AE 314

MODULE I

BASIC ATTITUDE MECHANICS

EULER ANGLES; EULER RATES

Any rotation can be specified by 3 independent parameters.It is often convenient to use the so–called *EULER ANGLES*. Suppose the rotation in question transforms the right–handed orthogonal triad of unit vectors $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$ into the right–handed triad $\{\mathbf{i}_\xi, \mathbf{i}_\eta, \mathbf{i}_\zeta\}$. This can be broken down into a sequence of three elementary rotations:–

(1) A counter–clockwise rotation about the z–axis through Ω, which transforms $\mathbf{i_x}$ into a unit vector $\mathbf{i_n}$. The angle ω is determined by the requirement that $\mathbf{i_n}$ be perpendicular to the plane of $\mathbf{i_z}$ and $\mathbf{i}_\zeta$. This yields a new triad $\{\mathbf{i_n}, \mathbf{i_z} \times \mathbf{i_n}, \mathbf{i_z}\}$.

(2) A counter–clockwise rotation about the n–axis through i, which transforms $\mathbf{i}_z$ into $\mathbf{i}_\zeta$. This yields a new triad $\{\mathbf{i}_n, \mathbf{i}_\zeta \times \mathbf{i}_n, \mathbf{i}_\zeta\}$.

(3) A counter–clockwise rotation about the ζ–axis through ω, which transforms $\mathbf{i}_n$ into $\mathbf{i}_\xi$. This yields the triad $\{\mathbf{i}_\xi, \mathbf{i}_\eta, \mathbf{i}_\zeta\}$.

The first rotation gives:–

$$\left.\begin{aligned} \mathbf{i}_n &= \cos\Omega\,\mathbf{i}_x + \sin\Omega\,\mathbf{i}_y \\ \mathbf{i}_z \times \mathbf{i}_n &= -\sin\Omega\,\mathbf{i}_x + \cos\Omega\,\mathbf{i}_y \\ \mathbf{i}_z &= \mathbf{i}_z \end{aligned}\right\} \tag{1}$$

The second rotation gives:–

$$\left.\begin{aligned} \mathbf{i}_n &= \mathbf{i}_n \\ \mathbf{i}_\zeta \times \mathbf{i}_n &= \cos i\,\mathbf{i}_z \times \mathbf{i}_n + \sin i\,\mathbf{i}_z \\ \mathbf{i}_\zeta &= -\sin i\,\mathbf{i}_z \times \mathbf{i}_n + \cos i\,\mathbf{i}_z \end{aligned}\right\} \tag{2}$$

The third rotation gives:–

$$\left.\begin{aligned} \mathbf{i}_\xi &= \cos\omega\,\mathbf{i}_n + \sin\omega\,\mathbf{i}_\zeta \times \mathbf{i}_n \\ \mathbf{i}_\eta &= -\sin\omega\,\mathbf{i}_n + \cos\omega\,\mathbf{i}_\zeta \times \mathbf{i_n} \\ \mathbf{i}_\zeta &= \mathbf{i}_\zeta \end{aligned}\right\} \tag{3}$$

It follows that

$$\left.\begin{aligned} l_1 &= \mathbf{i}_x \cdot \mathbf{i}_\xi = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ l_2 &= \mathbf{i}_x \cdot \mathbf{i}_\eta = -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ l_3 &= \mathbf{i}_x \cdot \mathbf{i}_\zeta = \sin\Omega\sin i \\ m_1 &= \mathbf{i}_y \cdot \mathbf{i}_\xi = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ m_2 &= \mathbf{i}_y \cdot \mathbf{i}_\eta = -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ m_3 &= \mathbf{i}_y \cdot \mathbf{i}_\zeta = -\cos\Omega\sin i \\ n_1 &= \mathbf{i}_z \cdot \mathbf{i}_\xi = \sin\omega\sin i \\ n_2 &= \mathbf{i}_z \cdot \mathbf{i}_\eta = \cos\omega\sin i \\ n_3 &= \mathbf{i}_z \cdot \mathbf{i}_\zeta = \cos i \end{aligned}\right\} \tag{4}$$

Suppose that the Euler angles vary with time. Our next task is to express the components $\omega_\xi, \omega_\eta,$ andω_ζ in terms of the Euler angles and their time–derivatives. We shall do this via the chain rule. We begin by noting the fact that

$$\left.\begin{aligned} \frac{\partial \mathbf{i}_\zeta}{\partial \Omega} &= \sin\, i\, \mathbf{i}_n = \sin\, i \cos\, \omega\, \mathbf{i}_\xi + \sin\, i \sin\, \omega\, \mathbf{i}_\eta \\ \frac{\partial \mathbf{i}_\zeta}{\partial i} &= -\cos\, i(\mathbf{i}_z \times \mathbf{i}_n) - \sin\, i\, \mathbf{i}_z = -\mathbf{i}_\zeta \times \mathbf{i}_n = -\sin\, \omega\, \mathbf{i}_\xi - \cos\, \omega\, \mathbf{i}_\eta \\ \frac{\partial \mathbf{i}_\zeta}{\partial \omega} &= 0 \end{aligned}\right\} \tag{5}$$

Moreover,

$$\left.\begin{aligned} \frac{\partial \mathbf{i}_\xi}{\partial \Omega} &= \cos\, \omega\, \frac{\partial \mathbf{i}_n}{\partial \Omega} + \sin\, \omega \left\{ \frac{\partial \mathbf{i}_\zeta}{\partial \Omega} \times \mathbf{i}_n + \mathbf{i}_\zeta \times \frac{\partial \mathbf{i}_n}{\partial \Omega} \right\} \\ &= \cos\, \omega\, \mathbf{i}_z \times \mathbf{i}_n + \sin\, \omega \Big\{ -\cos\, i\, \mathbf{i}_n \Big\} \\ &= \cos\, \omega \Big\{ \sin\, \omega \cos\, i\, \mathbf{i}_\xi + \cos\, \omega \cos\, i\, \mathbf{i}_\eta - \sin\, \mathbf{i}_\zeta \Big\} - \sin\, \omega \cos\, i \Big\{ \cos\, \omega\, \mathbf{i}_\xi - \sin\, \omega\, \mathbf{i}_\eta \Big\} \\ &= \cos\, i\, \mathbf{i}_\eta - \cos\, \omega \sin\, i\, \mathbf{i}_\zeta \\ \frac{\partial \mathbf{i}_\xi}{\partial i} &= \sin\, \omega\, \frac{\partial \mathbf{i}_\zeta}{\partial i} \times \mathbf{i}_n = -\sin\, \omega \Big[\Big(\mathbf{i}_\zeta \times \mathbf{i}_n \Big) \times \mathbf{i}_n \Big] = \sin\, \omega\, \mathbf{i}_\zeta \\ \frac{\partial \mathbf{i}_\xi}{\partial \omega} &= \mathbf{i}_\eta \end{aligned}\right\} \tag{6}$$

We now use the fact that

$$\frac{d\{\mathbf{i}_\zeta\}}{dt} = \omega_\eta\, \mathbf{i}_\xi - \omega_\xi\, \mathbf{i}_\eta = \dot{\Omega} \frac{\partial \mathbf{i}_\zeta}{\partial \Omega} + \frac{di}{dt} \frac{\partial \mathbf{i}_\zeta}{\partial i} + \dot{\omega} \frac{\partial \mathbf{i}_\zeta}{\partial \omega} \tag{7}$$

in combination with (5) to conclude that

$$\omega_\eta = \cos\, \omega \sin\, i\, \dot{\Omega} - \sin\, \omega\, \frac{di}{dt}$$

$$\omega_\xi = \sin\, \omega \sin\, i\, \dot{\Omega} + \cos\, \omega\, \frac{di}{dt}$$

