GEOPHYSICS, VOL. 60, NO. 1 (JANUARY-FEBRUARY 1995); P. 204–211, 2 FIGS., 1 TABLE.

Seismic anisotropy of fractured rock

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ABSTRACT

A simple method for including the effects of geologically realistic fractures on the seismic propagation through fractured rocks can be obtained by writing the effective compliance tensor of the fractured rock as the sum of the compliance tensor of the unfractured background rock and the compliance tensors for each set of parallel fractures or aligned fractures. The compliance tensor of each fracture set is derivable from a second rank fracture compliance tensor. For a rotationally symmetric set of fractures, the fracture compliance tensor depends on only two fracture compliances, one controlling fracture compliance normal, the other, tangential, to the plane of the fractures. The stiffness tensor, which is more useful in the consideration of elastic wave propagation through rocks, can then be obtained by inversion. The components of the excess fracture compliance tensor represent the maximum amount of information that can be obtained from seismic data. If the background rock is isotropic and the normal and shear compliance of each fracture are equal, although different from those of other fractures, the effective elastic behavior of the fractured rock is orthorhombic for any orientation distribution of fractures. A comparison of the theory with recent ultrasonic experiments on a simulated fractured medium shows near equality of the normal and shear compliance for the case of air-filled fractures.

INTRODUCTION

Hydrocarbon reservoirs are often layered, with high porosity zones interleaved with horizontal, relatively impermeable shale beds. Vertical fracturing in reservoirs, and in the caprock overlying the reservoir, significantly affects the flow characteristics of the reservoir, and hence the density and orientation of sets of fractures is of great interest. For example, significant permeability anisotropy can originate from the presence of oriented sets of fractures, and the use of seismic anisotropy to determine the orientation of fracture sets is of considerable interest.

In sedimentary rocks, the fracture orientations are determined by the stress history of the rock. However, any fractures open at depth will tend to be oriented normal to the direction of the minimum in-situ stress. For such rocks, observations of the seismic anisotropy have the potential of providing the orientation of the in-situ stress field. Because of the importance of fracture-induced anisotropy for seismic wave propagation, several theoretical studies of crack-induced anisotropy have been reported in the literature. In many of these studies (O'Connell and Budiansky, 1976; Budiansky and O'Connell, 1976; Hoenig, 1979; Bruner, 1976; Henyey and Pomphrey, 1982; Hudson, 1980, 1981, 1986) the fractures are modeled as ellipsoidal cavities of low aspect ratio. Real fractures, however, do not resemble isolated low-aspect ratio voids in a solid matrix. Borehole pictures, examination of outcrops, and rock fractured in the laboratory all indicate that fractures have many points of contact along their length (Reiss, 1980). Furthermore, minerals such as quartz and calcite may be deposited within the fracture and can appreciably stiffen its mechanical response.

In this paper, a flexible yet simple way to include the qualitative effects of geologically realistic fracturing on the long wavelength (compared to fracture spacing) elastic behavior of such systems is described. Subject to the simplifying assumption of linear, loss-free, elastic behavior, it is the elastic moduli and the density that determine the behavior of seismic waves that propagate through, and are reflected from, the reservoir. Thus the issue to be discussed here is how the presence of the fracture systems affect the elastic moduli of the fractured rock. Density is unchanged

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from that of the unfractured background rock as a result of the assumption of fractures of infinitesimal total volume.

MULTIPLE FRACTURE SYSTEMS IN AN ANISOTROPIC BACKGROUND

The effective elastic compliance tensor $s_{ijk\ell}$ of a rock containing fractures relates the average strain ε_{ij} over a representative volume V to the average stress components σ_{ij}:

$$\varepsilon_{ij} = s_{ijk\ell}\sigma_{k\ell}. \tag{1}$$

For fractures, ε_{ij} may be written in the form,

$$\varepsilon_{ij} = s_{ijk\ell_b}\sigma_{k\ell} + \frac{1}{2V}\sum_q \int_{S_q} ([u_i]n_j + [u_j]n_i)\, dS, \tag{2}$$

