)\big) - F(u_{\leq N}), u_{\leq N}\}_P \\ &\quad = \{F(u) - F(u_{\leq N}), u_{\leq N}\}_P - \{P_{>N}\big(F(u)\big), u_{\leq N}\}_P \\ &\quad =: I + II. \end{aligned}$$

Writing

$$I = \emptyset\big\{[F(u) - F(u_{\leq N})]\nabla u_{\leq N} + u_{\leq N}\nabla[F(u) - F(u_{\leq N})]\big\}$$

and integrating by parts in the second term, we find that the contribution of I to (8.9) is controlled by

$$\|\nabla u_{\leq N}\big(F(u) - F(u_{\leq N})\big)\|_{L_{t,x}^1} \tag{8.10}$$

$$+ \|\tfrac{1}{|x|}u_{\leq N}\big(F(u) - F(u_{\leq N})\big)\|_{L_{t,x}^1}. \tag{8.11}$$

Similarly, writing

$$II = \emptyset\big\{P_{>N}\big(F(u)\big)\nabla u_{\leq N} + \nabla P_{>N}\big(F(u)\big)u_{\leq N}\big\}$$

and integrating by parts in the second term, we find that the contribution of II to (8.9) is controlled by

$$\|\nabla u_{\leq N} P_{>N}\big(F(u)\big)\|_{L_{t,x}^1} \tag{8.12}$$

$$+ \|\tfrac{1}{|x|}u_{\leq N} P_{>N}\big(F(u)\big)\|_{L_{t,x}^1} \tag{8.13}$$

To complete the proof of (8.9), it therefore suffices to show that the error terms (8.10) through (8.13) are acceptable, in the sense that they can be controlled by $\eta(N^{1-2s_c}+K_I)$.

We first turn to (8.10). Using Hölder, (4.1), (8.2), and (8.4), we estimate

$$\begin{aligned}
(8.10) &\lesssim \|\nabla u_{\leq N}\|_{L_t^2 L_x^6} \|u_{>N} \emptyset(u^p)\|_{L_t^2 L_x^{6/5}} \\
&\lesssim \|\nabla u_{\leq N}\|_{L_t^2 L_x^6} \|u_{>N}\|_{L_t^2 L_x^6} \|u\|^p_{L_t^\infty L_x^{3p/2}} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}$$

which is acceptable.

We next turn to (8.11). We first write

$$(8.11) \lesssim \|\tfrac{1}{|x|} u_{\leq N}(u_{>N})^{p+1}\|_{L^1_{t,x}} + \|\tfrac{1}{|x|}(u_{\leq N})^{p+1} u_{>N}\|_{L^1_{t,x}}$$

For the first piece, we use Hölder, Hardy, Bernstein, (4.1), (8.3), and (8.4) to estimate

$$\begin{aligned}
\|\tfrac{1}{|x|} u_{\leq N}(u_{>N})^{p+1}\|_{L^1_{t,x}} &\lesssim \|\tfrac{1}{|x|} u_{\leq N}\|_{L_t^\infty L_x^{3p/2}} \|u_{>N}\|^2_{L_t^2 L_x^6} \|u\|^{p-1}_{L_t^\infty L_x^{3p/2}} \\
&\lesssim_u \|\nabla u_{\leq N}\|_{L_t^\infty L_x^{3p/2}} \|u_{>N}\|^2_{L_t^2 L_x^6} \\
&\lesssim_u N^{s_c} \|\nabla u_{\leq N}\|_{L_t^\infty L_x^2} \|u_{>N}\|^2_{L_t^2 L_x^6} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}$$

which is acceptable.

For the second piece, we use Hölder, Hardy, the chain rule, (4.1), (8.2), and (8.4) to estimate

$$\begin{aligned}
\|\tfrac{1}{|x|}(u_{\leq N})^{p+1} u_{>N}\|_{L^1_{t,x}} &\lesssim \|\tfrac{1}{|x|}(u_{\leq N})^{p+1}\|_{L_t^2 L_x^{6/5}} \|u_{>N}\|_{L_t^2 L_x^6} \\
&\lesssim \|\nabla (u_{\leq N})^{p+1}\|_{L_t^2 L_x^{6/5}} \|u_{>N}\|_{L_t^2 L_x^6} \\
&\lesssim \|u\|^p_{L_t^\infty L_x^{3p/2}} \|\nabla u_{\leq N}\|_{L_t^2 L_x^6} \|u_{>N}\|_{L_t^2 L_x^6} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}$$

which is acceptable. This completes the estimation of (8.11).

We next turn to (8.12). We first write

$$(8.12) \lesssim \|\nabla u_{\leq N} P_{>N}\big(F(u_{\leq \eta^2 N})\big)\|_{L^1_{t,x}} + \|\nabla u_{\leq N} P_{>N}\big(F(u) - F(u_{\leq \eta^2 N})\big)\|_{L^1_{t,x}}.$$

For the first piece, we use Hölder, Bernstein, the chain rule, (4.1), and (8.2) to estimate

$$\begin{aligned}
\|\nabla u_{\leq N} P_{>N}\big(F(u_{\leq \eta^2 N})\big)\|_{L^1_{t,x}} &\lesssim N^{-1}\|\nabla u_{\leq N}\|_{L^2_t L^6_x}\|\nabla F(u_{\leq \eta^2 N})\|_{L^2_t L^{6/5}_x} \\
&\lesssim N^{-1}\|\nabla u_{\leq N}\|_{L^2_t L^6_x}\|u\|^p_{L^\infty_t L^{3p/2}_x}\|\nabla u_{\leq \eta^2 N}\|_{L^2_t L^6_x} \\
&\lesssim_u \eta N^{1-2s_c}(1 + N^{2s_c - 1}K_I),
\end{aligned}$$

which is acceptable.

For the second piece, we use Hölder, (4.1), (8.2), and (8.6) to estimate

$$\begin{aligned}
\|\nabla u_{\leq N} P_{>N}\big(F(u) - F(u_{\leq \eta^2 N})\big)\|_{L^1_{t,x}} &\lesssim \|\nabla u_{\leq N}\|_{L^2_t L^6_x}\|u_{>\eta^2 N}\|_{L^2_t L^6_x}\|u\|^p_{L^\infty_t L^{3p/2}_x} \\
&\lesssim_u \eta N^{1-2s_c}(1 + N^{2s_c - 1}K_I),
\end{aligned}$$

which is acceptable. This completes the estimation of (8.12).

Finally, we turn to (8.13). We first write

$$(8.13) \lesssim \|\tfrac{1}{|x|} u_{\leq N} P_{>N}\big(F(u_{\leq \frac{N}{4}})\big)\|_{L^1_{t,x}} + \|\tfrac{1}{|x|} u_{\leq N} P_{>N}\big(F(u) - F(u_{\leq \frac{N}{4}})\big)\|_{L^1_{t,x}}.$$

For the first piece, we begin by noting that

$$P_{>N}\big(F(u_{\leq \frac{N}{4}})\big) = P_{>N}\big(P_{>\frac{N}{2}}(|u_{\leq \frac{N}{4}}|^p) u_{\leq \frac{N}{4}}\big).$$

Thus, using Cauchy–Schwarz, Hölder, Hardy, maximal function estimates (cf. (2.5)), Bernstein, Sobolev embedding, (4.1), (8.2), and (8.5), we can estimate

$$
\begin{aligned}
&\big\|\tfrac{1}{|x|}u_{\leq N}P_{>N}\big(F(u_{\leq \frac{N}{4}})\big)\big\|_{L^1_{t,x}} \\
&\lesssim \big\|\tfrac{1}{|x|}u_{\leq N}M\big(P_{>\frac{N}{2}}(|u_{\leq\frac{N}{4}}|^p)u_{\leq\frac{N}{4}}\big)\big\|_{L^1_{t,x}} \\
&\lesssim \big\|\tfrac{1}{|x|}u_{\leq N}\big[M\big(\big|P_{>\frac{N}{2}}(|u_{\leq\frac{N}{4}}|^p)\big|^2\big)\big]^{1/2}\big[M\big(|u_{\leq\frac{N}{4}}|^2\big)\big]^{1/2}\big\|_{L^1_{t,x}} \\
&\lesssim \big\|\tfrac{1}{|x|^{1/2}}u_{\leq N}\big\|_{L^4_tL^{\frac{12p}{4+p}}_x}\big\|M\big(\big|P_{>\frac{N}{2}}(|u_{\leq\frac{N}{4}}|^p)\big|^2\big)\big\|^{1/2}_{L^1_tL^{\frac{3p}{5p-4}}_x}\big\|\tfrac{1}{|x|}M\big(|u_{\leq\frac{N}{4}}|^2\big)\big\|^{1/2}_{L^2_tL^{\frac{6p}{4+p}}_x} \\
&\lesssim \big\||\nabla|^{1/2}u_{\leq N}\big\|_{L^4_tL^{\frac{12p}{4+p}}_x}\big\|P_{>\frac{N}{2}}(|u_{\leq\frac{N}{4}}|^p)\big\|_{L^2_tL^{\frac{6p}{5p-4}}_x}\big\|\nabla M(|u_{\leq\frac{N}{4}}|^2)\big\|^{1/2}_{L^2_tL^{\frac{6p}{4+p}}_x} \\
&\lesssim N^{-1}\big\||\nabla|^{(1+s_c)/2}u_{\leq N}\big\|_{L^4_tL^3_x}\big\|\nabla(|u_{\leq\frac{N}{4}}|^p)\big\|_{L^2_tL^{\frac{6p}{5p-4}}_x}\big\|\nabla(|u_{\leq\frac{N}{4}}|^2)\big\|^{1/2}_{L^2_tL^{\frac{6p}{4+p}}_x} \\
&\lesssim N^{-1}\big\||\nabla|^{(1+s_c)/2}u_{\leq N}\big\|_{L^4_tL^3_x}\|u\|^{p-1}_{L^\infty_tL^{3p/2}_x}\|\nabla u_{\leq\frac{N}{4}}\|_{L^2_tL^6_x}\|u\|^{1/2}_{L^\infty_tL^{3p/2}_x}\|\nabla u_{\leq\frac{N}{4}}\|^{1/2}_{L^2_tL^6_x} \\
&\lesssim_u N^{-1}\big\||\nabla|^{(1+s_c)/2}u_{\leq N}\big\|_{L^4_tL^3_x}\|\nabla u_{\leq\frac{N}{4}}\|_{L^2_tL^6_x}\|\nabla u_{\leq\frac{N}{4}}\|^{1/2}_{L^2_tL^6_x} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}
$$