Similarly,

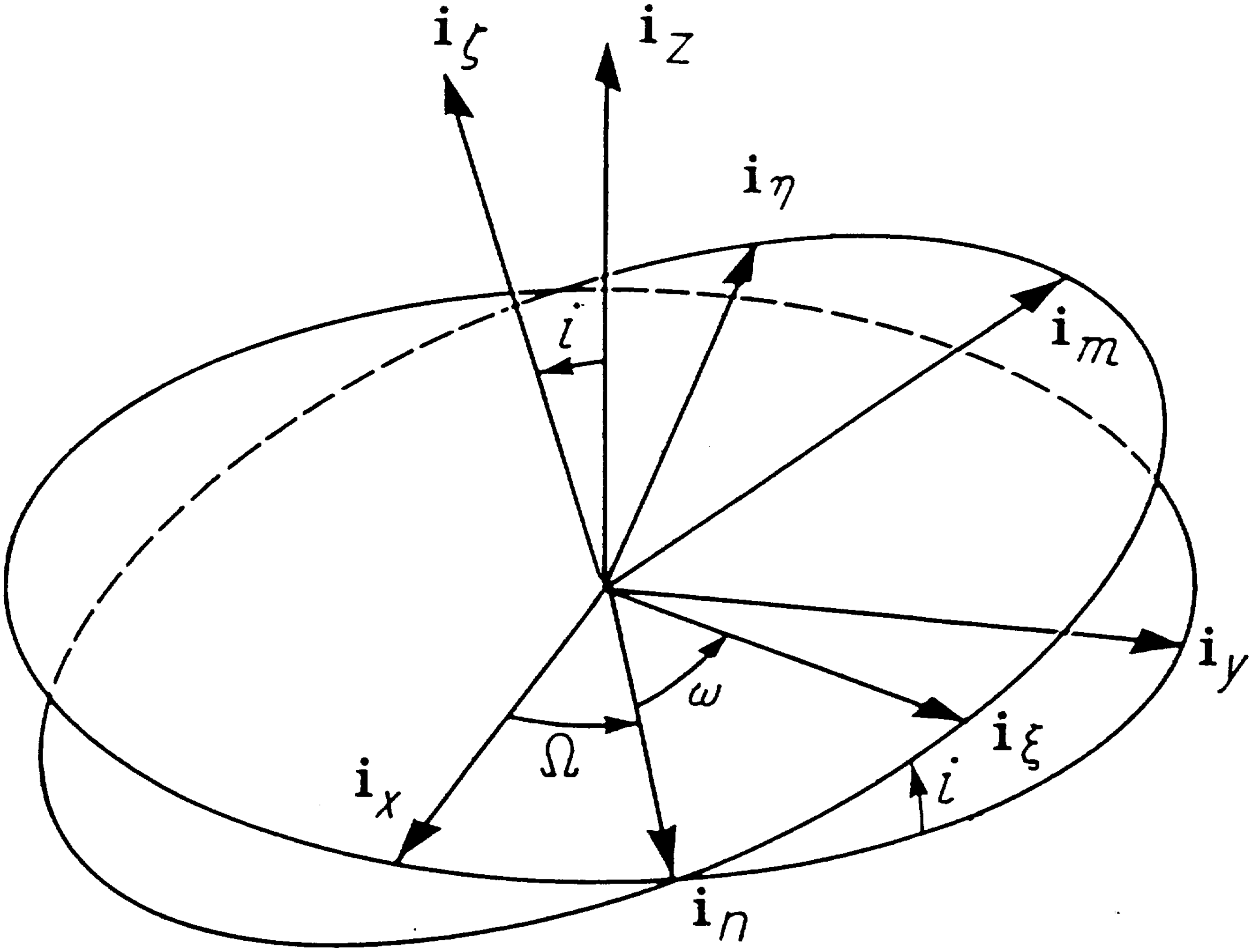
$$\frac{d\{\mathbf{i}_\xi\}}{dt} = -\omega_\eta\, \mathbf{i}_\zeta + \omega_\zeta\, \mathbf{i}_\eta = \dot{\Omega} \frac{\partial \mathbf{i}_\xi}{\partial \Omega} + \frac{di}{dt} \frac{\partial \mathbf{i}_\xi}{\partial i} + \dot{\omega} \frac{\partial \mathbf{i}_\xi}{\partial \omega} \tag{8}$$

so that

$$\omega_\zeta = \dot{\omega} + \dot{\Omega}\, \cos\, i.$$

Notice that

$$\omega_\xi\, \mathbf{i}_\xi + \omega_\eta\, \mathbf{i}_\eta + \omega_\zeta\, \mathbf{i}_\zeta = \dot{\Omega}\, \mathbf{i}_z + \frac{di}{dt}\, \mathbf{i}_n + \dot{\omega}\, \mathbf{i}_\zeta. \tag{9}$$



$\mathbf{i}_\zeta$
$\mathbf{i}_Z$
$\mathbf{i}_\eta$
$\mathbf{i}_m$
i
$\mathbf{i}_y$
ω
$\mathbf{i}_\xi$
Ω
i
$\mathbf{i}_x$
$\mathbf{i}_n$

JACOBIAN ELLIPTIC FUNCTIONS

Consider the differential equation

$$\left\{\frac{dy}{dt}\right\}^2 = \{1-y^2\}\{1-k^2y^2\} \qquad (0<k<1) \tag{1}$$

Consider, first, the limiting case k=0:

$$t = \int_0^{y(t)} \frac{dy}{\sqrt{\{1-y^2\}}} + c_1 = \arcsin(y(t)) + c_1 \tag{2}$$

so that

$$y = \sin(t - c_1) \tag{3}$$

Consider, now, the limiting case k=1:

$$t = \int_0^{y(t)} \frac{dy}{\{1-y^2\}} + c_1 = \frac{1}{2}\ln\left\{\frac{1+y(t)}{1-y(t)}\right\} + c_1 \tag{4}$$

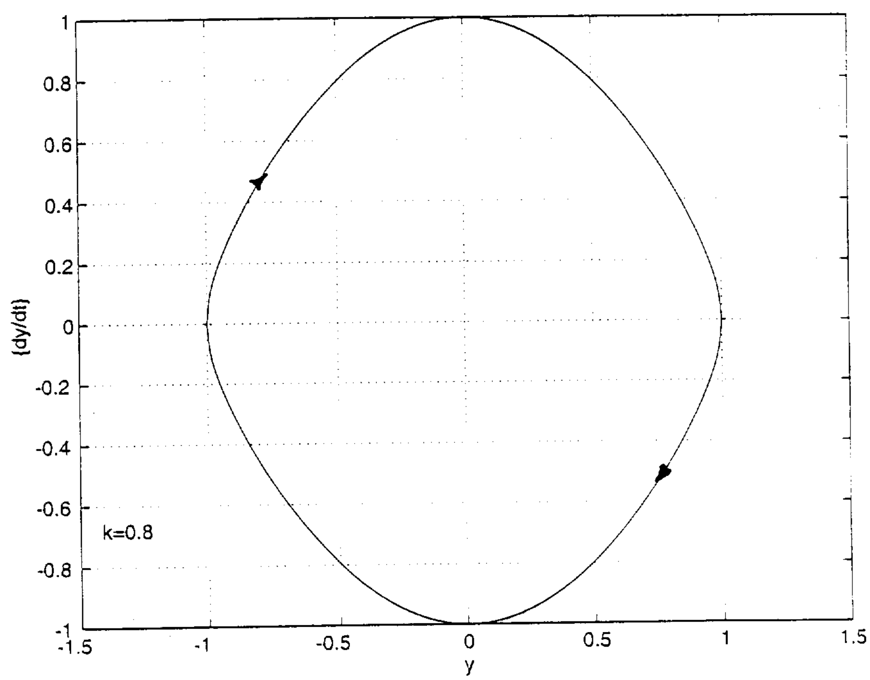
so that

$$y = \tanh(t - c_1) \tag{5}$$

For $0 < k < 1$:

$$t = \int_0^{y(t)} \frac{dy}{\sqrt{\{1-y^2\}\{1-k^2y^2\}}} + c_1 \tag{6}$$

The phase-plane diagram for (1) has the form



Clearly, solutions to (1) are periodic and oscillate between the values -1 and +1. The period of oscillation is 4K, where

$$K = \int_0^1 \frac{dy}{\sqrt{\{1-y^2\}\{1-k^2y^2\}}} \tag{7}$$

An expression of the form (7) is called a *complete elliptic integral.*

For each value $0 < k < 1$ of the parameter k, the Jacobian Elliptic Functions sn(t,k), cn(t,k), and dn(t,k) are defined as follows:

The function sn(t, k) is the periodic solution y(t) of (1) such that $y(0) = 0$ and $\left\{\frac{dy}{dt}\right\}_{t=0} = +1$.

$$cn^2(t,k) = 1 - sn^2(t,k) \qquad cn(0) = +1 \tag{8}$$

$$dn^2(t,k) = 1 - k^2 sn^2(t,k) \qquad dn(0) = +1 \tag{9}$$

In the limiting case k=0, cn(t,k) becomes $\cos t$, $dn(t,k) \to 1$. In the limiting case k=1, $cn(t,k) \to \operatorname{sech} t$ $dn(t,k) \to \operatorname{sech} t$.