where $s_{ijk\ell_b}$ is the compliance tensor of the unfractured background rock which may be of arbitrary anisotropy, S_q is the surface of the qth fracture lying within V (see Figure 1), n_i are the components of the local unit normal to the fracture surface which may in general be curved, and brackets [ ] denote jump discontinuities in the displacement; see, for example, Sayers and Kachanov (1991). Note that equation (2) is applicable to finite, nonplanar fractures in the long wavelength limit, i.e., the applied stress is assumed to be constant over the representative volume V. In the following, it will be assumed that fracture interactions may be neglected so that $[u_i]$ is determined by σ_{ij}. This assumption is exact for a set of infinite flat parallel fractures subject to a uniform stress field, as will be discussed below. Note that this assumption of noninteraction does not imply that the excess compliance as a result of the fractures is small relative to the unfractured background compliance.

Naturally occurring fractures can often be divided into sets based on their orientation. An example is given in Figure 2, which shows joint traces exposed on a horizontal 1-m-thick limestone unit of the Pennsylvanian-Permian Rico Formation (Olson and Pollard, 1989). When the fractures are approximately planar and parallel, and their unit normal is denoted by n_i (see Figure 1), a linearity assumption is conveniently introduced through a "fracture system compliance tensor" **Z** with components Z_{ij} such that,

$$\frac{1}{V}\sum_q \int_{S_q} [u_i]\, dS \equiv Z_{ij}\sigma_{jk}n_k, \tag{3}$$

where Z_{ij} is symmetric and nonnegative definite. Z_{ij} is similar to the crack compliance tensor recently used by Kachanov (1992). Let the extra strain as a result of the fractures be $s_{ijk\ell_f}\sigma_{k\ell}$, so that $s_{ijk\ell_f}$ may be thought of as an excess compliance tensor as a result of the presence of the parallel fractures. Substitution of equation (3) into equation (2) yields,

$$s_{ijk\ell_f}\sigma_{k\ell} = \tfrac{1}{2}(Z_{ir}\sigma_{rs}n_s n_j + Z_{jr}\sigma_{rs}n_s n_i)$$

$$= \tfrac{1}{2}(Z_{ir}n_s n_j + Z_{jr}n_s n_i)\,\frac{\delta_{rk}\delta_{s\ell} + \delta_{r\ell}\delta_{sk}}{2}\,\sigma_{k\ell}, \tag{4}$$

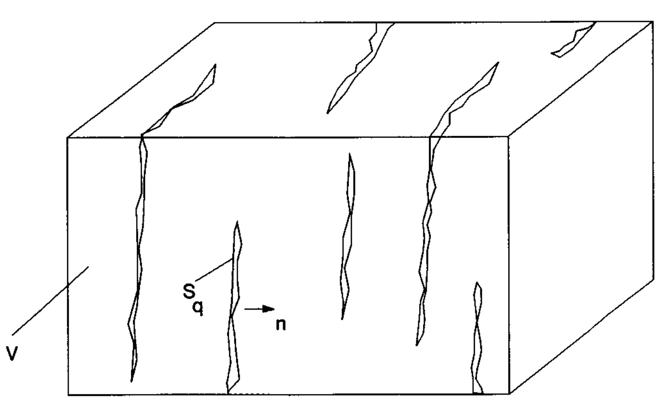

FIG. 1. A diagramatic view of a vertically fractured medium. The linearity assumption of equation (3) says that the average of the displacement discontinuity across the parallel fractures in volume V is linearly related to the stress traction acting on the fracture planes.

