1. In the following system, a motor is used to control the height of a massive structure. The height $h$ is detected as a voltage signal $v_h$ by the use of a variable resistor, such that $v_h = k_h h$. Obtain the detailed block diagram of the system, where $v_r$ is the input and $h$ is the output, and show the variables $v_r$, $i_r$, $v_h$, $i_h$, $i_v$, $i_a$, $i_b$, $\tau$, $\theta_1$, $\theta_2$, and $h$ on the block diagram. (30pts)

![Block Diagram](image)

2. Consider the following block diagram of a control system.

![Block Diagram](image)

Determine the transfer function of the system. Show all your work. (25pts)

3. The acceleration measured by a sensing device that is placed along a flexible beam is given by

$$a_m(t) = \begin{cases} 
-a^2(t) \sin \left( \frac{\pi}{2l} x(t) \right), & \text{if } a(t) < 0; \\
+a^2(t) \sin \left( \frac{\pi}{2l} x(t) \right), & \text{if } a(t) \geq 0;
\end{cases}$$

where $a$ and $x$ are the actual acceleration and the displacement along the flexible beam of length $l$, respectively.

(a) Obtain an affine approximation of the measured acceleration about $a = 0$ and $x = (l/2)$. (10pts)

(b) Obtain an affine approximation of the measured acceleration about $a = 10$ and $x = l$. (10pts)
4. A control system is described in state-space representation, such that

\[
\dot{x}(t) = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(t),
\]
\[
y(t) = \begin{bmatrix} -5 & 3 \end{bmatrix} x(t) + u(t),
\]

where \( u, x, \) and \( y \) are the input, the state, and the output variables, respectively.

(a) Solve \( y(t) \) for \( t \geq 0 \), when \( u(t) = 0 \) for \( t \geq 0 \) and \( x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \).

(b) Determine the transfer function or the transfer matrix of the system.
1. In the following system, a motor is used to control the height of a massive structure. The height $h$ is detected as a voltage signal $v_h$ by the use of a variable resistor, such that $v_h = k_h h$. Obtain the detailed block diagram of the system, where $v_r$ is the input and $h$ is the output, and show the variables $v_r, i_r, v_h, i_h, i, v_a, i_a, v_b, \theta_1, \theta_2$, and $h$ on the block diagram.

Solution: To determine the block diagram of the system, we first separate it into simpler components.

Since the input variable is $v_r$, we write $i_r$ in terms $v_r$, such that

$$I_r(s) = \frac{1}{R_r} V_r(s),$$

since the operational amplifier is assumed to be ideal.
Similarly, we have

\[ I_h(s) = \frac{1}{R_h} V_h(s). \]

For an ideal operational amplifier,

\[ i(t) = -(i_r(t) + i_h(t)). \]

Again for an ideal operational amplifier,

\[ v_a(t) = \frac{1}{C} \int^t i(\tau) \, d\tau, \]

or

\[ V_a(s) = \frac{1}{C s} I(s). \]
The armature current of the motor can be obtained from the Kirchhoff's Voltage Law, where

\[ L_a \frac{di_a(t)}{dt} + R_a i_a(t) + v_b(t) = v_a(t), \]

or

\[ I_a(s) = \frac{1}{L_a s + R_a} (V_a(s) - V_b(s)). \]

From the armature-controlled motor,

\[ \tau(t) = K_a i_a(t). \]

The back-emf voltage of the motor

\[ v_b(t) = K_b \dot{\theta}_1(t), \]

or

\[ V_b(s) = (K_b s) \Theta_1(s). \]
The torque equation for $\theta_1$ is

$$J\ddot{\theta}_1(t) = \tau(t) + rf_r(t) - B(\dot{\theta}_1(t) - \dot{\theta}_2(t)),$$

where $f_r$ is the internal tension of the rope. So,

$$\Theta_1(s) = \frac{1}{Js^2 + Bs} \left( T(s) + rf_r(s) + Bs\Theta_2(s) \right).$$

The torque equation for $\theta_2$ is

$$0 = -B(\dot{\theta}_2(t) - \dot{\theta}_1(t)) - K\theta_2(t).$$

So,

$$\Theta_2(s) = \frac{Bs}{Bs + K} \Theta_1(s).$$

The disc with the inertia $J$ changes the rotational motion to translational motion, where

$$h(t) = -r\theta_1(t).$$
The force acting on the mass is only due to the gravity.

\[ f_r(t) = mg \mathbb{1}(t). \]

And, finally the given relationship

\[ v_h(t) = K_h h(t). \]

When we connect all the individual blocks together, we get the following block diagram.
2. Consider the following block diagram of a control system.

Determine the transfer function of the system. Show all your work.