which is acceptable.

For the second piece, we write

$$
\begin{aligned}
\big\|\tfrac{1}{|x|}u_{\leq N}P_{>N}\big(F(u)-F(u_{\leq\frac{N}{4}})\big)\big\|_{L^1_{t,x}} &\lesssim \big\|\tfrac{1}{|x|}u_{\leq N}M\big(u_{>\frac{N}{4}}(u_{\leq\frac{N}{4}})^p\big)\big\|_{L^1_{t,x}} && (8.14)\\
&\quad+\big\|\tfrac{1}{|x|}u_{\leq N}M\big((u_{>\frac{N}{4}})^{p+1}\big)\big\|_{L^1_{t,x}}. && (8.15)
\end{aligned}
$$

For (8.14), we use Cauchy–Schwarz, Hölder, Hardy, the maximal function estimate, Sobolev embedding, (4.1), (8.2), (8.4), and (8.5) to estimate

$$
\begin{aligned}
&(8.14) \\
&\lesssim \big\|\tfrac{1}{|x|}u_{\leq N}\big[M\big(|u_{>\frac{N}{4}}|^2|u_{\leq\frac{N}{4}}|^{2(p-1)}\big)\big]^{1/2}\big[M\big(|u_{\leq\frac{N}{4}}|^2\big)\big]^{1/2}\big\|_{L^1_{t,x}} \\
&\lesssim \big\|\tfrac{1}{|x|^{1/2}}u_{\leq N}\big\|_{L^4_tL^{\frac{12p}{4+p}}_x}\big\|M\big(|u_{>\frac{N}{4}}|^2|u_{\leq\frac{N}{4}}|^{2(p-1)}\big)\big\|^{1/2}_{L^1_tL^{\frac{3p}{5p-4}}_x}\big\|\tfrac{1}{|x|}M\big(|u_{\leq\frac{N}{4}}|^2\big)\big\|^{1/2}_{L^2_tL^{\frac{6p}{4+p}}_x} \\
&\lesssim \big\||\nabla|^{1/2}u_{\leq N}\big\|_{L^4_tL^{\frac{12p}{4+p}}_x}\|u_{>\frac{N}{4}}\|_{L^2_tL^6_x}\|u\|^{p-1}_{L^\infty_tL^{3p/2}_x}\big\|\nabla M\big(|u_{\leq\frac{N}{4}}|^2\big)\big\|^{1/2}_{L^2_tL^{\frac{6p}{4+p}}_x} \\
&\lesssim \big\||\nabla|^{(1+s_c)/2}u_{\leq N}\big\|_{L^4_tL^3_x}\|u_{>\frac{N}{4}}\|_{L^2_tL^6_x}\|u\|^{p-1/2}_{L^\infty_tL^{3p/2}_x}\|\nabla u_{\leq\frac{N}{4}}\|^{1/2}_{L^2_tL^6_x} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}
$$

which is acceptable.

Finally, we use Hölder, Hardy, Bernstein, (4.1), (8.3), and (8.4) to estimate

$$\begin{aligned}(8.15) &\lesssim \big\|\tfrac{1}{|x|}u_{\leq N}\big\|_{L_t^\infty L_x^{3p/2}} \|u_{>\frac{N}{4}}\|_{L_t^2 L_x^6}^2 \|u\|_{L_t^\infty L_x^{3p/2}}^{p-1} \\ &\lesssim_u \|\nabla u_{\leq N}\|_{L_t^\infty L_x^{3p/2}} \|u_{>\frac{N}{4}}\|_{L_t^2 L_x^6}^2 \\ &\lesssim_u N^{s_c} \|\nabla u_{\leq N}\|_{L_t^\infty L_x^2} \|u_{>\frac{N}{4}}\|_{L_t^2 L_x^6}^2 \\ &\lesssim_u \eta N^{1-2s_c}(1 + N^{2s_c-1}K_I),\end{aligned}$$

which is acceptable. This completes the estimation of the final error term (8.13), which in turn completes the proof of Proposition 8.1.1. ∎

8.2 Frequency-localized Lin–Strauss Morawetz, $s_c > 1/2$

In this section, we use Proposition 6.2.1 to prove a frequency-localized Lin–Strauss Morawetz inequality, which we use to rule out the quasi-soliton scenario. As $s_c > 1/2$, we prove an estimate that is localized to high frequencies.

The main result of this section is the following.

Proposition 8.2.1 (Frequency-localized Morawetz) *Let $u : [0, T_{max}) \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.3. Let $I \subset [0, T_{max})$ be a compact time interval, which is a contiguous union of characteristic subintervals J_k. Then for any $\eta > 0$, there exists $N_0 = N_0(\eta) > 0$ such that for $N < N_0$, we have*

$$\iint_{I\times\mathbb{R}^3} \frac{|u_{>N}(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim_u \eta(N^{1-2s_c} + K_I), \tag{8.16}$$

where K_I is as in (6.20).

To prove Proposition 8.2.1, we begin as in the proof of the standard Lin–Strauss Morawetz inequality (1.7). We truncate the low frequencies of the solution and work with $u_{>N}$ for some $N > 0$. As $u_{>N}$ is not a true solution to (1.1), we need to control error terms arising from this frequency projection. To do this, we choose N small enough to capture 'most' of the solution and use the estimates proved in Chapter 6. We make these notions precise in the following lemma.

Lemma 8.2.2 (High and low frequency control) *Let u, I, K_I be as above. With all spacetime norms over $I \times \mathbb{R}^3$, we have the following.*

For any $N > 0$, we have

$$\|u_{>N}\|_{L_t^2 L_x^6} \lesssim_u N^{-s_c}(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.17}$$

For any $\eta > 0$, there exists $N_1 = N_1(\eta)$ so that for $N < N_1$, we have

$$\||\nabla|^{1/2} u_{>N}\|_{L_t^\infty L_x^2} \lesssim_u \eta N^{1/2-s_c}. \tag{8.18}$$

For any $\eta > 0$, there exists $N_2 = N_2(\eta)$ so that for $N < N_2$, we have

$$\||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^6} \lesssim_u \eta(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.19}$$

Proof. For (8.17), we use Bernstein and (6.22) to estimate

$$\begin{aligned}
\|u_{>N}\|_{L_t^2 L_x^6} &\lesssim \sum_{M>N} M^{-s_c}\||\nabla|^{s_c} u_M\|_{L_t^2 L_x^6} \\
&\lesssim_u \sum_{M>N} M^{-s_c}(1 + M^{2s_c-1}K_I)^{1/2} \\
&\lesssim_u N^{-s_c}(1 + N^{2s_c-1}K_I)^{1/2}.
\end{aligned}$$

For (8.18), we let $\eta > 0$. Using almost periodicity and the fact that $\inf N(t) \geq 1$, we may find $c(\eta) > 0$ so that $\||\nabla|^{s_c} u_{\leq c(\eta)}\|_{L_t^\infty L_x^2} < \eta$. Thus Bernstein gives

$$\begin{aligned}
&\||\nabla|^{1/2} u_{>N}\|_{L_t^\infty L_x^2} \\
&\quad \lesssim c(\eta)^{1/2-s_c}\||\nabla|^{s_c} u_{>c(\eta)}\|_{L_t^\infty L_x^2} + N^{1/2-s_c}\||\nabla|^{s_c} u_{N<\cdot\leq c(\eta)}\|_{L_t^\infty L_x^2} \\
&\quad \lesssim_u c(\eta)^{1/2-s_c} + \eta N^{1/2-s_c}.
\end{aligned}$$

Choosing $N_1 \ll \eta^{1/(s_c-1/2)} c(\eta)$, we recover (8.18).