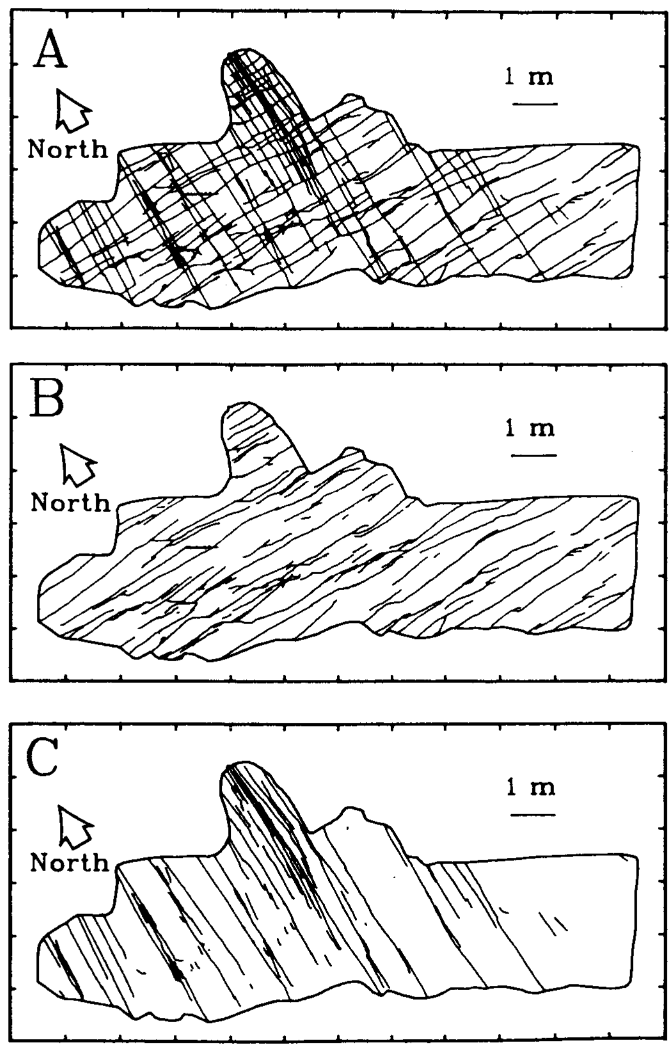

FIG. 2. Maps of joint traces on bedding surface of Rico Formation, Monument upwarp, southeastern Utah. (a) Both joint sets. (b) East-west set. (c) North-south set (Reproduced with permission from Olson and Pollard, 1989).

so one obtains,

$$s_{ijk\ell_f} = \tfrac{1}{4}(Z_{ik}n_\ell n_j + Z_{jk}n_\ell n_i + Z_{i\ell}n_k n_j + Z_{j\ell}n_k n_i). \quad (5)$$

For multiple sets of fractures, each set with its own normal $n_j^{(m)}$ and its own fracture system compliance tensor $Z_{ij}^{(m)}$ contributes an excess compliance tensor $s_{ijk\ell_f}^{(m)}$ additively according to equation (2), so that, for several different sets of aligned fractures,

$$s_{ijk\ell} = s_{ijk\ell_b} + \sum_m s_{ijk\ell_f}^{(m)}$$

$$= s_{ijk\ell_b} + \frac{1}{4}\sum_m (Z_{ik}^{(m)}n_\ell^{(m)}n_j^{(m)} + Z_{jk}^{(m)}n_\ell^{(m)}n_i^{(m)}$$

$$+ Z_{i\ell}^{(m)}n_k^{(m)}n_j^{(m)} + Z_{j\ell}^{(m)}n_k^{(m)}n_i^{(m)}). \quad (6)$$

It is for this reason that the insertion of fractures is carried out so easily in terms of compliances. Afterwards, the elastic stiffness tensor, from which the seismic velocities may be calculated, is obtained by inverting the compliance tensor.

This formulation for fractures is exactly equivalent to the introduction of linear slip deformation (LSD) in an elastic background medium. The LSD assumption is that the additional deformation consists of the sum of the displacement discontinuity, or slip, across "planes of weakness" taken to be parallel to the plane of the aligned fractures or microfractures. Further, the total slip discontinuity across all the parallel planes of weakness, per unit length in the direction denoted by unit normal n_i, is assumed to be linearly dependent, through the same symmetric, nonnegative definite Z_{ij} on the traction acting on the plane perpendicular to n_i. This condition on the excess displacement gradient as a result of the presence of the planes of weakness may be written as,

$$n_j \frac{\partial u_i}{\partial x_j} = Z_{ij}\sigma_{jk}n_k, \quad (7)$$

which is the indicial notation form of the theory presented by Schoenberg (1983) and Schoenberg and Douma (1988). Note that, in general, inclusion of several fracture sets, each with arbitrary orientation, will result in a triclinic medium, even when the background is isotropic.