**Solution:** If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.
If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.

\[
G_1 \left( \left( \frac{G_2}{1 - G_2 H_2} + G_4 \right) \left( \frac{G_3}{1 - G_3 H_1} \right) + \frac{G_5}{1 - G_2 H_2} \right)
\]

In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

\[
F_1 = G_1 l_1 l_2 G_2 l_3 l_4 G_3 l_5 l_6 = G_1 G_2 G_3,
\]
\[
F_2 = G_1 G_4 l_4 G_3 l_5 l_6 = G_1 G_4 G_3,
\]
\[
F_3 = G_2 l_1 l_2 G_5 l_6 = G_1 G_5.
\]

The loop gains are

\[
L_1 = l_2 G_2 H_2 = G_2 H_2,
\]
\[
L_2 = G_3 H_1.
\]

From the forward path and the loop gains, we determine the touching loops and the forward paths.

**Touching Loops**

<table>
<thead>
<tr>
<th>L₁</th>
<th>L₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>L₁</td>
<td>✗</td>
</tr>
<tr>
<td>L₂</td>
<td>✗</td>
</tr>
</tbody>
</table>

**Loops on Forward Paths**

<table>
<thead>
<tr>
<th>L₁</th>
<th>L₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>F₁</td>
<td>✗</td>
</tr>
<tr>
<td>F₂</td>
<td>✗</td>
</tr>
<tr>
<td>F₃</td>
<td>✗</td>
</tr>
</tbody>
</table>
Therefore,
\[
\Delta = 1 - (L_1 + L_2) + (L_1 L_2)
\]
\[
= 1 - ((G_2 H_2) + (G_3 H_1)) + ((G_2 H_2)(G_3 H_1))
\]
\[
= 1 - G_2 H_2 - G_3 H_1 + G_2 H_2 G_3 H_1,
\]
and
\[
\Delta_1 = \Delta|_{L_1=L_2=0} = 1,
\]
\[
\Delta_2 = \Delta|_{L_2=0} = 1 - L_1 = 1 - G_2 H_2,
\]
\[
\Delta_3 = \Delta|_{L_1=0} = 1 - L_2 = 1 - G_3 H_1.
\]

So,
\[
\frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_{i=1}^{3} F_i \Delta_i
\]
\[
= \frac{(G_1 G_2 G_3)(1) + (G_1 G_4 G_3)(1 - G_2 H_2) + (G_1 G_5)(1 - G_3 H_1)}{1 - G_2 H_2 - G_3 H_1 + G_2 H_2 G_3 H_1},
\]
or
\[
\frac{Y(s)}{U(s)} = \frac{G_1 G_2 G_3 + G_1 G_4 G_3(1 - G_2 H_2) + G_1 G_5(1 - G_3 H_1)}{1 - G_2 H_2 - G_3 H_1 + G_2 H_2 G_3 H_1}.
\]

3. The acceleration measured by a sensing device that is placed along a flexible beam is given by
\[
a_m(t) = \begin{cases} 
-a^2(t) \sin \left(\frac{\pi}{(2l)} x(t)\right), & \text{if } a(t) < 0; \\
a^2(t) \sin \left(\frac{\pi}{(2l)} x(t)\right), & \text{if } a(t) \geq 0;
\end{cases}
\]
where \(a\) and \(x\) are the actual acceleration and the displacement along the flexible beam of length \(l\), respectively.

(a) Obtain an affine approximation of the measured acceleration about \(a = 0\) and \(x = (l/2)\).

**Solution:** The affine approximation is obtained from the taylor series expansion of \(a_m(t)\) about \(a = 0\) and \(x = (l/2)\).
\[
a_m = \left[ a_m \right]_{a=0, x=(l/2)} + \left[ \frac{\partial}{\partial a} (a_m) \right]_{a=0, x=(l/2)} (a - 0) + \left[ \frac{\partial}{\partial x} (a_m) \right]_{a=0, x=(l/2)} (x - l/2) + O(a^2, x^2)
\]
\[
= \left[ -a^2 \sin \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} + \left[ -2a \sin \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} (a - 0)
\]
\[
+ \left[ -a^2 \left(\frac{\pi}{(2l)}\right) \cos \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} (x - l/2) + O(a^2, x^2), \quad \text{if } a < 0;
\]
\[
= \left[ a^2 \sin \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} + \left[ 2a \sin \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} (a - 0)
\]
\[
+ \left[ a^2 \left(\frac{\pi}{(2l)}\right) \cos \left(\frac{\pi}{(2l)} x\right) \right]_{a=0, x=(l/2)} (x - l/2) + O(a^2, x^2), \quad \text{if } a \geq 0;
\]
\[
= [0] + [0](a - 0) + [0](x - l/2) + O(a^2, x^2)
\]
\[
\approx 0.
\]
As a result, the affine approximation of the measured acceleration about \(a = 0\) and \(x = (l/2)\) is
\[
a_m(t) = 0.
\]
(b) Obtain an affine approximation of the measured acceleration about \( a = 10 \) and \( x = l \).