Finally, we note that (8.19) is just a restatement of (6.23). ∎

We turn to the proof of Proposition 8.2.1.

Proof of Proposition 8.2.1 Throughout the proof, we take all spacetime norms over $I \times \mathbb{R}^3$.

We let $0 < \eta \ll 1$ and choose

$$N < \min\{N_1(\eta), \eta^2 N_2(\eta^{2s_c})\},$$

where N_1 and N_2 are as in Lemma 8.2.2. In particular, we note that (8.17) gives

$$\|u_{>N/\eta^2}\|_{L_t^2 L_x^6} \lesssim_u \eta N^{-s_c}(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.20}$$

Moreover, as $N/\eta^2 < N_2(\eta^{2s_c})$, we can apply (8.19) to get

$$\||\nabla|^{s_c} u_{\leq N/\eta^2}\|_{L_t^2 L_x^6} \lesssim_u \eta(1 + N^{2s_c-1}K_I)^{1/2}. \tag{8.21}$$

We define the Morawetz action

$$\mathrm{Mor}(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot \nabla u_{>N}(t,x) \bar{u}_{>N}(t,x)\,dx.$$

A standard computation using $(i\partial_t + \Delta)u_{>N} = P_{>N}\big(F(u)\big)$ gives

$$\partial_t \mathrm{Mor}(t) \gtrsim \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot \{P_{>N}\big(F(u)\big), u_{>N}\}_P\,dx,$$

where the *momentum bracket* $\{\cdot,\cdot\}_P$ is defined by $\{f,g\}_P := \mathrm{Re}(f\nabla\bar{g} - g\nabla\bar{f})$. Thus, by the fundamental theorem of calculus, we get

$$\iint_{I\times\mathbb{R}^3} \frac{x}{|x|} \cdot \{P_{>N}\big(F(u)\big), u_{>N}\}_P\,dx \lesssim \|\mathrm{Mor}\|_{L_t^\infty(I)}. \tag{8.22}$$

Noting that $\{F(u), u\}_P = -\frac{p}{p+2}\nabla(|u|^{p+2})$, we may write

$$\begin{aligned}
&\{P_{>N}\big(F(u)\big), u_{>N}\}_P \\
&\quad = \{F(u), u\}_P - \{F(u_{\leq N}), u_{\leq N}\}_P \\
&\qquad - \{F(u) - F(u_{\leq N}), u_{\leq N}\}_P - \{P_{\leq N}\big(F(u)\big), u_{>N}\}_P \\
&\quad = -\tfrac{p}{p+2}\nabla(|u|^{p+2} - |u_{\leq N}|^{p+2}) - \{F(u) - F(u_{\leq N}), u_{\leq N}\}_P \\
&\qquad - \{P_{\leq N}\big(F(u)\big), u_{>N}\}_P \\
&\quad =: I + II + III.
\end{aligned}$$

Integrating by parts, we see that I contributes to the left-hand side of (8.22) a multiple of

$$\iint_{I\times\mathbb{R}^3} \frac{|u_{>N}(t,x)|^{p+2}}{|x|}\,dx\,dt$$

and to the right-hand side of (8.22) a multiple of

$$\|\tfrac{1}{|x|}(|u|^{p+2}-|u_{>N}|^{p+2}-|u_{\leq N}|^{p+2})\|_{L^1_{t,x}}. \tag{8.23}$$

For term II, we use $\{f,g\}_P = \nabla\O(fg)+\O(f\nabla g)$. When the derivative hits the product, we integrate by parts. We find that II contributes to the right-hand side of (8.22) a multiple of

$$\|\tfrac{1}{|x|}u_{\leq N}[F(u)-F(u_{\leq N})]\|_{L^1_{t,x}} \tag{8.24}$$

$$+\|\nabla u_{\leq N}[F(u)-F(u_{\leq N})]\|_{L^1_{t,x}}. \tag{8.25}$$

Finally, for III, we integrate by parts when the derivative hits $u_{>N}$. We find that III contributes to the right-hand side of (8.22) a multiple of

$$\|\tfrac{1}{|x|}u_{>N}P_{\leq N}\big(F(u)\big)\|_{L^1_{t,x}} \tag{8.26}$$

$$+\|u_{>N}\nabla P_{\leq N}\big(F(u)\big)\|_{L^1_{t,x}}. \tag{8.27}$$

Thus, continuing from (8.22), we see that to complete the proof of Proposition 8.2.1 it will suffice to show that

$$\|\mathrm{Mor}\|_{L^\infty_t(I)} \lesssim_u \eta N^{1-2s_c} \tag{8.28}$$

and that the error terms (8.23) through (8.27) are acceptable, in the sense that they can be controlled by $\eta(N^{1-2s_c}+K_I)$.

To prove (8.28), we use Bernstein, (2.4), (8.18) to estimate

$$\begin{aligned}\|\mathrm{Mor}\|_{L^\infty_t(I)} &\lesssim \||\nabla|^{-1/2}\nabla u_{>N}\|_{L^\infty_t L^2_x}\||\nabla|^{1/2}(\tfrac{x}{|x|}u_{>N})\|_{L^\infty_t L^2_x}\\ &\lesssim \||\nabla|^{1/2}u_{>N}\|^2_{L^\infty_t L^2_x} \lesssim_u \eta N^{1-2s_c}.\end{aligned}$$

We next turn to the estimation of the error terms (8.23) through (8.27).

For (8.23), we first write

$$(8.23) \lesssim \|\tfrac{1}{|x|}(u_{\leq N})^{p+1}u_{>N}\|_{L^1_{t,x}} \tag{8.29}$$

$$+ \|\tfrac{1}{|x|}u_{\leq N}(u_{>N})^{p+1}\|_{L^1_{t,x}}. \tag{8.30}$$

For (8.29), we use Hölder, Hardy, the chain rule, Bernstein, (4.1), (8.17), and (8.19) to estimate

$$\begin{aligned}
\|\tfrac{1}{|x|}(u_{\leq N})^{p+1}u_{>N}\|_{L^1_{t,x}} &\lesssim \|\tfrac{1}{|x|}(u_{\leq N})^{p+1}\|_{L^2_tL^{6/5}_x}\|u_{>N}\|_{L^2_tL^6_x} \\
&\lesssim \|\nabla(u_{\leq N})^{p+1}\|_{L^2_tL^{6/5}_x}\|u_{>N}\|_{L^2_tL^6_x} \\
&\lesssim \|u\|^p_{L^\infty_tL^{3p/2}_x}\|\nabla u_{\leq N}\|_{L^2_tL^6_x}\|u_{>N}\|_{L^2_tL^6_x} \\
&\lesssim_u \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),
\end{aligned}$$

which is acceptable.

For (8.30), we consider two cases. If $|u_{\leq N}| \ll |u_{>N}|$, then we can absorb this term into the left-hand side of (8.22), provided we can show

$$\|\tfrac{1}{|x|}|u_{>N}|^{p+2}\|_{L^1_{t,x}} < \infty. \tag{8.31}$$

Otherwise, we are back in the situation of (8.29), which we have already handled. Thus, to render (8.30) an acceptable error term it suffices to establish (8.31). To this end, we use Hardy, Sobolev embedding, Bernstein, and Lemma 4.1.5 to estimate

$$\begin{aligned}
\|\tfrac{1}{|x|}|u_{>N}|^{p+2}\|_{L^1_{t,x}} &\lesssim \||x|^{-\frac{1}{p+2}}u_{>N}\|^{p+2}_{L^{p+2}_{t,x}} \lesssim \||\nabla|^{\frac{1}{p+2}}u_{>N}\|^{p+2}_{L^{p+2}_{t,x}} \\
&\lesssim \||\nabla|^{\frac{3p-2}{2(p+2)}}u_{>N}\|^{p+2}_{L^{p+2}_tL_x^{\frac{6(p+2)}{3p+2}}} \lesssim N^{1-2s_c}\||\nabla|^{s_c}u\|^{p+2}_{L^{p+2}_tL_x^{\frac{6(p+2)}{3p+2}}} \\
&\lesssim_u N^{1-2s_c}(1+\textstyle\int_I N(t)^2\,dt) < \infty.
\end{aligned}$$

We next turn to (8.24). Writing

$$\begin{aligned}
\|\tfrac{1}{|x|}u_{\leq N}[F(u)-F(u_{\leq N})]\|_{L^1_{t,x}} &\lesssim \|\tfrac{1}{|x|}(u_{\leq N})^{p+1}u_{>N}\|_{L^1_{t,x}} \\
&\quad + \|\tfrac{1}{|x|}u_{\leq N}(u_{>N})^{p+1}\|_{L^1_{t,x}},
\end{aligned}$$

we recognize the error terms that we just estimated, namely (8.29) and (8.30). Thus (8.24) is acceptable.