ROTATIONALLY INVARIANT FRACTURE SETS

The simplest assumption concerning the behavior as a result of fracturing is to let the normal compliance of the fractures be given by Z_N and the tangential compliance by Z_T. This causes the fracture behavior to be invariant with respect to rotation about an axis normal to the fractures. Under this condition,

$$Z_{ij} = Z_N n_i n_j + Z_T(\delta_{ij} - n_i n_j) = Z_T\delta_{ij}$$

$$+ (Z_N - Z_T)n_i n_j, \quad (8)$$

so that the excess compliance tensor of a single rotationally invariant fracture set becomes,

$$s_{ijk\ell_f} = \frac{Z_T}{4}(\delta_{ik}n_\ell n_j + \delta_{jk}n_\ell n_i + \delta_{i\ell}n_k n_j + \delta_{j\ell}n_k n_i)$$

$$+ (Z_N - Z_T)n_i n_j n_k n_\ell. \quad (9)$$

Note that effective elastic behavior of an isotropic medium containing several fracture sets may be triclinic even if the fracture sets are assumed to be rotationally invariant.

As an example, consider a single set of rotationally invariant vertical fractures whose normal is parallel to the x_1-axis, i.e., $n_1 = \{1, 0, 0\}$. In this case,

$$s_{1111_f} = Z_N,$$

$$s_{1212_f} = s_{2121_f} = s_{1221_f} = s_{2112_f} = s_{1313_f} = s_{3131_f}$$

$$= s_{1331_f} = s_{3113_f} = \frac{Z_T}{4}, \quad (10)$$

with all other compliance components equal to zero. In conventional (2-subscript) condensed 6×6 matrix notation, $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$, $12 \to 6$, while factors 2 and 4 are introduced as follows (Nye, 1985):

$$s_{ijk\ell} \to S_{pq} \text{ when both } p, q \text{ are 1, 2, or 3;}$$

$$2s_{ijk\ell} \to S_{pq} \text{ when one of } p, q \text{ are 4, 5, or 6;}$$

$$4s_{ijk\ell} \to S_{pq} \text{ when both } p, q \text{ are 4, 5, or 6.}$$

These factors of 2 or 4 are absent in the condensation of the stiffness tensor components. Thus the excess compliance [equation (10)] may be written in the following 6×6 matrix form:

$$\begin{bmatrix} Z_N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_T & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_T \end{bmatrix}. \quad (11)$$

in agreement with Nichols et al. (1989).

For a single set of rotationally invariant fractures in an isotropic background, the medium is transversely isotropic (TI) with its symmetry axis perpendicular to the fractures. This is, however, a restricted subset of all possible TI media since it depends only on the two background moduli, say Lamé parameters μ_b and λ_b, and two nonnegative fracture compliances Z_N and Z_T. Such a TI medium may be called TI(LSD). From equation (11), the 6×6 compliance matrix of a TI(LSD) medium whose symmetry axis is parallel to the 1-direction is given by

$$\underset{\sim}{S} = \begin{bmatrix} \frac{\lambda_b + \mu_b}{\mu_b(3\lambda_b + 2\mu_b)} + Z_N & -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & 0 & 0 & 0 \\ -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & \frac{\lambda_b + \mu_b}{\mu_b(3\lambda_b + 2\mu_b)} & -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & 0 & 0 & 0 \\ -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & -\frac{\lambda_b}{2\mu_b(3\lambda_b + 2\mu_b)} & \frac{\lambda_b + \mu_b}{\mu_b(3\lambda_b + 2\mu_b)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_b} + Z_T & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_b} + Z_T \end{bmatrix} \quad (12)$$