DERIVATIVES OF JACOBIAN ELLIPTIC FUNCTIONS

It follows from (1) that

$$\frac{d}{dt}\{sn(t,k)\} = cn(t,k)\, dn(t,k) \tag{10}$$

Differentiation of (8), (9), and the use of (10) then yields

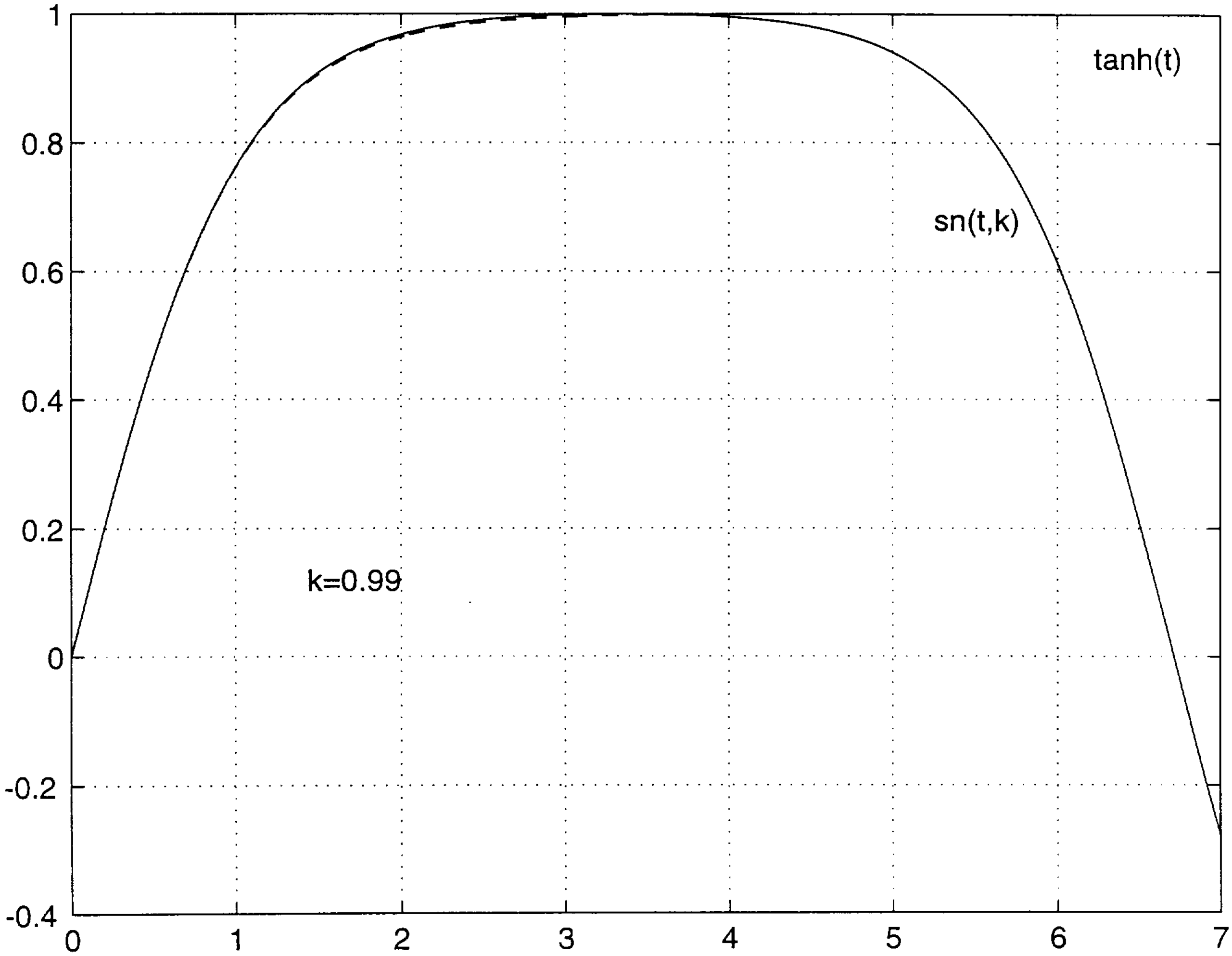
$$\frac{d}{dt}\{cn(t,k)\} = -sn(t,k)\, dn(t,k)$$

$$\frac{d}{dt}\{dn(t,k)\} = -k^2\, sn(t,k)\, cn(t,k)$$

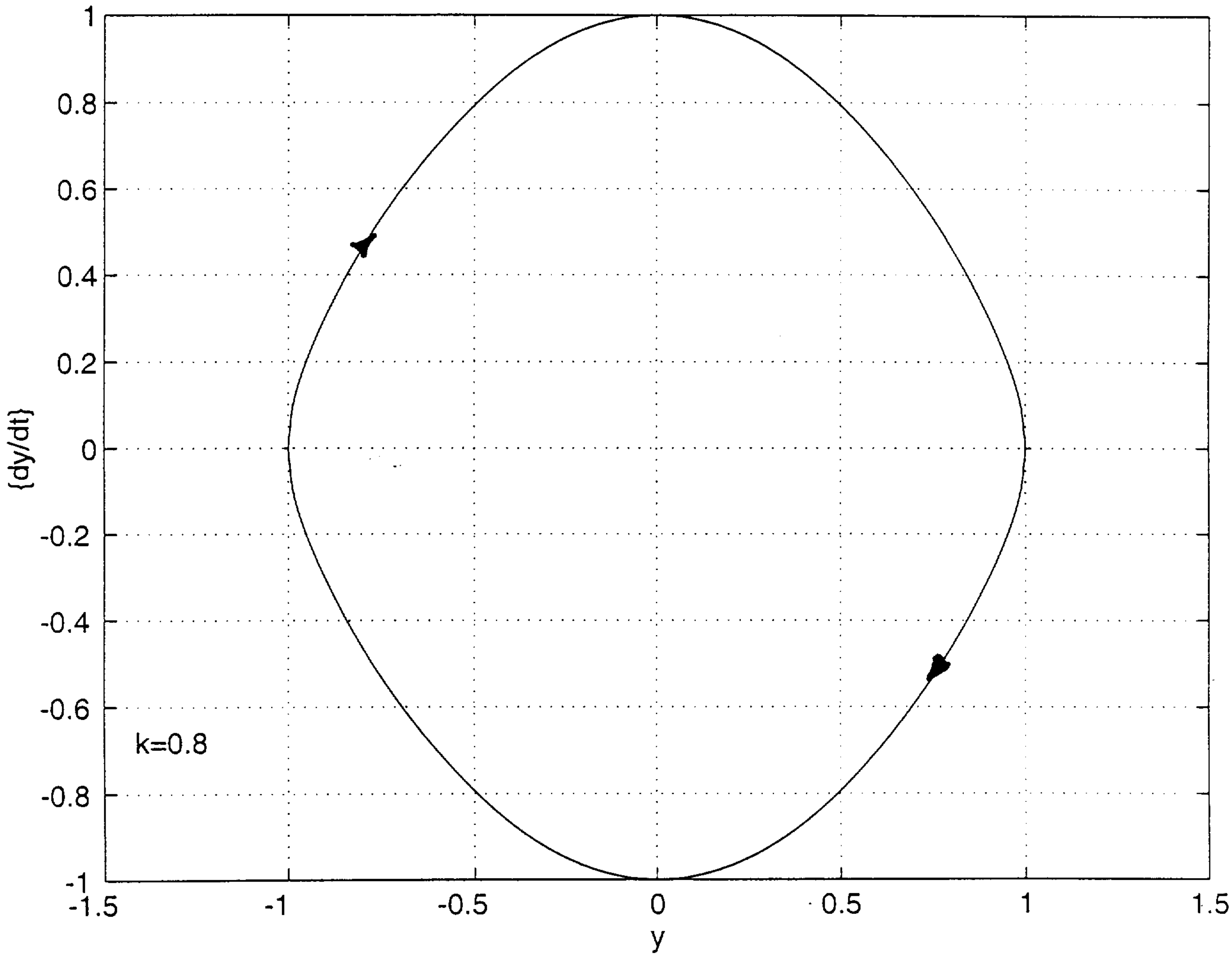
The first graph shows values of the Jacobian Elliptic Functions for k=0.8. The corresponding value of the quarter–period K is 1.9953, so the period is approximately equal to 8.

The second graph compares the functions sin(t) and sn(t,k) for k=0.6. The corresponding value of K is 1.75075, so the period is approximately 7.