For (8.25), we use Hölder, (4.1) (8.17), and (8.19) to estimate

$$\begin{aligned}(8.25) &\lesssim \|\nabla u_{\le N}\|_{L_t^2 L_x^6}\|u_{>N}\|_{L_t^2 L_x^6}\|u\|^p_{L_t^\infty L_x^{3p/2}} \\ &\lesssim \eta N^{1-2s_c}(1+N^{2s_c-1}K_I),\end{aligned}$$

which is acceptable.

Finally, for (8.26) and (8.27), we first use Hardy to estimate

$$\begin{aligned}&(8.26)+(8.27) \\ &\quad\lesssim \|u_{>N}\|_{L_t^2 L_x^6}\|\tfrac{1}{|x|}P_{\le N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}} + \|u_{>N}\|_{L_t^2 L_x^6}\|\nabla P_{\le N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}} \\ &\quad\lesssim \|u_{>N}\|_{L_t^2 L_x^6}\|\nabla P_{\le N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}}\end{aligned}$$

Thus, in light of (8.17) it suffices to prove

$$\|\nabla P_{\le N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}} \lesssim_u \eta N^{1-s_c}(1+N^{2s_c-1}K_I)^{1/2}.$$

To this end, we use Hölder, Bernstein, the fractional chain rule, (4.1), (8.20), and (8.21) to estimate

$$\begin{aligned}&\|\nabla P_{\le N}\big(F(u)\big)\|_{L_t^2 L_x^{6/5}} \\ &\quad\lesssim N\|F(u)-F(u_{\le N/\eta^2})\|_{L_t^2 L_x^{6/5}} + N^{1-s_c}\||\nabla|^{s_c}F(u_{\le N/\eta^2})\|_{L_t^2 L_x^{6/5}}. \\ &\quad\lesssim N\|u\|^p_{L_t^\infty L_x^{3p/2}}\|u_{>N/\eta^2}\|_{L_t^2 L_x^6} + N^{1-s_c}\|u\|^p_{L_t^\infty L_x^{3p/2}}\||\nabla|^{s_c}u_{\le N/\eta^2}\|_{L_t^2 L_x^6} \\ &\quad\lesssim_u \eta N^{1-s_c}(1+N^{2s_c-1}K_I)^{1/2}.\end{aligned}$$

This completes the proof of Proposition 8.2.1. ∎

8.3 Frequency-localized interaction Morawetz

In this section, we use Proposition 6.3.1 to prove a frequency-localized interaction Morawetz estimate inequality, which we use to rule out the quasi-soliton scenario of Theorem 4.5.2.

We will prove an estimate that is localized to high frequencies.

The main result of this section is the following.

Proposition 8.3.1 (Frequency-localized interaction Morawetz) *Let $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ be an almost periodic solution as in Theorem 4.5.2. Let $I \subset [0, T_{max})$ be a compact time interval, which is a union of contiguous subintervals J_k. Then for any $\eta > 0$, there exists $N_0 = N_0(\eta)$ such that for any $N \leq N_0$, we have*

$$-\int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u_{>N}(t,y)|^2 \Delta(\tfrac{1}{|x-y|}) |u_{>N}(t,x)|^2 \, dx\, dy\, dt \lesssim_u \eta(N^{1-4s_c} + K_I), \tag{8.32}$$

where K_I is as in (6.29)*.*

Before we begin the proof of Proposition 8.3.1, we recall a general form of the interaction Morawetz inequality, introduced originally in [14] (for more discussion, see also [38] and the references cited therein). We will essentially follow the presentation in [69, Section 5].

For a fixed function $a : \mathbb{R}^d \to \mathbb{R}$ and φ solving $(i\partial_t + \Delta)\varphi = \mathcal{N}$, we define the interaction Morawetz action by

$$M(t) = 2\,\mathrm{Im} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2 a_k(x-y)(\varphi_k \bar{\varphi})(t,x)\, dx\, dy,$$

where subscripts denote spatial derivatives and repeated indices are summed. If we define the mass bracket

$$\{f, g\}_m := \mathrm{Im}(f\bar{g})$$

and the momentum bracket

$$\{f, g\}_{\mathcal{P}} := \mathrm{Re}(f\nabla\bar{g} - g\nabla\bar{f}),$$

then one can show

$$\begin{aligned}\partial_t M(t) = &- \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2 a_{jjkk}(x-y)|\varphi(t,x)|^2\,dx\,dy \\ &+ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2 4a_{jk}(x-y)\mathrm{Re}(\bar\varphi_j\varphi_k)(t,x)\,dx\,dy \qquad (8.33)\\ &- \iint_{\mathbb{R}^d\times\mathbb{R}^d} 2\,\mathrm{Im}(\bar\varphi\varphi_k)(t,y)a_{jk}(x-y)2\,\mathrm{Im}(\bar\varphi\varphi_j)(t,x)\,dx\,dy \qquad (8.34)\\ &+ \iint_{\mathbb{R}^d\times\mathbb{R}^d} 2\{\mathcal{N},\varphi\}_m(t,y)a_j(x-y)\;2\,\mathrm{Im}(\bar\varphi\varphi_j)(t,x)\,dx\,dy \\ &+ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2\;2\,\nabla a(x-y)\cdot\{\mathcal{N},\varphi\}_{\mathcal{P}}(t,x)\,dx\,dy.\end{aligned}$$

To prove Proposition 8.3.1, we will use $a(x)=|x|$. Note that in this case, we have

$$\begin{cases} a_j(x) = \frac{x_j}{|x|}, \\ a_{jk}(x) = \frac{\delta_{jk}}{|x|} - \frac{x_jx_k}{|x|^3}, \\ \Delta a(x) = \frac{d-1}{|x|}, \\ \Delta\Delta a(x) = -(d-1)\Delta(\frac{1}{|x|}). \end{cases}$$

For this choice of a, one can also show that (8.33) + (8.34) ≥ 0. For details, see [69, Lemma 5.4]. Thus, integrating $\partial_t M$ over I, we arrive at the following

Lemma 8.3.2 (Interaction Morawetz inequality)

$$\begin{aligned} &-\int_I\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2\Delta(\tfrac{1}{|\cdot|})(x-y)|\varphi(t,x)|^2\,dx\,dy\,dt \\ &+\int_I\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(t,y)|^2\tfrac{x-y}{|x-y|}\cdot\{\mathcal{N},\varphi\}_{\mathcal{P}}(t,x)\,dx\,dy\,dt \\ &\lesssim \sup_{t\in I}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\varphi(t,y)|^2\tfrac{x-y}{|x-y|}\cdot\nabla\varphi(t,x)\bar\varphi(t,x)\,dx\,dy \\ &\quad+\left|\int_I\iint_{\mathbb{R}^d\times\mathbb{R}^d}\{\mathcal{N},\varphi\}_m(t,y)\tfrac{x-y}{|x-y|}\cdot\nabla\varphi(t,x)\bar\varphi(t,x)\,dx\,dy\,dt\right|. \end{aligned}$$

To prove Proposition 8.3.1, we will apply this estimate with $\varphi = u_{>N}$, with N chosen small enough to capture 'most' of the solution. To make this idea more precise, we first need to record the following corollary of Proposition 6.3.1.

Lemma 8.3.3 (Low and high frequency control) *Let u, I, and K_I be as in Proposition 8.3.1.*

For any frequency $N > 0$, we have

$$\|u_{>N}\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim_u N^{-s_c}(1+N^{4s_c-1}K_I)^{\frac{1}{q}} \tag{8.35}$$

for all $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$ with $q > 4-\frac{2p}{dp-4}$.

For any $\eta > 0$, there exists $N_0 = N_0(\eta)$ such that for all $N \le N_0$, we have

$$\||\nabla|^{s_c} u_{\le N}\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim_u \eta(1+N^{4s_c-1}K_I)^{\frac{1}{q}} \tag{8.36}$$

for all $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$ with $q \ge 2$.

Proof of Lemma 8.3.3 We first show (8.35). For fixed $\alpha > s_c - \frac{1}{2}$, we can use Bernstein and (6.30) to see

$$\begin{aligned}
\||\nabla|^{-\alpha} u_{>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)} &\lesssim \sum_{M>N} M^{-\alpha-s_c}\||\nabla|^{s_c} u_M\|_{L_t^2 L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)} \\
&\lesssim_u \sum_{M>N} M^{-\alpha-s_c}(1+M^{2s_c-\frac{1}{2}}K_I^{\frac{1}{2}}) \\
&\lesssim_u N^{-\alpha-s_c}(1+N^{4s_c-1}K_I)^{\frac{1}{2}}. \qquad (8.37)
\end{aligned}$$

Now, take (q,r) with $2 < q \le \infty$ and $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$, and define $\alpha = \frac{(q-2)(dp-4)}{4p}$. Notice that $\alpha > s_c - \frac{1}{2}$ exactly when $q > 4 - \frac{2p}{dp-4}$. Thus, in this case, we get by interpolation and (8.37) that

$$\begin{aligned}
\|u_{>N}\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} &\lesssim \||\nabla|^{-\alpha} u_{>N}\|^{\frac{2}{q}}_{L_t^2 L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)} \||\nabla|^{s_c} u_{>N}\|^{1-\frac{2}{q}}_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \\
&\lesssim_u \left[N^{-\frac{qs_c}{2}}(1+N^{4s_c-1}K_I)^{\frac{1}{2}}\right]^{\frac{2}{q}},
\end{aligned}$$

which gives (8.35). As for (8.36), we first note that since $\inf_{t\in I} N(t) \ge 1$, for any $\eta > 0$ we may find $N_0(\eta)$ so that

$$\||\nabla|^{s_c} u_{\le N}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \le \eta$$

for all $N \le N_0$. The estimate (8.36) then follows by interpolating with (6.31). ∎

We are now ready for the proof of Proposition 8.3.1.