Note that Poisson's ratio ν_b and Young's modulus E_b for the isotropic background medium are given by,

$$\nu_b \equiv -S_{13}/S_{33} = \lambda_b/2(\lambda_b + \mu_b), \quad 1/E_b \equiv S_{33},$$

respectively. From equation (12) and the fact that Z_T and Z_N are positive, the compliances of a TI(LSD) medium satisfy the following constraints:

$$S_{44} \le S_{55}, \; -2S_{33} < -2S_{13} < S_{33} \le S_{11},$$

$$S_{13} = S_{23} \equiv S_{33} - \frac{S_{44}}{2}, \quad (13)$$

where the second constraint becomes $0 < -2S_{13} < S_{33} \le S_{11}$ for $\nu_b > 0$. Note that any TI medium satisfying constraints [equation (13)] is equivalent to a TI(LSD) medium, and further, the background parameters and the presumed fracture compliances and background moduli are recoverable from a full knowledge of the TI(LSD) compliances (Hsu and Schoenberg, 1993).

Inverting equation (12) yields the elastic stiffness matrix,

The constraints on the stiffness matrix components, analogous to equation (13), for the medium to be TI(LSD) are,

$$C_{55} \le C_{44}, \; -\frac{C_{11}}{2} < C_{13} < C_{11} \le C_{33},$$

$$C_{11}C_{33} - C_{13}^2 = 2C_{44}(C_{11} + C_{13}), \quad (15)$$

where, as above, the second constraint becomes $0 < C_{13} < C_{11} \le C_{33}$ for $\nu_b > 0$.

Shear waves propagating in a direction parallel to the fractures, taken here to be the x_3-direction (which is usually vertical when the normal to the fractures, the x_1-direction, is horizontal), can propagate at two different velocities depending on their polarization. The higher velocity is for polarization parallel to the fractures, the x_2-direction, and is controlled by C_{44}. The lower velocity is for polarization perpendicular to the fractures and is controlled by C_{55}. For this choice of reference axes, the difference of the two shear-wave velocities from equation (14) is,

$$\underset{\sim}{C} = \begin{bmatrix} M_b(1-\delta_N) & \lambda_b(1-\delta_N) & \lambda_b(1-\delta_N) & 0 & 0 & 0 \\ \lambda_b(1-\delta_N) & M_b(1-r_b^2\delta_N) & \lambda_b(1-r_b\delta_N) & 0 & 0 & 0 \\ \lambda_b(1-\delta_N) & \lambda_b(1-r_b\delta_N) & M_b(1-r_b^2\delta_N) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_b & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_b(1-\delta_T) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_b(1-\delta_T) \end{bmatrix}, \quad (14)$$

where

$$M_b = \lambda_b + 2\mu_b, \; r_b = \frac{\lambda_b}{M_b} \equiv \frac{\nu_b}{1-\nu_b},$$

$$0 \le \delta_T = \frac{Z_T\mu_b}{1 + Z_T\mu_b} < 1, \; 0 \le \delta_N = \frac{Z_N M_b}{1 + Z_N M_b} < 1.$$

$$\sqrt{C_{44}/\rho_b} - \sqrt{C_{55}/\rho_b} = \sqrt{\mu_b/\rho_b}(1 - \sqrt{1-\delta_T}). \quad (16)$$

It is interesting to note that the form of equation (14) is *identical* to that of the stiffness matrix shown by Crampin (1984) for an isotropic medium with a vertical array of parallel, penny-shaped fractures, even though his expressions include second-order terms in the fracture density.

However, an expansion of the elastic stiffness tensor to second-order in the crack density may be unnecessarily precise for geophysical purposes since real fractures do not resemble ellipsoids. For example, the assumption of dry ellipsoidal cracks of a given aspect ratio will fix the ratio of Z_N to Z_T, but this, in a real rock is a rough approximation, even to first order in crack density. Roughness, asperities, and infilling debris will affect those results. That the form of the result agrees with the form given here suggests that the general behavior of a solid with any kind of fractures will be the same, independent of the details of the actual fracture-like voids. Further, this means that shape and texture details are not recoverable from long wavelength or quasi-static data. But no matter where estimates of fracture parameters are obtained, the assumption of rotationally invariant fractures and pure elastic behavior implies their contribution to the elastic compliance can be modeled with two nonnegative real compliances Z_N and Z_T, which will, in general, be stress-dependent.