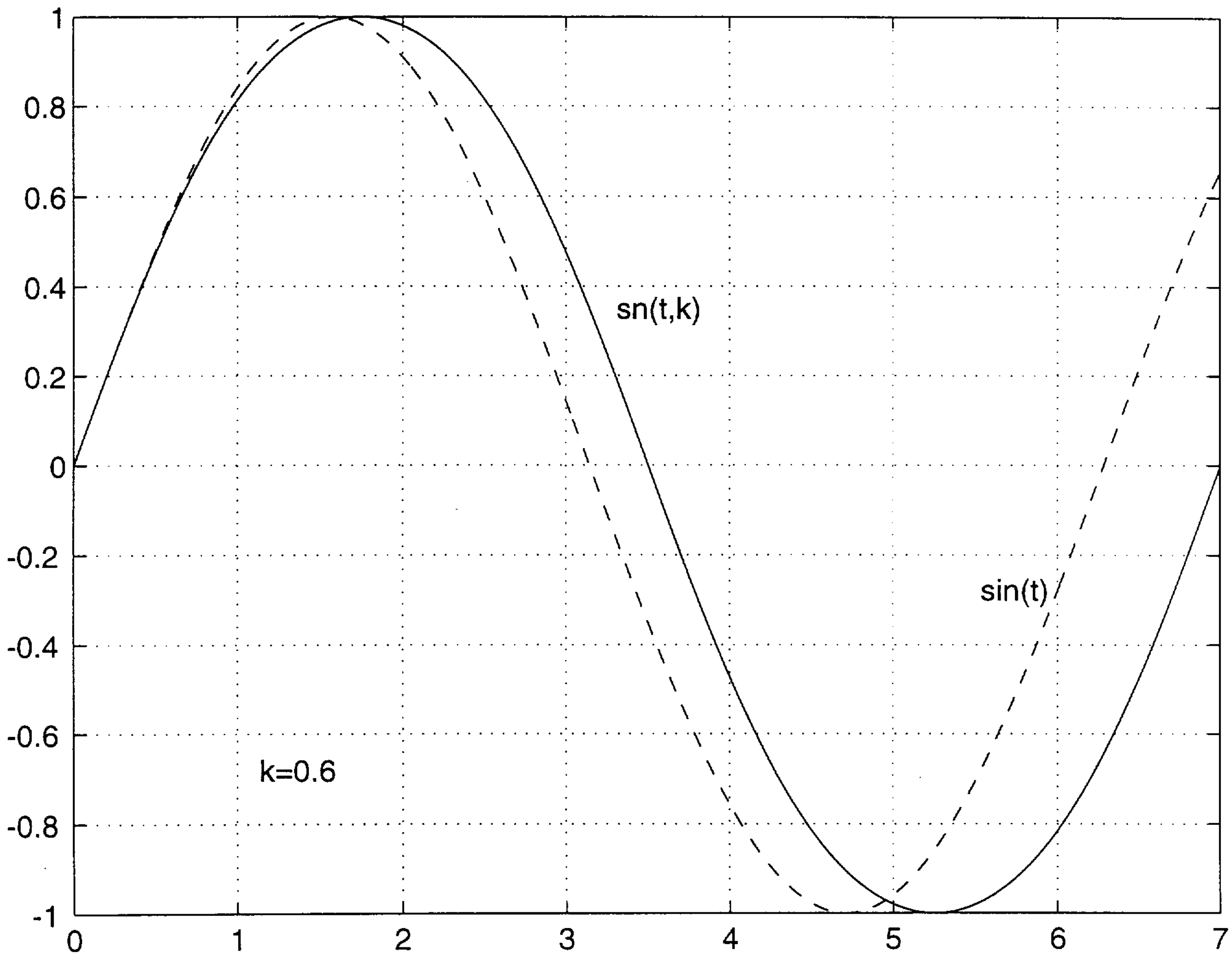
The third graph compares the functions tanh(t) and sn(t,k) for k=0.99. The corresponding value of K is 3.3566, and the period is approximately 13.4264.

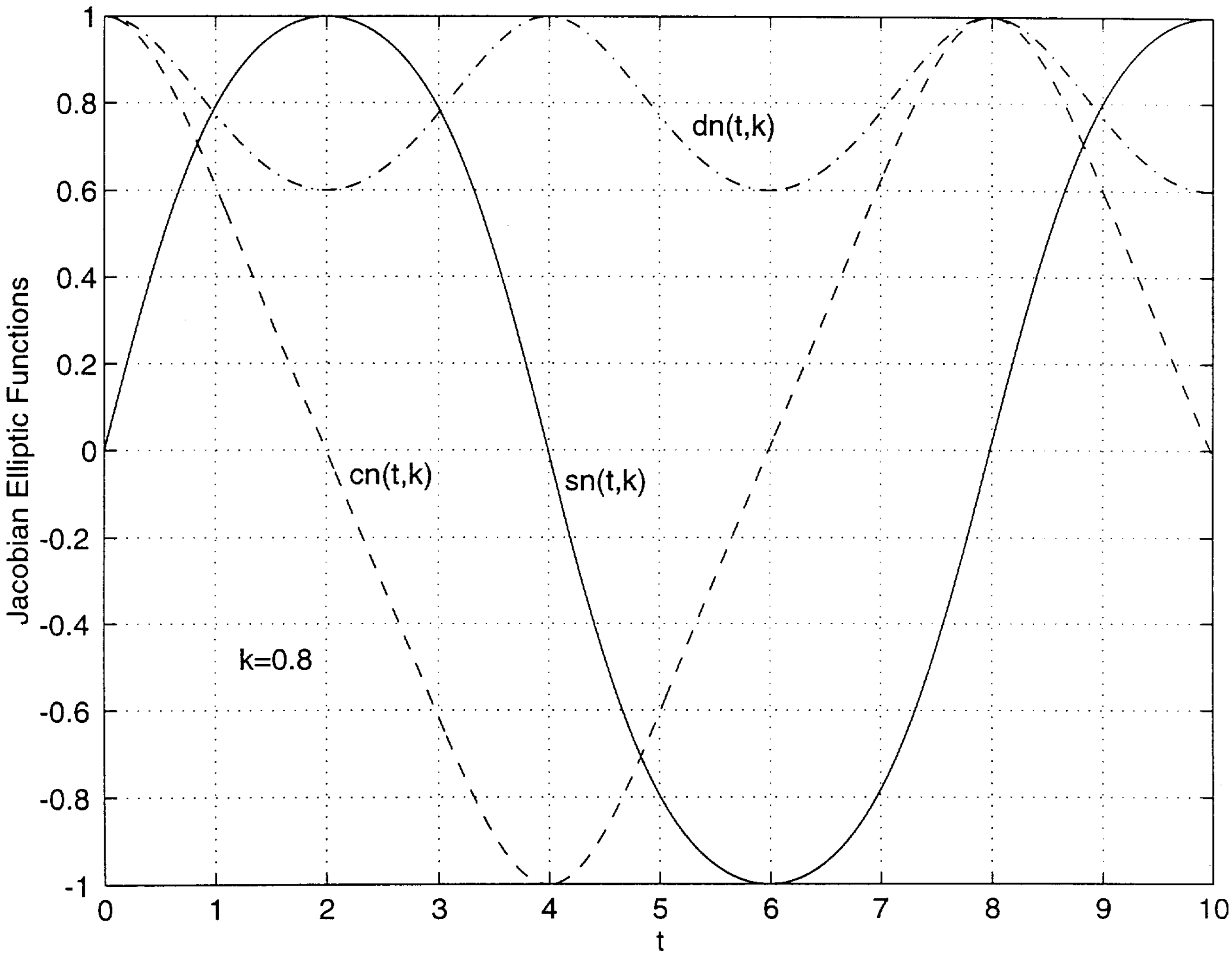


tanh(t)
sn(t,k)
k=0.99
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
2
3
4
5
6
7



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
{dy/dt}
-1.5
-1
-0.5
0
0.5
1
1.5
y
k=0.8


sn(t,k)
sin(t)
k=0.6
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7



Jacobian Elliptic Functions
t
dn(t,k)
cn(t,k)
sn(t,k)
k=0.8
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7
8
9
10

TORQUE–FREE MOTION

The principles governing the three–dimensional motion of a rigid body are:–
the Principle of Linear Momentum,

$$m\vec{a}_G = \sum \vec{F} \tag{1}$$

and the Principle of Angular Momentum, in either the form

$$\dot{\vec{H}}_G = \sum \vec{M}_G \tag{2}$$

or

$$\dot{\vec{H}}_A = \sum \vec{M}_A\,, \tag{2'}$$

G being the center-of-mass of the body, and A any fixed point in *space.*
In terms of a coordinate system Gxyz fixed in the body , and one GXYZ fixed in space, which instantaneously coincides with Gxyz,

$$\dot{\vec{H}}_G = (\dot{\vec{H}}_G)_{Gxyz} + \vec{\omega} \times \vec{H}_G \tag{3}$$

Thus, equation (2) may be written in component form as:–

$$\dot{H}_{Gx} + \omega_y H_{Gz} - \omega_z H_{Gy} = \sum M_{Gx}$$

$$\dot{H}_{Gy} + \omega_z H_{Gx} - \omega_x H_{Gz} = \sum M_{Gy}$$

$$\dot{H}_{Gz} + \omega_x H_{Gy} - \omega_y H_{Gx} = \sum M_{Gz}$$

PRINCIPAL AXES FORMULATION

If Gxyz are the principal axes of inertia of the body, so that $I_{xy} = I_{xz} = I_{yz} = 0$, then

$H_{Gx} = I_x\omega_x$, $H_{Gy} = I_y\omega_y$,$H_{Gz} = I_z\omega_z$. Equation (2) may therefore be written in component form as:–

$$I_x\dot{\omega}_x + [I_z - I_y]\omega_y\omega_z = \sum M_{Gx} \tag{4}$$

$$I_y\dot{\omega}_y + [I_x - I_z]\omega_x\omega_z = \sum M_{Gy} \tag{5}$$

$$I_z\dot{\omega}_z + [I_y - I_x]\omega_x\omega_y = \sum M_{Gz}\,. \tag{6}$$

Consider, now, the torque-free motion of a rigid body. If Gxyz are principal axes of inertia, then equations (4)–(6) reduce to

$$\mathrm{I_x}\dot{\omega}_\mathrm{x} + [\mathrm{I_z} - \mathrm{I_y}]\omega_\mathrm{y}\omega_\mathrm{z} = 0\,, \tag{7}$$

$$\mathrm{I_y}\dot{\omega}_\mathrm{y} + [\mathrm{I_x} - \mathrm{I_z}]\omega_\mathrm{x}\omega_\mathrm{z} = 0 \tag{8}$$

and,

$$\mathrm{I_z}\dot{\omega}_\mathrm{z} + [\mathrm{I_y} - \mathrm{I_x}]\omega_\mathrm{x}\omega_\mathrm{y} = 0 \tag{9}$$