Proof of Proposition 8.3.1. Throughout the proof, all spacetime norms will be taken over $I \times \mathbb{R}^d$.

Fix $\eta > 0$, and choose $N_0 = N_0(\eta)$ small enough that (8.36) holds; recall that (8.35) holds without any restriction on N. Next, we claim that for N_0 possibly even smaller, we can guarantee that for $N \leq N_0$, we have

$$\|u_{>N}\|_{L_t^\infty L_x^2} \lesssim_u \eta^{10} N^{-s_c} \tag{8.38}$$

and

$$\||\nabla|^{1-s_c} u_{>N}\|_{L_t^\infty L_x^2} \lesssim_u \eta^{10} N^{1-2s_c}. \tag{8.39}$$

Indeed, using the fact that $\inf_{t\in I} N(t) \geq 1$, we may find $c(\eta) > 0$ so that

$$\||\nabla|^{s_c} u_{\leq c(\eta)}\|_{L_t^\infty L_x^2} \leq \eta^{10};$$

combining this inequality with Bernstein, we get

$$\begin{aligned}
N^{s_c}\|u_{>N}\|_{L_t^\infty L_x^2} &\lesssim N^{s_c}\|u_{N\leq\cdot\leq c(\eta)}\|_{L_t^\infty L_x^2} + N^{s_c}\|u_{>c(\eta)}\|_{L_t^\infty L_x^2} \\
&\lesssim \||\nabla|^{s_c} u_{\leq c(\eta)}\|_{L_t^\infty L_x^2} + \tfrac{N^{s_c}}{c(\eta)^{s_c}}\||\nabla|^{s_c} u_{>c(\eta)}\|_{L_t^\infty L_x^2} \\
&\lesssim_u \eta^{10} + N^{s_c}.
\end{aligned}$$

Thus, taking N sufficiently small, we recover (8.38). A similar argument yields (8.39).

Next, we record the following inequality that will be useful below:

$$\sup_{y\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \tfrac{x-y}{|x-y|}\cdot\nabla\varphi(x)\bar{\varphi}(x)\,dx\right| \lesssim \||\nabla|^s\varphi\|_2\||\nabla|^{1-s}\varphi\|_2 \tag{8.40}$$

for $0 \leq s \leq 1$. Indeed, for fixed $y \in \mathbb{R}^d$, we can first write

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \tfrac{x-y}{|x-y|}\cdot\nabla\varphi(x)\bar{\varphi}(x)\,dx\right| &\lesssim \||\nabla|^s \tfrac{x-y}{|x-y|}\varphi\|_2\||\nabla|^{-s}\nabla\varphi\|_2 \\
&\sim \||\nabla|^s \tfrac{x-y}{|x-y|}\varphi\|_2\||\nabla|^{1-s}\varphi\|_2.
\end{aligned}$$

Thus we can complete the proof of (8.40) with an application of (2.4).

We now wish to apply the interaction Morawetz inequality (Lemma 8.3.2) with $\varphi = u_{>N}$ and $\mathcal{N} = P_{>N}(|u|^p u)$, with $N \leq N_0$. Together with (8.38), (8.39), (8.40), Bernstein, and the fact that $u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^d)$, an application of Lemma 8.3.2 gives

$$\begin{aligned}
&- \iiint |u_{>N}(t,y)|^2 \Delta(\tfrac{1}{|\cdot|})(x-y)|u_{>N}(t,x)|^2 \, dx\, dy\, dt \\
&\quad + \iiint |u_{>N}(t,y)|^2 \tfrac{x-y}{|x-y|} \cdot \{P_{>N}(|u|^p u), u_{>N}\}_{\mathcal{P}}(t,x)\, dx\, dy\, dt \\
&\quad \lesssim \|u_{>N}\|_{L_t^\infty L_x^2}^2 \||\nabla|^{1/2} u_{>N}\|_{L_t^\infty L_x^2}^2 \\
&\qquad + \|\{P_{>N}(|u|^p u), u_{>N}\}_m\|_{L^1_{t,x}} \||\nabla|^{s_c} u_{>N}\|_{L_t^\infty L_x^2} \||\nabla|^{1-s_c} u_{>N}\|_{L_t^\infty L_x^2} \\
&\quad \lesssim_u \eta^{20} N^{1-4s_c} + \eta^{10} N^{1-2s_c} \|\{P_{>N}(|u|^p u), u_{>N}\}_m\|_{L^1_{t,x}}.
\end{aligned} \tag{8.41}$$

Thus, to prove Proposition 8.3.1, we need to get sufficient control over the mass and momentum bracket terms appearing above.

To begin, we consider the contribution of the momentum bracket term. We can write

$$\begin{aligned}
&\{P_{>N}(|u|^p u), u_{>N}\}_{\mathcal{P}} \\
&\quad = \{|u|^p u, u\}_{\mathcal{P}} - \{|u_{\leq N}|^p u_{\leq N}, u_{\leq N}\}_{\mathcal{P}} \\
&\qquad - \{|u|^p u - |u_{\leq N}|^p u_{\leq N}, u_{\leq N}\}_{\mathcal{P}} - \{P_{\leq N}(|u|^p u), u_{>N}\}_{\mathcal{P}} \\
&\quad = -\tfrac{p}{p+2} \nabla(|u|^{p+2} - |u_{\leq N}|^{p+2}) - \{|u|^p u - |u_{\leq N}|^p u_{\leq N}, u_{\leq N}\}_{\mathcal{P}} \\
&\qquad - \{P_{\leq N}(|u|^p u), u_{>N}\}_{\mathcal{P}} \\
&\quad =: I + II + III.
\end{aligned}$$

After an integration by parts, we see that term I contributes to the left-hand side of (8.41) a multiple of

$$\begin{aligned}
&\iiint \frac{|u_{>N}(t,y)|^2 |u_{>N}(t,x)|^{p+2}}{|x-y|} \, dx\, dy\, dt \\
&- \iiint \frac{|u_{>N}(t,y)|^2 (|u|^{p+2} - |u_{>N}|^{p+2} - |u_{\leq N}|^{p+2})(t,x)}{|x-y|} \, dx\, dy\, dt.
\end{aligned}$$

For term II, we use $\{f,g\}_{\mathcal{P}} = \nabla \mathcal{O}(fg) + \mathcal{O}(f \nabla g)$; when the derivative hits the product, we integrate by parts, while for the second term we simply bring absolute values inside the integral. In this way, we find that term II contributes to the right-hand side of (8.41) a multiple of

$$\iiint \frac{|u_{>N}(t,y)|^2 |(|u|^p u - |u_{\le N}|^p u_{\le N}) u_{\le N}(t,x)|}{|x-y|} \, dx\, dy\, dt$$
$$+ \iiint |u_{>N}(t,y)|^2 |(|u|^p u - |u_{\le N}|^p u_{\le N})(t,x)| \; |\nabla u_{\le N}(t,x)| \, dx\, dy\, dt.$$

Finally, for term III, we integrate by parts when the derivative falls on $u_{>N}$; in this way, we see that term III contributes to the right-hand side of (8.41) a multiple of

$$\iiint \frac{|u_{>N}(t,y)|^2 |u_{>N}(t,x)| \, |P_{\le N}(|u|^p u)(t,x)|}{|x-y|} \, dx\, dy\, dt$$
$$+ \iiint |u_{>N}(t,y)|^2 |u_{>N}(t,x)| \, |\nabla P_{\le N}(|u|^p u)(t,x)| \, dx\, dy\, dt.$$

We next consider the mass bracket term in (8.41). Exploiting the fact that

$$\{|u_{>N}|^p u_{>N}, u_{>N}\}_m = 0,$$

we can write

$$\begin{aligned}
\{P_{>N}(|u|^p u), u_{>N}\}_m &= \{P_{>N}(|u|^p u) - |u_{>N}|^p u_{>N}, u_{>N}\}_m \\
&= \{P_{>N}(|u|^p u - |u_{>N}|^p u_{>N} - |u_{\le N}|^p u_{\le N}), u_{>N}\}_m \\
&\quad + \{P_{>N}(|u_{\le N}|^p u_{\le N}), u_{>N}\}_m - \{P_{\le N}(|u_{>N}|^p u_{>N}), u_{>N}\}_m.
\end{aligned}$$