The two non-negative dimensionless fracture parameters, δ_N, $\delta_T < 1$, have simple physical interpretations. They relate the fracture compliance to the total compliance of the fractured medium, i.e., δ_N is the part of the strain ε_{11} that is a result of the normal fracture compliance divided by the total ε_{11}. Similarly, δ_T corresponds to the part of total shear compliance (on a vertical plane) that is a result of the vertical tangential fracture compliance. This allows the seismic response of a fractured reservoir to be modeled by varying the two parameters δ_N and δ_T independently.

The relative magnitudes of Z_N and Z_T control the anellipticity of this medium. The anellipticity for a TI medium whose symmetry axis is in the 1-direction is given by (Gassmann, 1964):

$$(c_{11} - c_{55})(c_{33} - c_{55}) - (c_{13} + c_{55})^2.$$

From equation (14), the anellipticity in terms of μ_b, M_b, Z_T, and Z_N can be calculated. It is given by

$$\frac{4\mu_b^2(M_b - \mu_b)}{(1 + Z_T\mu_b)(1 + Z_N M_b)}(Z_T - Z_N).$$

If $Z_N < Z_T$, the medium has positive anellipticity, the usual geological situation. If $Z_N = Z_T$, the anellipticity vanishes and the medium is elliptical. If $Z_N > Z_T$, the medium has negative anellipticity. An elliptical medium is a special case of a TI medium in that the quasi-P sheet of the slowness surface, and hence the wave surface also, is ellipsoidal, while the quasi-S sheet for shear waves with polarizations lying in the plane of propagation, and hence the corresponding wave surface also, is spherical.

SCALAR FRACTURE SETS

In a medium with many sets of aligned fractures, a particular set, say the mth set, is called "scalar" when in addition to being rotationally invariant, the normal and tangential compliances satisfy $Z_N^{(m)} = Z_T^{(m)} \equiv Z^{(m)}$, i.e., from equation (8), $Z_{ij}^{(m)} \equiv Z^{(m)}\delta_{ij}$ so that the displacement discontinuity vector is parallel to the traction. The behavior of such a fracture set is prescribed by a single positive scalar. When the background medium is isotropic and each of the fracture sets is scalar, the lowest possible symmetry of the fractured medium is orthorhombic (Sayers and Kachanov, 1991) for arbitrary values of scalar compliances, the $Z^{(m)}$, and arbitrary orientations, the $n^{(m)}$. This may be proved as follows:

Define α_{ij} for multiple scalar fracture sets, each with scalar compliance $Z^{(m)}$ and normal unit vector $n_i^{(m)}$, to be the following symmetric second rank tensor,

$$\alpha_{ij} = \sum_m Z^{(m)} n_i^{(m)} n_j^{(m)}. \tag{17}$$

Since all the $Z^{(m)} > 0$, α_{ij} is nonnegative definite. It follows from (6) and (8) that,

$$\begin{aligned} s_{ijk\ell_f} &= \frac{1}{4}\sum_m (Z^{(m)}\delta_{ik} n_\ell^{(m)} n_j^{(m)} + Z^{(m)}\delta_{jk} n_\ell^{(m)} n_i^{(m)} \\ &\quad + Z^{(m)}\delta_{i\ell} n_k^{(m)} n_j^{(m)} + Z^{(m)}\delta_{j\ell} n_k^{(m)} n_i^{(m)}) \\ &= \tfrac{1}{4}(\delta_{ik}\alpha_{\ell j} + \delta_{jk}\alpha_{\ell i} + \delta_{i\ell}\alpha_{kj} + \delta_{j\ell}\alpha_{ki}). \end{aligned} \tag{18}$$