Moreover, angular momentum and rotational kinetic energy are conserved, i. e.,

$$\vec{\mathrm{H}}_\mathrm{G} = \mathrm{I_x}\omega_\mathrm{x}\hat{\imath} + \mathrm{I_y}\omega_\mathrm{y}\hat{\jmath} + \mathrm{I_z}\omega_\mathrm{z}\hat{\mathrm{k}} = \mathrm{H}\hat{\mathrm{K}} = \mathrm{const} \tag{10}$$

for some fixed direction $\hat{\mathrm{K}}$ in space [the invariant line], and

$$\mathrm{H}^2 = (\mathrm{I_x}\omega_\mathrm{x})^2 + (\mathrm{I_y}\omega_\mathrm{y})^2 + (\mathrm{I_z}\omega_\mathrm{z})^2 = \mathrm{const} \tag{11}$$

$$2\mathrm{T_{rot}} = \mathrm{I_x}(\omega_\mathrm{x})^2 + \mathrm{I_y}(\omega_\mathrm{y})^2 + \mathrm{I_z}(\omega_\mathrm{z})^2 = \mathrm{const} \tag{12}$$

Suppose, now, that $\mathrm{I_x} > \mathrm{I_y} > \mathrm{I_z}$. Then

$$2\mathrm{I_x}\mathrm{T_{rot}} - \mathrm{H}^2 = \mathrm{I_y}(\mathrm{I_x} - \mathrm{I_y})\omega_\mathrm{y}^2 + \mathrm{I_z}(\mathrm{I_x} - \mathrm{I_z})\omega_\mathrm{z}^2 = \mathrm{const} \tag{13}$$

$$\mathrm{H}^2 - 2\mathrm{I_z}\mathrm{T_{rot}} = \mathrm{I_y}(\mathrm{I_y} - \mathrm{I_z})\omega_\mathrm{y}^2 + \mathrm{I_x}(\mathrm{I_x} - \mathrm{I_z})\omega_\mathrm{x}^2 = \mathrm{const} \tag{14}$$

so that

$$\omega_\mathrm{x}^2 = \mathrm{P} - \mathrm{Q}\omega_\mathrm{y}^2 \tag{15}$$

$$\omega_\mathrm{z}^2 = \mathrm{R} - \mathrm{S}\omega_\mathrm{y}^2 \tag{16}$$

where

$$\mathrm{P} = \frac{\mathrm{H}^2 - 2\mathrm{I_z}\mathrm{T_{rot}}}{\mathrm{I_x}(\mathrm{I_x} - \mathrm{I_z})} \qquad \mathrm{Q} = \frac{\mathrm{I_y}(\mathrm{I_y} - \mathrm{I_z})}{\mathrm{I_x}(\mathrm{I_x} - \mathrm{I_z})} \qquad \mathrm{R} = \frac{2\mathrm{I_x}\mathrm{T_{rot}} - \mathrm{H}^2}{\mathrm{I_z}(\mathrm{I_x} - \mathrm{I_z})} \qquad \mathrm{S} = \frac{\mathrm{I_y}(\mathrm{I_x} - \mathrm{I_y})}{\mathrm{I_z}(\mathrm{I_x} - \mathrm{I_z})}$$

It follows that

$$\dot{\omega}_\mathrm{y}^2 = \mathrm{D}^2\{\mathrm{P} - \mathrm{Q}\omega_\mathrm{y}^2\}\{\mathrm{R} - \mathrm{S}\omega_\mathrm{y}^2\} \tag{20}$$

where

$$\mathrm{D} = \frac{\mathrm{I_x} - \mathrm{I_z}}{\mathrm{I_y}}$$

Also,

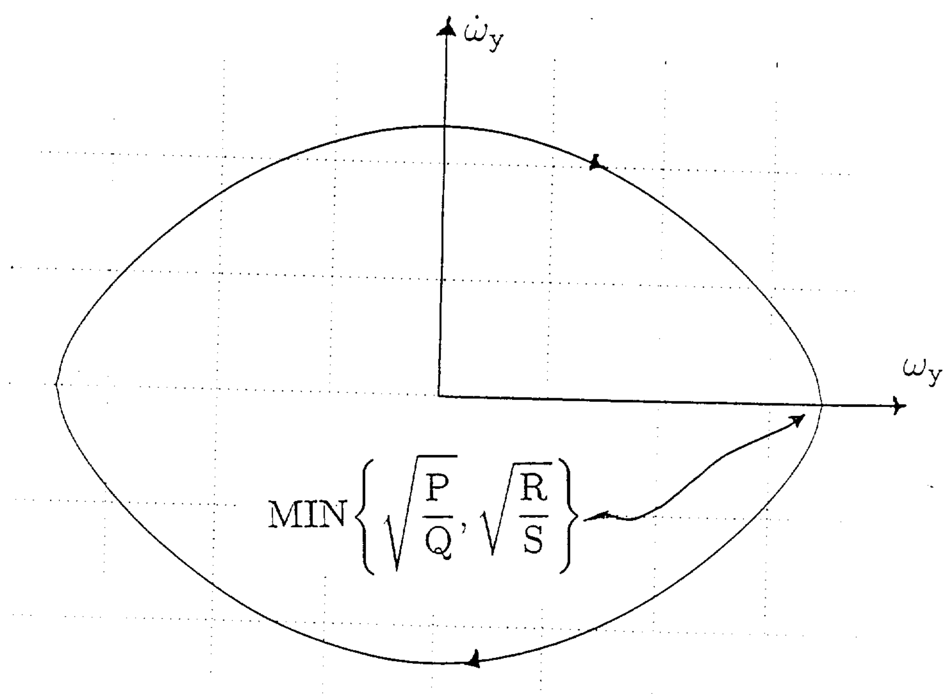
$$S\omega_x^2 - Q\omega_z^2 = SP - QR\,. \tag{18}$$

and

$$SP - QR = \frac{I_y}{I_x I_z [I_x - I_z]}\left\{H^2 - 2I_y T_{rot}\right\} \tag{19}$$

Suppose $SP - QR > 0$. Then, ω_y oscillates between the values $-\sqrt{\frac{R}{S}}$ and $\sqrt{\frac{R}{S}}$, ω_z oscillates between the values $-\sqrt{R}$ and $\sqrt{R}$, and ω_x oscillates either between the values $\sqrt{\frac{SP-QR}{S}}$ and $\sqrt{P}$ or, between the values $-\sqrt{P}$ and $-\sqrt{\frac{SP-QR}{S}}$.

If $QR - SP > 0$. Then, ω_y oscillates between the values $-\sqrt{\frac{P}{Q}}$ and $\sqrt{\frac{P}{Q}}$, ω_x oscillates between the values $-\sqrt{P}$ and $\sqrt{P}$, and ω_z oscillates either between the values $\sqrt{\frac{QR-SP}{Q}}$ and $\sqrt{P}$ or, between the values $-\sqrt{P}$ and $-\sqrt{\frac{QR-SP}{S}}$.



Suppose $SP - QR > 0$ Then (17) may be written as

$$\left\{\frac{d}{dt}\left\{\sqrt{\frac{S}{R}}\omega_y\right\}\right\}^2 = D^2PS\left\{1 - \left(\frac{QR}{PS}\right)\left(\frac{S}{R}\right)\omega_y^2\right\}\left\{1 - \left(\frac{S}{R}\right)\omega_y^2\right\} \tag{20}$$

Define

$$L = \sqrt{\frac{R}{S}} \qquad k = \sqrt{\frac{QR}{SP}} \qquad p = D\sqrt{SP}$$

It follows from (20) that

$$\omega_y = L\,\mathrm{sn}(p(t - t_0), k)$$

for some constant of integration t_0. It then follows from (15), (16), that

$$\omega_x^2 = P\,\mathrm{dn}^2(p(t - t_0), k) \qquad \omega_z^2 = R\,\mathrm{cn}^2(p(t - t_0), k)$$

In order to satisfy (8), take

$$\omega_x = \pm\sqrt{P}\,\mathrm{dn}(p(t - t_0), k) \qquad \omega_z = \mp\sqrt{R}\,\mathrm{cn}(p(t - t_0), k) \tag{21}$$

that is, take opposite signs in the square roots.

Suppose $SP - QR < 0$ Then (17) may be written as

$$\left\{\frac{d}{dt}\left\{\sqrt{\frac{Q}{P}}\omega_y\right\}\right\}^2 = D^2QR\left\{1 - \left(\frac{SP}{QR}\right)\left(\frac{Q}{P}\right)\omega_y^2\right\}\left\{1 - \left(\frac{Q}{P}\right)\omega_y^2\right\} \tag{22}$$

Define

$$L = \sqrt{\frac{P}{Q}} \qquad k = \sqrt{\frac{SP}{QR}} \qquad p = D\sqrt{QR}$$

It follows from (22) that

$$\omega_y = L\,\mathrm{sn}(p(t - t_0), k)$$

for some constant of integration t_0. It then follows from (15), (16), that

$$\omega_x^2 = P\,\mathrm{cn}^2(p(t - t_0), k) \qquad \omega_z^2 = R\,\mathrm{dn}^2(p(t - t_0), k)$$

In order to satisfy (8), take

$$\omega_x = \pm\sqrt{P}\,\mathrm{cn}(p(t - t_0), k) \qquad \omega_z = \mp\sqrt{R}\,\mathrm{dn}(p(t - t_0), k) \tag{23}$$

that is, take opposite signs in the square roots.