We will now collect the contributions of the mass and momentum bracket terms and insert them back into (8.41). We will also make use of the pointwise inequalities

$$\begin{aligned}
\big||f+g|^p(f+g) - |f|^p f\big| &\lesssim |g|^{p+1} + |g| \; |f|^p, \\
\big||f+g|^{p+2} - |f|^{p+2} - |g|^{p+2}\big| &\lesssim |f| \; |g|^{p+1} + |f|^{p+1}|g|.
\end{aligned}$$

In this way, (8.41) becomes

$$-\iiint |u_{>N}(t,y)|^2\Delta(\tfrac{1}{|\cdot|})(x-y)|u_{>N}(t,x)|^2\,dx\,dy\,dt \tag{8.42}$$
$$+\iiint \frac{|u_{>N}(t,y)|^2|u_{>N}(t,x)|^{p+2}}{|x-y|}\,dx\,dy\,dt$$

$$\lesssim_u \eta^{20}N^{1-4s_c} \tag{8.43}$$
$$+\eta^{10}N^{1-2s_c}\||u_{\leq N}|^p u_{>N}^2\|_{L^1_{t,x}} \tag{8.44}$$
$$+\eta^{10}N^{1-2s_c}\||u_{>N}|^{p+1}u_{\leq N}\|_{L^1_{t,x}} \tag{8.45}$$
$$+\eta^{10}N^{1-2s_c}\|P_{>N}(|u_{\leq N}|^p u_{\leq N})u_{>N}\|_{L^1_{t,x}} \tag{8.46}$$
$$+\eta^{10}N^{1-2s_c}\|P_{\leq N}(|u_{>N}|^p u_{>N})u_{>N}\|_{L^1_{t,x}} \tag{8.47}$$
$$+\iiint \frac{|u_{>N}(t,y)|^2|u_{>N}(t,x)|\,|u_{\leq N}(t,x)|^{p+1}}{|x-y|}\,dx\,dy\,dt \tag{8.48}$$
$$+\iiint \frac{|u_{>N}(t,y)|^2|u_{>N}(t,x)|^{p+1}|u_{\leq N}(t,x)|}{|x-y|}\,dx\,dy\,dt \tag{8.49}$$
$$+\iiint \frac{|u_{>N}(t,y)|^2|P_{\leq N}(|u|^p u)(t,x)|\,|u_{>N}(t,x)|}{|x-y|}\,dx\,dy\,dt \tag{8.50}$$
$$+\iiint |u_{>N}(t,y)|^2|u_{>N}(t,x)|\,|u_{\leq N}(t,x)|^p|\nabla u_{\leq N}(t,x)|\,dx\,dy\,dt \tag{8.51}$$
$$+\iiint |u_{>N}(t,y)|^2|u_{>N}(t,x)|^{p+1}|\nabla u_{\leq N}(t,x)|\,dx\,dy\,dt \tag{8.52}$$
$$+\iiint |u_{>N}(t,y)|^2|u_{>N}(t,x)|\,|\nabla P_{\leq N}(|u|^p u)(t,x)|\,dx\,dy\,dt. \tag{8.53}$$

To complete the proof of Proposition 8.3.1, we need to show that the error terms (8.43) through (8.53) are acceptable, in the sense that they can be controlled by $\eta(N^{1-4s_c}+K_I)$. Clearly, (8.43) is acceptable.

Next, we consider (8.44). Using Hölder, Sobolev embedding, (8.35), and (8.36), we get

$$\begin{aligned}
\||u_{\leq N}|^p u_{>N}^2\|_{L^1_{t,x}} &\lesssim \|u_{\leq N}\|^p_{L^{2p}_t L^{dp}_x}\|u_{>N}\|^2_{L^4_t L_x^{\frac{2d}{d-1}}}\\
&\lesssim \||\nabla|^{s_c}u_{\leq N}\|^p_{L^{2p}_t L_x^{\frac{2dp}{dp-2}}}\|u_{>N}\|^2_{L^4_t L_x^{\frac{2d}{d-1}}}\\
&\lesssim_u \eta^p N^{-2s_c}(1+N^{4s_c-1}K_I),
\end{aligned}$$

which renders (8.44) acceptable.

We now turn to (8.45). For this term, we can again use Hölder, Sobolev embedding, (8.35), and (8.36) to see

$$\begin{aligned}\||u_{>N}|^{p+1}u_{\leq N}\|_{L^1_{t,x}} &\lesssim \|u_{>N}\|^2_{L^3_t L_x^{\frac{6d}{3d-4}}}\|u_{>N}\|^{p-1}_{L^\infty_t L_x^{\frac{dp}{2}}}\|u_{\leq N}\|_{L^3_t L_x^{\frac{3dp}{6-2p}}} \\ &\lesssim \|u_{>N}\|^2_{L^3_t L_x^{\frac{6d}{3d-4}}}\||\nabla|^{s_c}u\|^{p-1}_{L^\infty_t L^2_x}\||\nabla|^{s_c}u_{\leq N}\|_{L^3_t L_x^{\frac{6d}{3d-4}}} \\ &\lesssim_u \eta N^{-2s_c}(1+N^{4s_c-1}K_I).\end{aligned}$$

Thus this term is acceptable as well. Before proceeding, however, we note that it is this term that has forced us to exclude the cases $(d, s_c) \in \{3\} \times (\frac{3}{4}, 1)$ from this paper; we postpone further discussion until Remark 8.3.4 below.

We next turn to (8.46); using Hölder, Bernstein, the fractional chain rule, Sobolev embedding, (8.35), and (8.36), we see

$$\begin{aligned}&\|P_{>N}(|u_{\leq N}|^p u_{\leq N})u_{>N}\|_{L^1_{t,x}} \\ &\quad\lesssim N^{-s_c}\|u_{>N}\|_{L^4_t L_x^{\frac{2d}{d-1}}}\||\nabla|^{s_c}(|u_{\leq N}|^p u_{\leq N})\|_{L_t^{\frac{4}{3}} L_x^{\frac{2d}{d+1}}} \\ &\quad\lesssim N^{-s_c}\|u_{>N}\|_{L^4_t L_x^{\frac{2d}{d-1}}}\|u_{\leq N}\|^p_{L^{4p}_t L_x^{\frac{2dp}{3}}}\||\nabla|^{s_c}u_{\leq N}\|_{L^2_t L_x^{\frac{2d}{d-2}}} \\ &\quad\lesssim N^{-s_c}\|u_{>N}\|_{L^4_t L_x^{\frac{2d}{d-1}}}\||\nabla|^{s_c}u_{\leq N}\|^p_{L^{4p}_t L_x^{\frac{2dp}{dp-1}}}\||\nabla|^{s_c}u_{\leq N}\|_{L^2_t L_x^{\frac{2d}{d-2}}} \\ &\quad\lesssim_u \eta^{p+1}N^{-2s_c}(1+N^{4s_c-1}K_I),\end{aligned}$$

so that (8.46) is also acceptable.

For the final term originating from the mass bracket, (8.47), we use Hölder, Bernstein, Sobolev embedding, (8.35), and (8.38) to see

$$\begin{aligned}
\|P_{\le N}(|u_{>N}|^p u_{>N})u_{>N}\|_{L^1_{t,x}} &\lesssim \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|P_{\le N}(|u_{>N}|^p u_{>N})\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d+4}}} \\
&\lesssim N^{s_c}\|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||u_{>N}|^p u_{>N}\|_{L_t^{\frac{3}{2}} L_x^{\frac{3dp}{3dp+2p-6}}} \\
&\lesssim N^{s_c}\|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_{>N}\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim N^{s_c}\|u_{>N}\|^3_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||\nabla|^{s_c} u\|^{p-1}_{L_t^\infty L_x^2} \\
&\lesssim_u N^{-2s_c}(1 + N^{4s_c-1}K_I),
\end{aligned}$$

which shows that (8.47) is acceptable.

We now turn to the terms originating from the momentum bracket. First, consider (8.48). By Hölder, Hardy–Littlewood–Sobolev, Sobolev embedding, Bernstein, (8.35), (8.36), and (8.38), we can estimate

$$\begin{aligned}
(8.48) &\lesssim \|\tfrac{1}{|x|} * |u_{>N}|^2\|_{L_t^3 L_x^{3d}} \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\quad \times \|u_{\le N}\|_{L_t^6 L_x^{\frac{3dp}{6-p}}} \|u_{\le N}\|_{L_t^6 L_x^{\frac{6d}{3d-8}}} \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim \||u_{>N}|^2\|_{L_t^3 L_x^{\frac{3d}{3d-2}}} \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\quad \times \||\nabla|^{s_c} u_{\le N}\|_{L_t^6 L_x^{\frac{6d}{3d-2}}} \|\nabla u_{\le N}\|_{L_t^6 L_x^{\frac{6d}{3d-2}}} \||\nabla|^{s_c} u\|^{p-1}_{L_t^\infty L_x^2} \\
&\lesssim_u N^{1-s_c}\|u_{>N}\|_{L_t^\infty L_x^2} \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||\nabla|^{s_c} u_{\le N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \\
&\lesssim_u \eta^{12} N^{1-4s_c}(1 + N^{4s_c-1}K_I),
\end{aligned}$$

so that (8.48) is acceptable.