In the coordinate system in which α_{ij} is diagonal, with diagonal components $\alpha_1, \alpha_2, \alpha_3$, from equation (18) it is seen that the components of $s_{ijk\ell_f}$ vanish whenever any one index is different from the other three. This is the defining condition of orthorhombic symmetry. To see this, assume our coordinate system is the one in which α_{ij} is diagonal, so that,

$$\begin{aligned} s_{1111_f} &= \alpha_1 \ge 0, \\ s_{2222_f} &= \alpha_2 \ge 0, \\ s_{3333_f} &= \alpha_3 \ge 0, \\ s_{2323_f} = s_{3232_f} = s_{2332_f} = s_{3223_f} &= \frac{\alpha_2 + \alpha_3}{4} \ge 0, \\ s_{1313_f} = s_{3131_f} = s_{1331_f} = s_{3113_f} &= \frac{\alpha_1 + \alpha_3}{4} \ge 0, \\ s_{1212_f} = s_{2121_f} = s_{1221_f} = s_{2112_f} &= \frac{\alpha_1 + \alpha_2}{4} \ge 0. \end{aligned} \tag{19}$$

When this fourth rank compliance tensor is added to any isotropic compliance tensor, the result is a positive definite tensor with orthorhombic symmetry, completing the proof.

In conventional 2-subscript condensed 6×6 matrix notation, then, the excess compliance matrix, from equation (19), may be written,

$$\underset{\sim}{S}_f = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 + \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 + \alpha_2 \end{bmatrix}. \tag{20}$$

If the background medium is orthorhombic, then for the fractured rock to be orthorhombic, the background's natural coordinate axes must coincide with the principal directions of α_{ij}. Such an excess compliance matrix is equal to the excess compliance matrix of three mutually perpendicular scalar fracture sets, one fracture set perpendicular to the 1-direction with its scalar compliance equal to α_1, the second perpendicular to the 2-direction with its scalar compliance equal to α_2, and the third perpendicular to the 3-direction with its scalar compliance equal to α_3. Thus any number of arbitrary scalar fracture sets in a medium is long wavelength equivalent to three mutually perpendicular scalar fracture sets at most.

If this compliance matrix is added to an isotropic background compliance matrix, then, not only may it be seen that the fractured medium is orthorhombic, but further, such an orthorhombic medium depends on only four parameters, not five (μ_b, λ_b, α_1, α_2, α_3) one might expect. We call an orthorhombic medium equivalent to such a fractured medium, a "scalar orthorhombic" medium. To see how there are only four parameters for a scalar orthorhombic medium, consider the form of such a fractured medium, although for simplicity, we shall use Young's modulus E_b, and Poisson's ratio ν_b, and note then that,

$$\frac{1}{\mu_b} \equiv \frac{2(1+\nu_b)}{E_b}.$$

Then, the 6×6 compliance matrix $\underset{\sim}{S} = \underset{\sim}{S}_b + \underset{\sim}{S}_f$ may be written,

$$\begin{bmatrix} \frac{1}{E_b}+\alpha_1 & -\frac{\nu_b}{E_b} & -\frac{\nu_b}{E_b} & 0 & 0 & 0 \\ -\frac{\nu_b}{E_b} & \frac{1}{E_b}+\alpha_2 & -\frac{\nu_b}{E_b} & 0 & 0 & 0 \\ -\frac{\nu_b}{E_b} & -\frac{\nu_b}{E_b} & \frac{1}{E_b}+\alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_b)}{E_b}+\alpha_2+\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu_b)}{E_b}+\alpha_1+\alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu_b)}{E_b}+\alpha_1+\alpha_2 \end{bmatrix},$$

and letting dimensionless $\beta_i \equiv 1 + E_b\alpha_i$, $i = 1, 2, 3$, $\underset{\sim}{S}$ may be written in terms of the β_i/E_b and ν_b/E_b only, as:

$$\begin{bmatrix} \frac{\beta_1}{E_b} & -\frac{\nu_b}{E_b} & -\frac{\nu_b}{E_b} & 0 & 0 & 0 \\ -\frac{\nu_b}{E_b} & \frac{\beta_2}{E_b} & \frac{\nu_b}{E_b} & 0 & 0 & 0 \\ -\frac{\nu_b}{E_b} & -\frac{\nu_b}{E_b} & \frac{\beta_1}{E_b} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\nu_b}{E_b}+\frac{\beta_2}{E_b}+\frac{\beta_3}{E_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\nu_b}{E_b}+\frac{\beta_1}{E_b}+\frac{\beta_3}{E_b} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\nu_b}{E_b}+\frac{\beta_1}{E_b}+\frac{\beta_2}{E_b} \end{bmatrix}. \qquad (21)$$