EULER ANGLES; EULER RATES

In Rigid–Body Mechanics, the attitude (orientation) of a rigid body is space is often specified by means of the Euler Angles relating an orthonormal frame { $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ } fixed in the body to an orthonormal frame { $\hat{I}$, $\hat{J}$, $\hat{K}$ } fixed in space. In this context, it is standard to denote the Euler angles by ψ, θ, and ϕ. The angle ψ is the angle between the $\hat{J}$–$\hat{K}$ and $\hat{k}$–$\hat{K}$ planes, the angle θ is the angle between the $\hat{k}$ and $\hat{K}$ directions, ϕ is the angle between the $\hat{k}$–$\hat{K}$ and $\hat{\jmath}$–$\hat{k}$ planes. In this notation, the unit vectors of the two triads are related as follows:–

$$\hat{\imath}= [\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta]\hat{I} + [\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta]\hat{J} + \sin\phi\sin\theta\,\hat{K}$$

$$\hat{\jmath}= [-\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta]\hat{I} + [-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta]\hat{J} + \cos\phi\sin\theta\,\hat{K}$$

$$\hat{k} = \sin\psi\sin\theta\hat{I} - \cos\psi\sin\theta\hat{J} + \cos\theta\,\hat{K}$$

The components of angular velocity are given in terms of the Euler rates, $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ by:–

$$\omega_x = \dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi$$

$$\omega_y = \dot{\psi}\cos\phi\sin\theta - \dot{\theta}\sin\phi$$

$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

The Euler rates are given in terms of the components of angular velocity by

$$\dot{\psi} = \frac{[\omega_x\ \sin\phi + \omega_y\ \cos\phi]}{\sin\theta}$$

$$\dot{\theta} = \omega_x\ \cos\phi - \omega_y\ \sin\phi$$

$$\dot{\phi} = -\omega_x\ \sin\phi\cot\theta - \omega_y\ \cos\phi\cot\theta + \omega_z$$

By taking the $\hat{K}$ direction to be the (fixed) direction of the angular momentum vector $\vec{H}_G$, and using the fact that

$$\hat{K}=\sin\phi\sin\theta\,\hat{\imath}+\cos\phi\sin\theta\,\hat{\jmath}+\cos\theta\,\hat{k}$$

it follows that

$$I_x\omega_x = H\ \sin\phi\sin\theta \qquad I_y\omega_y = H\ \cos\phi\sin\theta \qquad I_z\omega_z = H\ \cos\theta$$

$$\theta = \arccos\left\{\frac{I_z\omega_z}{H}\right\} \tag{24}$$

$$\phi = \arctan\left\{\frac{I_x\omega_x}{I_y\omega_y}\right\} \tag{25}$$

$$\dot{\psi} = H\left\{\frac{\left[I_x\omega_x^2 + I_y\omega_y^2\right]}{\left[(I_x\omega_x)^2 + (I_y\omega_y)^2\right]}\right\} \tag{26}$$

PROGRAM TFM.m

```
format long e;
Ix=5;
Iy=3;
Iz=2;
Wx=.05;
Wy=6;
Wz=-.05;
D=Ix*(Wx^2)+Iy*(Wy^2)+Iz*(Wz^2);
T=0.5*D;
Q=(Ix*Wx)^2+(Iy*Wy)^2+(Iz*Wz)^2;
H=sqrt(Q);
A=Ix*D-Q;
B=Q-Iy*D;
C=Q-Iz*D;
if B > 0
Bplus;
else
Bminus;
end
tmax=(4*K)/P;
deltat=.005;
t=[0:deltat:tmax];
n1=max(size(t));
x1=zeros(size(t));
x2=zeros(size(t));
x3=zeros(size(t));
x4=zeros(size(t));
x5=zeros(size(t));
g=zeros(size(t));
x6=zeros(size(t));
for j=1:n1
[a1 a2 a3]=ellipj(P*(t(j)-t0),m);
if B > 0
x1(j)=L*a3;
x2(j)=M*a1;
x3(j)=N*a2;
else
x1(j)=L*a2;
x2(j)=M*a1;
x3(j)=N*a3;
end
x5(j)=acos( (Iz*x3(j))/H);
x6(j)=atan((Iy*x2(j))/(Ix*x1(j)));
g(j)=H*( (D-Iz*(x3(j)^2))/(Q-(Iz*x3(j))^2));
end
for i=2:n1
x4(i)=x4(i-1)+deltat*g(i-1);
end
```

SUBROUTINE Bplus.m

```
if Wx > 0
L=sqrt(C/(Ix*(Ix-Iz)));
M=sqrt(A/(Iy*(Ix-Iy)));
N=-sqrt(A/(Iz*(Ix-Iz)));
else
L=-sqrt(C/(Ix*(Ix-Iz)));
M=sqrt(A/(Iy*(Ix-Iy)));
N=sqrt(A/(Iz*(Ix-Iz)));
end
P=sqrt((C*(Ix-Iy))/(Ix*Iy*Iz));
m=(A*(Iy-Iz))/(C*(Ix-Iy));
K=ellipke(m);
delta=.00000001;
r=Wy/M;
n=30;
R=[1:1:n];
y=zeros(size(R));
z=zeros(size(R));
w=zeros(size(R));
d=zeros(size(R));
if r > 0
y(1)=asin(r);
z(1)=K;
else
z(1)=asin(r);
y(1)=-K;
end
for k=2:n
if z(k - 1) - y(k - 1) < delta
l=k-1;
break;
else
l=k;
w(k)=0.5*(y(k-1)+z(k-1));
d(k)=ellipj(w(k),m);
end
if d(k) > r
y(k)=y(k-1);
z(k)=w(k);
else
z(k)=z(k-1);
y(k)=w(k);
end
end
f=z(l);
```

```
[s1 s2 s3]=ellipj(f,m);
if r > 0
if [r Wz/N Wx/L]== [s1 s2 s3]
t0=-f/P;
else
t0=(f-2*K)/P;
end
else
if [r Wz/N Wx/L]== [s1 s2 s3]
t0=-f/P;
else
t0=(f+2*K)/P;
end
end
```

SUBROUTINE Bminus.m

```
if Wz < 0
L=sqrt(C/(Ix*(Ix-Iz)));
M=sqrt(C/(Iy*(Iy-Iz)));
N=-sqrt(A/(Iz*(Ix-Iz)));
else
L=-sqrt(C/(Ix*(Ix-Iz)));
M=sqrt(C/(Iy*(Iy-Iz)));
N=sqrt(A/(Iz*(Ix-Iz)));
end
P=sqrt((A*(Iy-Iz))/(Ix*Iy*Iz));
m=(C*(Ix-Iy))/(A*(Iy-Iz));
K=ellipke(m);
delta=.00000001;
r=Wy/M;
n=30;
R=[1:1:n];
y=zeros(size(R));
z=zeros(size(R));
w=zeros(size(R));
d=zeros(size(R));
if r > 0
y(1)=asin(r);
z(1)=K;
else
z(1)=asin(r);
y(1)=-K;
end
```

```
for k=2:n
if z(k - 1) - y(k - 1) < delta
l=k-1;
break;
else
l=k;
w(k)=0.5*(y(k-1)+z(k-1));
d(k)=ellipj(w(k),m);
end
if d(k) > r
y(k)=y(k-1);
z(k)=w(k);
else
z(k)=z(k-1);
y(k)=w(k);
end
end
f=z(l);
[s1 s2 s3]=ellipj(f,m);
if r > 0
if [r Wx/L Wz/N]== [s1 s2 s3]
t0=-f/P;
else
t0=(f-2*K)/P;
end
else
if [r Wx/L Wz/N]==[s1 s2 s3]
t0=-f/P;
else
t0=(f+2*K)/P;
end
end
```

TORQUE–FREE MOTION OF A SYMMETRICAL BODY

Consider the torque-free motion of a rigid body. If Gxyz are principal axes of inertia, then governing equations are

$$I_x\dot{\omega}_x + [I_z - I_y]\omega_y\omega_z = 0\,, \tag{1}$$

$$I_y\dot{\omega}_y + [I_x - I_z]\omega_x\omega_z = 0 \tag{2}$$

and,

$$I_z\dot{\omega}_z + [I_y - I_x]\omega_x\omega_y = 0 \tag{3}$$

Moreover, angular momentum and rotational kinetic energy are conserved, i. e.,

$$\vec{H}_G = I_x\omega_x\hat{\imath} + I_y\omega_y\hat{\jmath} + I_z\omega_z\hat{k} = H\hat{K} = \text{const} \tag{4}$$

for some fixed direction $\hat{K}$ in space [the invariant line], and

$$H^2 = (I_x\omega_x)^2 + (I_y\omega_y)^2 + (I_z\omega_z)^2 = \text{const} \tag{5}$$

$$2T_{rot} = I_x(\omega_x)^2 + I_y(\omega_y)^2 + I_z(\omega_z)^2 = \text{const} \tag{6}$$