For (8.49), we consider two cases. If $|u_{\le N}| \le 10^{-100}|u_{>N}|$, then we can absorb this term into the left-hand side of the inequality, provided we can show

$$\iiint \frac{|u_{>N}(t,y)|^2 |u_{>N}(t,x)|^{p+2}}{|x-y|}\, dx\, dy\, dt < \infty. \tag{8.54}$$

On the other hand, if $|u_{>N}| \le 10^{100}|u_{\le N}|$, then we are back in the situation of (8.48), which we have already handled. Thus, to render (8.49) acceptable, it remains to prove (8.54). To this end, we define

$$\theta = \tfrac{4dp-16-3p}{2(dp-4)} \in (0, p+2),$$

and use Hölder, Hardy–Littlewood–Sobolev, Sobolev embedding, Lemma 4.1.5, and interpolation to estimate

$$\begin{aligned}
\mathrm{LHS}(8.54) &\lesssim \|\tfrac{1}{|x|} * |u_{>N}|^2\|_{L_t^4 L_x^{4d}} \|u_{>N}\|_{L_t^\infty L_x^2}^{\theta} \|u_{>N}\|^{p+2-\theta}_{L_t^{\frac{2p(2dp-5)}{3(dp-4)}} L_x^{\frac{dp(2dp-5)}{dp+2}}} \\
&\lesssim \||u_{>N}|^2\|_{L_t^4 L_x^{\frac{4d}{4d-3}}} \|u_{>N}\|_{L_t^\infty L_x^2}^{\theta} \||\nabla|^{s_c} u_{>N}\|^{p+2-\theta}_{L_t^{\frac{2p(2dp-5)}{3(dp-4)}} L_x^{\frac{2dp(2dp-5)}{2(dp)^2-11dp+24}}} \\
&\lesssim_u \|u_{>N}\|_{L_t^4 L_x^{\frac{4d}{2d-3}}} \|u_{>N}\|_{L_t^\infty L_x^2}^{1+\theta} \left(1 + \textstyle\int_I N(t)^2\,dt\right)^{\frac{(p+2-\theta)(3(dp-4))}{2p(2dp-5)}} \\
&\lesssim_u \||\nabla|^{1/4} u_{>N}\|_{L_t^4 L_x^{\frac{2d}{d-1}}} N^{-s_c(1+\theta)} \left(1 + \textstyle\int_I N(t)^2\,dt\right)^{\frac{(p+2-\theta)(3(dp-4))}{2p(2dp-5)}} \\
&\lesssim_u N^{1/4 - s_c(2+\theta)} \||\nabla|^{s_c} u_{>N}\|_{L_t^4 L_x^{\frac{2d}{d-1}}} \left(1 + \textstyle\int_I N(t)^2\,dt\right)^{\frac{(p+2-\theta)(3(dp-4))}{2p(2dp-5)}} \\
&\lesssim_u N^{1-4s_c} \left(1 + \textstyle\int_I N(t)^2\,dt\right)^{\frac14 + \frac{(p+2-\theta)(3(dp-4))}{2p(2dp-5)}} \\
&\lesssim_u N^{1-4s_c} \left(1 + \textstyle\int_I N(t)^2\,dt\right),
\end{aligned}$$

which gives (8.54), and thereby shows that (8.49) is acceptable.

Next, we turn to (8.50). Denoting

$$G = |u|^p u - |u_{\le N}|^p u_{\le N} - |u_{>N}|^p u_{>N},$$

we begin by writing

$$\begin{aligned}
&(8.50) \\
&\quad = \iiint \frac{|u_{>N}(t,y)|^2 |P_{\le N}(|u_{\le N}|^p u_{\le N})(t,x)|\,|u_{>N}(t,x)|}{|x-y|}\,dx\,dy\,dt && (8.55) \\
&\quad\quad + \iiint \frac{|u_{>N}(t,y)|^2 |P_{\le N}(|u_{>N}|^p u_{>N})(t,x)|\,|u_{>N}(t,x)|}{|x-y|}\,dx\,dy\,dt && (8.56) \\
&\quad\quad + \iiint \frac{|u_{>N}(t,y)|^2 |P_{\le N} G(t,x)|\,|u_{>N}(t,x)|}{|x-y|}\,dx\,dy\,dt. && (8.57)
\end{aligned}$$

For (8.55), we can write

$$\begin{aligned}
(8.55) &\lesssim \|\tfrac{1}{|x|} * |u_{>N}|^2\|_{L_t^3 L_x^{3d}} \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_{\le N}\|_{L_t^6 L_x^{\frac{3dp}{6-p}}} \|u_{\le N}\|_{L_t^6 L_x^{\frac{6d}{3d-8}}} \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim_u \eta^{12} N^{1-4s_c}(1 + N^{4s_c - 1} K_I)
\end{aligned}$$

by the same arguments that dealt with (8.48).

For (8.56), we can use Hölder, Hardy–Littlewood–Sobolev, Bernstein, Sobolev embedding, (8.35), (8.36), and (8.38) to estimate

$$
\begin{aligned}
(8.56) &\lesssim \|u_{>N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \big\| \tfrac{1}{|x|} * (P_{\leq N}(|u_{>N}|^p u_{>N}) u_{>N}) \big\|_{L_t^{\frac{3}{2}} L_x^{\frac{3d}{2}}} \\
&\lesssim \|u_{>N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \big\| P_{\leq N}(|u_{>N}|^p u_{>N}) u_{>N} \big\|_{L_t^{\frac{3}{2}} L_x^{\frac{3d}{3d-1}}} \\
&\lesssim \|u_{>N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \|u_{>N}\|_{L_t^\infty L_x^2} \big\| P_{\leq N}(|u_{>N}|^p u_{>N}) \big\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d-2}}} \\
&\lesssim \|u_{>N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \|u_{>N}\|_{L_t^\infty L_x^2} N^{1+s_c} \big\| P_{\leq N}(|u_{>N}|^p u_{>N}) \big\|_{L_t^{\frac{3}{2}} L_x^{\frac{3dp}{3dp+2p-6}}} \\
&\lesssim \|u_{>N}\|^2_{L_t^6 L_x^{\frac{6d}{3d-2}}} \|u_{>N}\|_{L_t^\infty L_x^2} N^{1+s_c} \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim_u \eta^{10} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

which renders (8.56) acceptable.

For (8.57), we first note

$$
|u|^p u - |u_{\leq N}|^p u_{\leq N} - |u_{>N}|^p u_{>N} = \emptyset(u_{>N} u_{\leq N} |u|^{p-1}).
$$

Thus, we can use Hölder, Hardy–Littlewood–Sobolev, Bernstein, Sobolev embedding, (8.35), (8.36), and (8.38), to get

$$
\begin{aligned}
(8.57) &\lesssim \|\tfrac{1}{|x|} * |u_{>N}|^2\|_{L_t^3 L_x^{3d}} \|P_{\leq N}(\emptyset(u_{>N} u_{\leq N}|u|^{p-1}))\|_{L_t^3 L_x^{\frac{6d}{3d+2}}} \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim N \| |u_{>N}|^2 \|_{L_t^3 L_x^{\frac{3d}{3d-2}}} \|\emptyset(u_{>N} u_{\leq N}|u|^{p-1})\|_{L_t^3 L_x^{\frac{6d}{3d+8}}} \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim N \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{\leq N}\|_{L_t^3 L_x^{\frac{3dp}{6-2p}}} \|u\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \\
&\lesssim N \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_{>N}\|^2_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||\nabla|^{s_c} u\|^{p-1}_{L_t^\infty L_x^2} \\
&\lesssim_u \eta^{21} N^{1-4s_c}(1 + N^{4s_c-1} K_I).
\end{aligned}
$$

Thus (8.57), and so (8.50), is acceptable.

We now turn to (8.51). By Hölder, Sobolev embedding, Bernstein, (8.36), and (8.38), we estimate

$$
\begin{aligned}
(8.51) &\lesssim \|u_{>N}\|^3_{L_t^\infty L_x^2} \|u_{\leq N}\|^p_{L_t^{2p} L_x^{dp}} \|\nabla u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\lesssim N^{1-s_c} \|u_{>N}\|^3_{L_t^\infty L_x^2} \||\nabla|^{s_c} u_{\leq N}\|^p_{L_t^{2p} L_x^{\frac{2dp}{dp-2}}} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\lesssim_u \eta^{p+31} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

so that (8.51) is acceptable.