Then the stiffness matrix, found by inversion of the compliance matrix, is,

$$E_b \begin{bmatrix} \frac{\beta_2\beta_3 - \nu_b^2}{\Delta} & \frac{\nu_b(\nu_b + \beta_3)}{\Delta} & \frac{\nu_b(\nu_b + \beta_2)}{\Delta} & 0 & 0 & 0 \\ \frac{\nu_b(\nu_b + \beta_3)}{\Delta} & \frac{\beta_1\beta_3 - \nu_b^2}{\Delta} & \frac{\nu_b(\nu_b + \beta_1)}{\Delta} & 0 & 0 & 0 \\ \frac{\nu_b(\nu_b + \beta_2)}{\Delta} & \frac{\nu_b(\nu_b + \beta_1)}{\Delta} & \frac{\beta_1\beta_2 - \nu_b^2}{\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\nu_b + \beta_2 + \beta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\nu_b + \beta_1 + \beta_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\nu_b + \beta_1 + \beta_2} \end{bmatrix}, \quad (22)$$

$$\Delta = \beta_1\beta_2\beta_3 - \nu_b^2(\beta_1 + \beta_2 + \beta_3) - 2\nu_b^3,$$

and evaluating the anellipticity in each of the coordinate planes yields the interesting result that a medium with a compliance matrix of the form of equation (21) has vanishing anellipticity in each of the coordinate planes (Kachanov, 1980, 1992).

For gas-saturated cracked porous sandstones, the assumption that $Z_N = Z_T$ for each set of cracks may not be unreasonable. For dry penny-shaped cracks $Z_N/Z_T = 1 - \nu_b/2$ [for example, see Hudson (1981), or Sayers and Kachanov (1991)] and from measurements on Berea sandstone, for example, $\nu_b = 0.11$ (Lo et al., 1986). This yields $(Z_T - Z_N)/(Z_T + Z_N) = 0.028$ as a measure of how much Z_N differs from Z_T. Hsu and Schoenberg (1993) have recently reported measurements of ultrasonic velocities made on a block composed of lucite plates with roughened surfaces pressed together with a static normal stress to simulate a fractured medium. Table 1 gives values of $(Z_T - Z_N)/(Z_T + Z_N)$ calculated using the values of c_{11}/ρ, c_{33}/ρ, c_{44}/ρ and c_{66}/ρ reported in Table 1 of Hsu and Schoenberg (1993). It is seen that the approximation $Z_N = Z_T$ is reasonable, particularly at the higher stress levels.

Table 1. $(Z_T - Z_N)/(Z_T + Z_N)$ calculated from the measurements of Hsu and Schoenberg (1993) as a function of the stress applied normal to the joints.

Stress (MPa)	$(Z_T - Z_N)/(Z_T + Z_N)$
6	0.1736
12	0.1177
18	0.0332
24	−0.0035

CONCLUSION

For seismic modeling of fractured media, when the fractures seem to be aligned in one or several directions, and wavelengths are much larger than the fracture spacing, it is convenient to formulate the equivalent anisotropic medium problem in terms of the elastic compliance. The assumption of linear slip deformation, essentially equation (3), or equivalently, equation (7), has some experimental validation, for example Cheadle et al. (1991) whose measurements were shown by Hood (Private Communication, 1991) to fit the VFTI assumption, and Hsu and Schoenberg (1993) who demonstrated that the behavior of roughened surfaces in Plexiglas fit the LSD model. What can be obtained from seismic data is the orientation of the dominant set of fractures and some estimate of the fracture compliances relative to the compliance of the complete fractured rock. The estimation of the shape and size distribution is beyond the capability of long wavelength seismic data.

The formulation presented in this paper is expected to be of wide applicability because of (a) its simplicity because of the additive property of the compliances, and (b) its utility since the theory mimics the behavior of sets of large parallel joints as well aligned micro-fractures. One might expect that the theory is at least approximately accurate in modeling the behavior of fractured rock for all the intermediate sizes (relative to wavelength) as well.

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