SYMMETRICAL BODY

Suppose the body has an axis of symmetry so that $I_x = I_y \neq I_z$. Equations (1)–(3) become

$$I_x\dot{\omega}_x + [I_z - I_x]\omega_y\omega_z = 0 \tag{7}$$

$$I_x\dot{\omega}_y + [I_x - I_z]\omega_x\omega_z = 0 \tag{8}$$

and

$$I_z\dot{\omega}_z = 0 \tag{9}$$

Thus, $\omega_z = n\,(\text{const})$, and

$$\ddot{\omega}_x + \left\{\frac{n[I_x - I_z]}{I_x}\right\}^2 \omega_x = \ddot{\omega}_x + q^2\omega_x = 0 \tag{10}$$

$$\omega_y = \frac{1}{q}\dot{\omega}_x \tag{11}$$

It follows from (10), (11) that

$$\omega_x = A\cos qt + B\sin qt \qquad\qquad \omega_y = B\cos qt - A\sin qt \tag{12}$$

for some constants A and B. It follows from (5), (6), (12) that

$$\omega_x^2 + \omega_y^2 = A^2 + B^2 = \frac{1}{I_x}\left\{2T_{rot} - I_z n^2\right\} = \frac{1}{I_x^2}\left\{H^2 - \left(I_z n\right)^2\right\} = \text{const} \tag{13}$$

$$\vec{\omega} = \omega_1 \hat{e} + n\hat{k} \qquad\qquad \vec{H} = \left(I_x\omega_1\right)\hat{e} + \left(I_z n\right)\hat{k}$$

$$\hat{e}=\cos\left(qt-\epsilon\right)\hat{\imath}-\sin\left(qt-\epsilon\right)\hat{\jmath} \qquad\qquad \omega_1 = \sqrt{\left(A^2+B^2\right)} \qquad\qquad \sin\epsilon = \frac{B}{\sqrt{\left(A^2+B^2\right)}}$$

It follows from the foregoing that the vectors $\vec{\omega}$ and $\vec{H}$ have constant magnitudes and lie in the $\hat{e}$–$\hat{k}$ plane. The vector $\vec{\omega}$ makes a constant angle γ with the (rotating) $\hat{k}$–direction, while the vector $\hat{k}$ makes a constant angle θ with the constant vector $\vec{H}$ $\left[\ \hat{K}\text{–direction}\ \right]$.

$$\tan\gamma = \frac{\omega_1}{n} \qquad\qquad \tan\theta = \frac{I_x\omega_1}{I_z n} = \left(\frac{I_x}{I_z}\right)\tan\gamma$$

In general, the components of angular velocity are given in terms of the Euler rates, $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ by:–

$$\omega_x = \dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi$$

$$\omega_y = \dot{\psi}\cos\phi\sin\theta - \dot{\theta}\sin\phi$$

$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

In this case $\dot{\theta} \equiv 0$, so that

$$\omega_x = \dot{\psi}\sin\phi\sin\theta = \omega_1 e_x$$

$$\omega_y = \dot{\psi}\cos\phi\sin\theta = \omega_1 e_y$$

$$n = \dot{\psi}\cos\theta + \dot{\phi}$$

Thus,

$$\dot{\psi} = \frac{\omega_1}{\sin\theta} = \frac{I_z n}{\left[I_x\cot\theta\right]} \qquad\qquad \dot{\phi} = n - \omega_1\cot\theta = \left\{\frac{I_x - I_z}{I_x}\right\}n$$

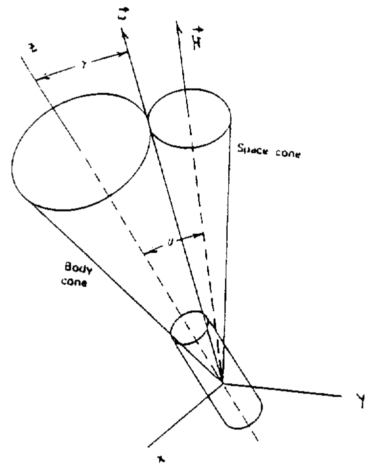
$$\dot{\psi} = \frac{I_z\dot{\phi}}{\left[I_x - I_z\right]\cos\theta}$$

SPACE CONE AND BODY CONE

The angular velocity vector $\vec{\omega}$ is constant in magnitude and makes constant angles with both the (fixed–in–the–body) $\hat{k}$–direction and the (fixed–in–space) $\hat{K}$– direction. As the motion unfolds, the vector $\vec{\omega}$ sweeps out cones about each of these directions, the body cone and space cone, respectively.

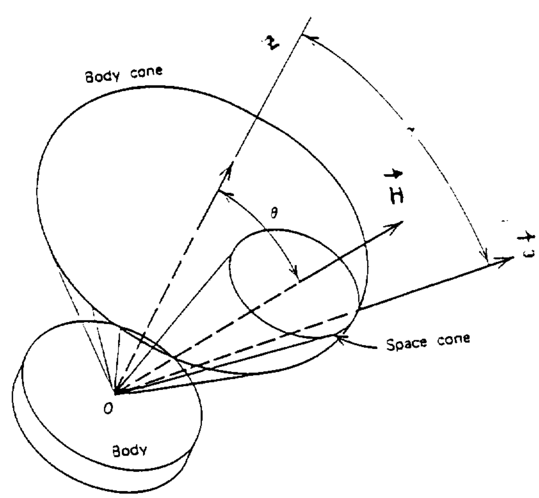
ROD–LIKE BODY $\left(I_x > I_z\right)$

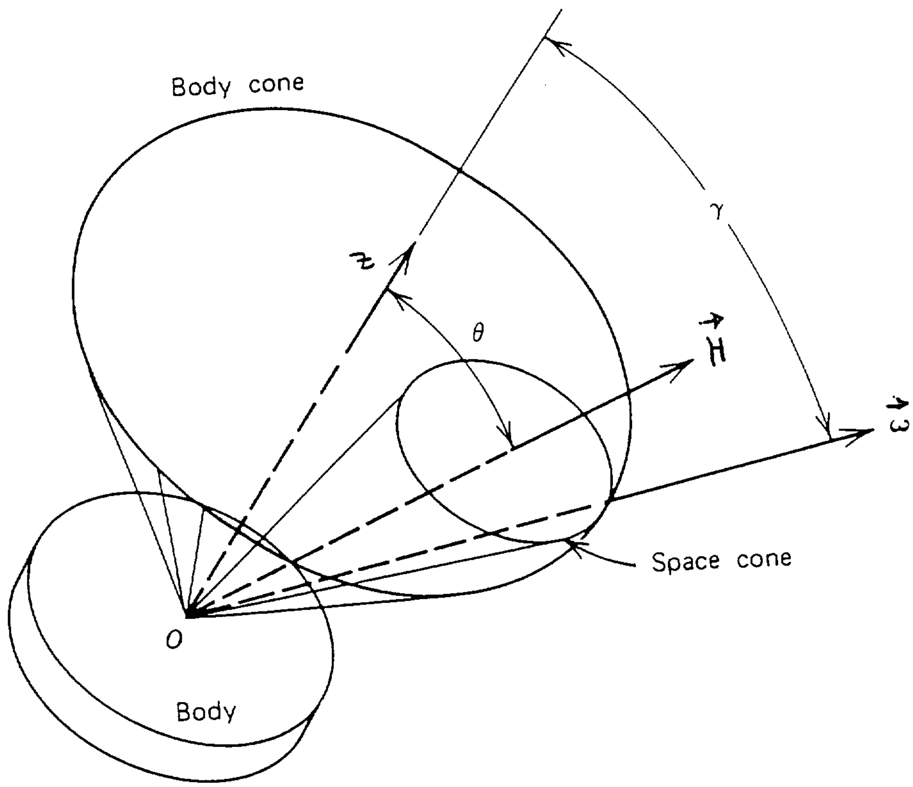
For a rodlike body, $\theta > \gamma$ and so the space cone is exterior to the body cone. Moreover, $\dot{\psi}$ and $\dot{\phi}$ have the same signs (*direct precession*).