For (8.52), we use Hölder, Sobolev embedding, Bernstein, (8.35), (8.36), and (8.38) to get

$$
\begin{aligned}
(8.52) &\lesssim \|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{>N}\|^{p-1}_{L_t^\infty L_x^{\frac{dp}{2}}} \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|\nabla u_{\leq N}\|_{L_t^3 L_x^{\frac{3dp}{6-2p}}} \\
&\lesssim_u N \|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{>N}\|^2_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||\nabla|^{s_c} u_{\leq N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim_u \eta^{21} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

which renders (8.52) acceptable.

Finally, we consider (8.53). We begin by writing

$$
\begin{aligned}
(8.53) \lesssim\, &\|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{>N} \nabla P_{\leq N}(|u_{\leq N}|^p u_{\leq N})\|_{L^1_{t,x}} && (8.58) \\
&+ \|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{>N} \nabla P_{\leq N}(|u_{>N}|^p u_{>N})\|_{L^1_{t,x}} && (8.59) \\
&+ \|u_{>N}\|^2_{L_t^\infty L_x^2} \|u_{>N} \nabla P_{\leq N}(|u|^p u - |u_{\leq N}|^p u_{\leq N} - |u_{>N}|^p u_{>N})\|_{L^1_{t,x}}. && (8.60)
\end{aligned}
$$

To begin, we use Hölder, the chain rule, and the arguments that gave (8.51) to see

$$
\begin{aligned}
(8.58) &\lesssim \|u_{>N}\|^3_{L_t^\infty L_x^2} \|u_{\leq N}\|^p_{L_t^{2p} L_x^{dp}} \|\nabla u_{\leq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\lesssim_u \eta^{p+31} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

so that (8.58) is acceptable.

For (8.59), we argue essentially as we did for (8.47). That is, we use Hölder, Bernstein, Sobolev embedding, (8.35), and (8.38) to estimate

$$
\begin{aligned}
(8.59) &\lesssim \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|\nabla P_{\leq N}(|u_{>N}|^p u_{>N})\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d+4}}} \\
&\lesssim N^{1+s_c} \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||u_{>N}|^p u_{>N}\|_{L_t^{\frac{3}{2}} L_x^{\frac{3dp}{3dp+2p-6}}} \\
&\lesssim N^{1+s_c} \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}^3 \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^{p-1} \\
&\lesssim_u \eta^{20} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

which gives that (8.59) is acceptable.

For (8.60), we argue similarly to the case of (8.57). In particular, we use Hölder, Bernstein, Sobolev embedding, (8.35), (8.36), and (8.38) to see

$$
\begin{aligned}
(8.60) &\lesssim N \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|\emptyset(u_{\leq N} u_{>N} |u|^{p-1})\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d+4}}} \\
&\lesssim N \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}^2 \|u_{\leq N}\|_{L_t^3 L_x^{\frac{3dp}{6-2p}}} \|u\|_{L_t^\infty L_x^{\frac{dp}{2}}}^{p-1} \\
&\lesssim N \|u_{>N}\|_{L_t^\infty L_x^2}^2 \|u_{>N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}^2 \||\nabla|^{s_c} u_{\leq N}\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^{p-1} \\
&\lesssim_u \eta^{21} N^{1-4s_c}(1 + N^{4s_c-1} K_I),
\end{aligned}
$$

which gives that (8.60). Collecting the estimates for (8.58), (8.59), and (8.60), we see that (8.53) is acceptable. This completes the proof of Proposition 8.3.1. ∎

Remark 8.3.4 Let us discuss why (8.45) has forced us to exclude the cases $(d, s_c) \in \{3\} \times (\frac{3}{4}, 1)$ from (1.3). As one can see in the proof above, in the cases we consider, this term is fairly harmless. However, once $s_c > \frac{3}{4}$ in dimension $d = 3$ (which corresponds to $p > \frac{8}{3}$), this term becomes a problem; put simply, we end up with too many copies of $u_{>N}$ to deal with.

This problem has already been encountered in the energy-critical setting ($s_c = 1$) in dimension $d = 3$; in this case, one can overcome the hurdle by applying a spatial truncation to the weight a. One can refer to [15] for the original argument, wherein spatial truncation is applied at various levels and subsequently averaged. The authors of [43] revisit the result of [15] in the context of minimal counterexamples; at this point in the argument, they choose to work with a more carefully designed spatial truncation, which removes the need for any subsequent averaging argument.

This discussion begs the question: why doesn't spatial truncation work in our setting? To answer this, we need to understand how spatial truncations affect the argument that leads to Proposition 8.3.1. What we find is that spatial truncations ruin the convexity properties of a that made some of the terms in the proof of Lemma 8.3.2 positive; thus, to establish Proposition 8.3.1 with a further spatial truncation, we have to control additional error terms. It turns out that one of these additional error terms requires uniform control over $\|u\|_{L_x^{p+2}}$, while another requires uniform control over $\|\nabla u\|_{L_x^2}$ (see [43, Lemma 6.5 and Lemma 6.6]). In the energy-critical case, one can use the conservation of energy to push the argument through, while in our cases, we cannot proceed without some significant new input. We have therefore abandoned the cases $(d, s_c) \in \{3\} \times (\frac{3}{4}, 1)$ in (1.3).

CHAPTER 9

Quasi-solitons

In this chapter, we preclude the existence of quasi-solitons. We will use frequency-localized Morawetz estimates proved in the previous chapter, as well as the lower bounds established in Chapter 4.

The results in this chapter appeared originally in [49, 51].

9.1 The radial setting, $s_c < 1/2$

In this section, we preclude the existence of solutions as in Theorem 4.5.4 for which

$$K_{[0,\infty)} = \int_0^\infty N(t)^{3-2s_c}\,dt = \infty. \tag{9.1}$$

We will rely on the frequency-localized Lin–Strauss Morawetz inequality established in Chapter 8, as well as the lower bounds given in Proposition 4.2.2.

Theorem 9.1.1 (No quasi-solitons) *There are no almost periodic solutions as in Theorem 4.5.4 such that* (9.1) *holds.*

Proof. Suppose u were such a solution. Let $\eta > 0$ and let $I \subset [0,\infty)$ be a compact time interval, which is a contiguous union of characteristic subintervals.

Combining (8.1) and (4.12), we find that for N sufficiently large, we have

$$K_I \lesssim_u \iint_{I\times\mathbb{R}^3} \frac{|u_{\leq N}(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim_u \eta(N^{1-2s_c} + K_I).$$

Choosing η sufficiently small, we deduce $K_I \lesssim_u N^{1-2s_c}$ uniformly in I. We now contradict (9.1) by taking I sufficiently large inside of $[0,\infty)$. This completes the proof of Theorem 9.1.1. ∎

9.2 The radial setting, $s_c > 1/2$

In this section, we preclude the existence of solutions as in Theorem 4.5.3 for which

$$K_{[0,T_{max})} = \int_0^{T_{max}} N(t)^{3-2s_c}\,dt = \infty. \tag{9.2}$$

We will again rely on the frequency-localized Lin–Strauss Morawetz inequality established in Chapter 8, as well as the lower bounds given in Proposition 4.2.2.

Theorem 9.2.1 (No quasi-solitons) *There are no almost periodic solutions as in Theorem 4.5.3 such that* (9.2) *holds.*

Proof. Suppose u were such a solution. Let $\eta > 0$ and let $I \subset [0,\infty)$ be a compact time interval, which is a contiguous union of characteristic subintervals.

Combining (8.16) and (4.11), we find that for N sufficiently small, we have

$$K_I \lesssim_u \iint_{I\times\mathbb{R}^3} \frac{|u_{>N}(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim_u \eta(N^{1-2s_c} + K_I).$$

Choosing η sufficiently small, we deduce $K_I \lesssim_u N^{1-2s_c}$ uniformly in I. We now contradict (9.2) by taking I sufficiently large inside of $[0,T_{max})$. This completes the proof of Theorem 9.2.1. ∎

9.3 The non-radial setting

In this section we preclude the existence of almost periodic solutions as in Theorem 4.5.2 for which

$$K_{[0,T_{max})} = \int_0^{T_{max}} N(t)^{3-4s_c}\,dt = \infty. \tag{9.3}$$

We will rely on the frequency-localized interaction Morawetz inequality established in Chapter 8, as well as the lower bounds in Proposition 4.2.2.

Theorem 9.3.1 (No quasi-solitons) *There are no almost periodic solutions as in Theorem 4.5.2 such that* (9.3) *holds.*

Proof. Suppose u were such a solution. Let $\eta > 0$ and let $I \subset [0, T_{max})$ be a compact time interval, which is a contiguous union of characteristic subintervals.

Combining (8.32) and (4.10), we find that for N sufficiently small, we have

$$K_I \lesssim_u - \int_I \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_{>N}(t,x)|^2 \Delta(\tfrac{1}{|x-y|}) |u_{>N}(t,y)|^2 \, dx \, dy \, dt \lesssim_u \eta (N^{1-4s_c} + K_I).$$

Choosing η sufficiently small, we deduce $K_I \lesssim_u N^{1-4s_c}$ uniformly in I. We now contradict (9.3) by taking I sufficiently large inside of $[0, T_{max})$. This completes the proof of Theorem 9.3.1. ∎

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