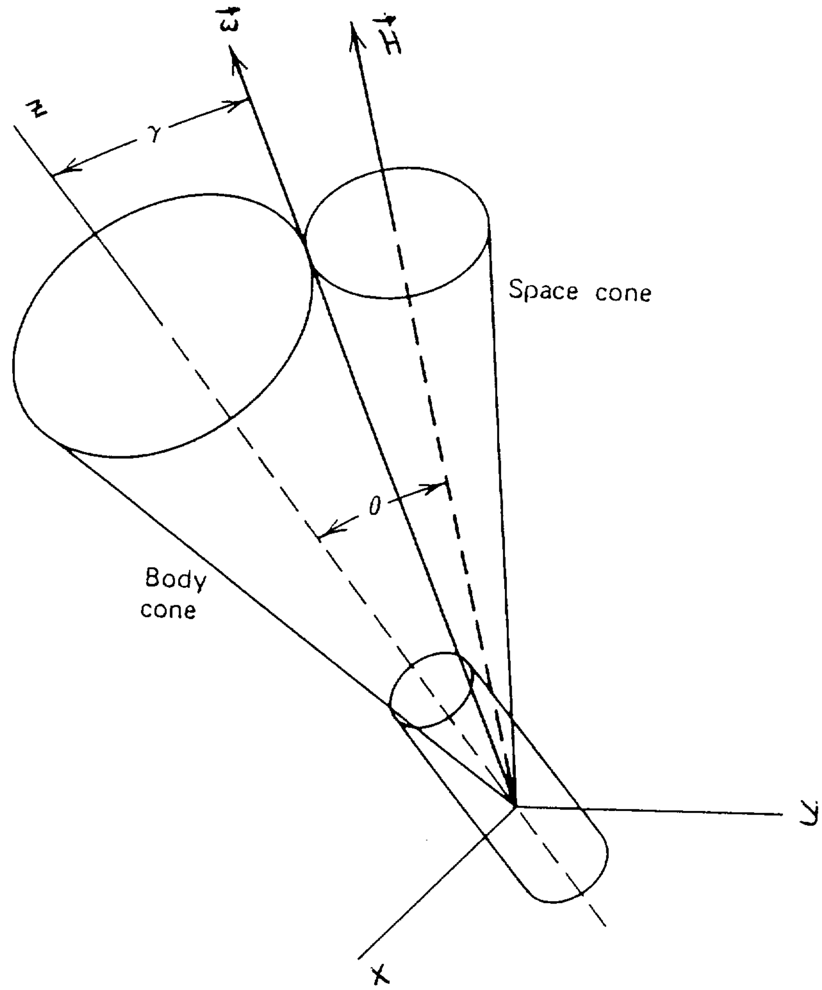


DISK–LIKE BODY $\left(I_x < I_z\right)$

For a disklike body, $\theta < \gamma$ and so the space cone is interior to the body cone. Moreover, $\dot{\psi}$ and $\dot{\phi}$ have opposite signs (*retrograde precession*).

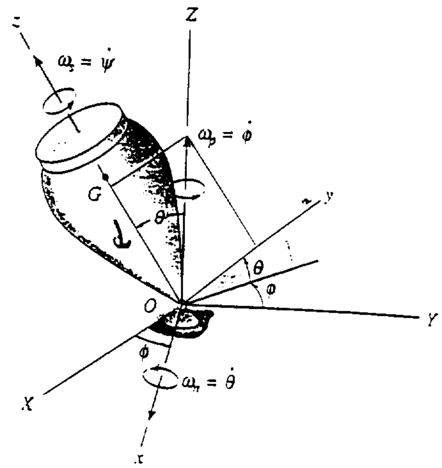



Body cone
γ
θ
Space cone
O
Body



Space cone
Body
cone
z
y
x

THE SPINNING TOP



The heavy spinning top shown is a body with an axis of symmetry which is mounted on a ball–and–socket joint at O. Its moment of inertia about its axis of symmetry z is I_z, its moment of inertia about any axis through O perpendicular to the z–axis is I_x, the distance from O to the center of mass G is d.

The rotating frame Oxyz is defined by requiring the x-axis to lie normal to the z–Z plane. [These axes are not fixed in the body, but always coincide with principal axes of inertia.] The *ANGLE OF PRECESSION*, ϕ, is defined as the angle between the X–Z and z–Z planes. The *ANGLE OF NUTATION*, θ, is the angle between the axis of symmetry and the vertical, and the *ANGLE OF SPIN*, ψ is the angle of rotation of the top about its axis of symmetry.

$$\hat{K} = \sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}$$

The components of the angular velocity $\vec{\Omega} = \dot\theta\,\hat{\imath} + \dot\phi\,\hat{K}$ of the frame Oxyz are therefore

$$\Omega_x = \dot\theta \qquad \Omega_y = \dot\phi\,\sin\theta \qquad \Omega_z = \dot\phi\,\cos\theta$$

The components of the angular velocity $\vec{\omega} = \dot\theta\,\hat{\imath} + \dot\phi\,\hat{K} + \dot\psi\,\hat{k}$ of the top are

$$\omega_x = \dot\theta \qquad \omega_y = \dot\phi\,\sin\theta \qquad \omega_z = \dot\phi\,\cos\theta + \dot\psi$$

The equation of motion is

$$(\dot{\vec{H}}_O)_{OXYZ} = (\dot{\vec{H}}_O)_{Oxyz} + \vec{\Omega}\times\vec{H}_O = \sum\vec{M}_O = d\,\hat{k}\times\{-Mg\,\hat{K}\} \tag{1}$$

Thus,

$$I_x\ddot\theta + I_z[\dot\psi + \dot\phi\,\cos\theta]\dot\phi\,\sin\theta - I_x\dot\phi^2\,\sin\theta\cos\theta = Mgd\,\sin\theta$$

$$I_x\frac{d}{dt}\{\dot\phi\,\sin\theta\} + I_x\dot\theta\dot\phi\,\cos\theta - I_z\dot\theta[\dot\psi + \dot\phi\,\cos\theta] = 0$$

$$I_z\frac{d}{dt}\{\dot\psi + \dot\phi\,\cos\theta\} = 0$$

The third of these implies that

$$\omega_z = \dot\psi + \dot\phi\,\cos\theta = n\,(\text{const}) \tag{2}$$

The energy of the spinning top is conserved. The kinetic energy is

$$T = \frac{1}{2}\left\{I_x[\omega_x^2 + \omega_y^2] + I_z\omega_z^2\right\} = \frac{1}{2}\left\{I_x[\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2] + I_z n^2\right\}$$

The potential energy is

$$V = Mgd\cos\theta$$

Thus,the energy identity takes the form

$$I_x[\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2] + 2Mgd\cos\theta = 2E - I_z n^2 \tag{3}$$

It follows from (1) that

$$\hat{K}\cdot(\dot{\vec{H}}_O)_{OXYZ} = \left(\frac{d}{dt}\left\{\hat{K}\cdot\vec{H}_O\right\}\right)_{OXYZ} = \dot{H}_Z = 0$$

Thus,

$$\hat{K}\cdot\vec{H}_O = I_x\omega_y\sin\theta + I_z\omega_z\cos\theta = I_x\dot{\phi}\sin^2\theta + I_z n\cos\theta = H_Z\,(\text{const}) \tag{4}$$

Combining this with (3) yields

$$I_x\dot{\theta}^2 + \frac{\left[H_Z - I_z n\cos\theta\right]^2}{I_x\sin^2\theta} + 2Mgd\cos\theta = 2E - I_z n^2$$

Define constants

$$a = \frac{1}{I_x}\left[2E - I_z n^2\right] \qquad w = \frac{2Mgd}{I_x} \qquad k = \frac{H_Z}{I_x} \qquad p = \frac{I_z n}{I_x}$$

Then,

$$\dot{\theta}^2\sin^2\theta + \left[k - p\cos\theta\right]^2 + w\cos\theta\sin^2\theta = a\sin^2\theta$$

Make the change of variable

$$u = \cos\theta \qquad \dot{u} = -\dot{\theta}\sin\theta$$

to arrive at

$$\dot{u}^2 = \left[a - wu\right]\left[1 - u^2\right] - \left[k - pu\right]^2 = f(u) \tag{5}$$

together with

$$\dot{\phi} = \frac{\left[k - pu\right]}{\left[1 - u^2\right]} \tag{6}$$

Consider, now, the cubic expression f(u). For the system considered, $0 \leq \theta \leq \frac{\pi}{2}$, so $a > 0$. Thus,

$$f(\pm 1) = -[k - p]^2 \leq 0 \qquad f(\infty) = \infty$$

One possible solution of (5) is $u \equiv 1$, $k = p$. For solutions to (5) with $u \neq 1$ to exist, the cubic expression f(u) must be positive in some range $0 < u_2 \leq u \leq u_1 < 1$, and the cubic must possess three real roots $0 < u_2 \leq u_1 \leq 1 \leq u_3$.

$$f(u) = w[u - u_2][u - u_1][u - u_3]$$

Let

$$\sigma = \sqrt{[u - u_2]} \qquad \dot{u} = 2\sigma\dot{\sigma}$$

Then

$$\dot{\sigma}^2 = \frac{w}{4}[u_1 - u_2 - \sigma^2][u_3 - u_2 - \sigma^2]$$

Now, define

$$\Gamma = \frac{\sigma}{\sqrt{u_1 - u_2}} \qquad k = \sqrt{\frac{[u_1 - u_2]}{[u_3 - u_2]}} \qquad \lambda = \frac{1}{2}\sqrt{w[u_3 - u_2]}$$

Then,

$$\dot{\Gamma}^2 = \lambda^2[1 - \Gamma^2][1 - k^2\Gamma^2]$$

and

$$\Gamma = \mathrm{sn}(\lambda(t - t_0), k)$$

Define

$$\theta_1 = \arccos(u_1) \qquad \theta_2 = \arccos(u_2)$$

Then

$$\cos\theta = \cos\theta_2 + [\cos\theta_1 - \cos\theta_2]\,\mathrm{sn}^2(\lambda(t - t_0), k)$$