**Solution:** The affine approximation is obtained from the taylor series expansion of \( a_m(t) \) about \( a = 10 \) and \( x = l \).

\[
a_m = \left[ \begin{array}{c} a_m \\ \frac{\partial a_m}{\partial a} \\ \frac{\partial a_m}{\partial x} \end{array} \right]_{a=10, x=l} (a-10) + \left[ \begin{array}{c} \frac{\partial^2 a_m}{\partial a^2} \\ \frac{\partial^2 a_m}{\partial a \partial x} \\ \frac{\partial^2 a_m}{\partial x^2} \end{array} \right]_{a=10, x=l} (x-l) + O(a^2, x^2)
\]

\[
\begin{align*}
&= \left\{ \begin{array}{l}
\left[ -a^2 \sin\left(\frac{\pi}{2}\right) \right]_{a=10} + \left[ -2a \sin\left(\frac{\pi}{2}\right) \right]_{a=10} (a-10) \\
+ \left[ a^2 \cos\left(\frac{\pi}{2}\right) \right]_{a=10} (x-l) + O(a^2, x^2), \quad \text{if } a < 0; \\
\left[ a^2 \sin\left(\frac{\pi}{2}\right) \right]_{a=10} + \left[ 2a \sin\left(\frac{\pi}{2}\right) \right]_{a=10} (a-10) \\
+ \left[ a^2 \cos\left(\frac{\pi}{2}\right) \right]_{a=10} (x-l) + O(a^2, x^2), \quad \text{if } a \geq 0; \\
\end{array} \right.
\end{align*}
\]

\[
\approx 100 + 20(a-10).
\]

As a result, the affine approximation of the measured acceleration about \( a = 10 \) and \( x = l \) is

\[ a_m(t) = 20a(t) - 100. \]

4. A control system is described in state-space representation, such that

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} -5 & 3 \end{bmatrix} x(t) + u(t),
\end{align*}
\]

where \( u, x, \) and \( y \) are the input, the state, and the output variables, respectively.

(a) Solve \( y(t) \) for \( t \geq 0 \), when \( u(t) = 0 \) for \( t \geq 0 \) and \( x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \).

**Solution:** The general solution to the state-space representation of a system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

is obtained from

\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) \, d\tau,
\]

where

\[ e^{At} = L^{-1}s \left[ (sI - A)^{-1} \right](t). \]

Here, \( I \) is the appropriately dimensioned identity matrix. In our case,

\[
A = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix},
\]
and \( u(t) = 0 \) for \( t \geq 0 \). As a result, the integral term in the solution of \( x \) is identically zero. So,

\[
y(t) = Ce^{At}x(0)
\]

\[
= [ -5 \ 3 ] \mathcal{L}_s^{-1} [(sI - A)^{-1}] (t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \mathcal{L}_s^{-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} \right)^{-1} (t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \mathcal{L}_s^{-1} \left( \begin{bmatrix} s - 5 & 2 \\ -8 & s + 3 \end{bmatrix} \right)^{-1} (t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \mathcal{L}_s^{-1} \left( \frac{1}{(s - 5)(s + 3) - (-8)(2)} \begin{bmatrix} s + 3 & -2 \\ 8 & s - 5 \end{bmatrix} \right) (t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \begin{bmatrix} \mathcal{L}_s^{-1} \left[ \frac{s + 3}{(s - 1)^2} \right] (t) & \mathcal{L}_s^{-1} \left[ \frac{-2}{(s - 1)^2} \right] (t) \\ \mathcal{L}_s^{-1} \left[ \frac{8}{(s - 1)^2} \right] (t) & \mathcal{L}_s^{-1} \left[ \frac{s - 5}{(s - 1)^2} \right] (t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \begin{bmatrix} \mathcal{L}_s^{-1} \left[ \frac{-2}{(s - 1)^2} \right] (t) \\ \mathcal{L}_s^{-1} \left[ \frac{-4}{(s - 1)^2} + \frac{1}{s - 1} \right] (t) \end{bmatrix}
\]

\[
= [ -5 \ 3 ] \begin{bmatrix} -2te^t \\ e^t - 4te^t \end{bmatrix}
\]

Or,

\[
y(t) = -2te^t + 3e^t \text{ for } t \geq 0.
\]

(b) Determine the transfer function or the transfer matrix of the system.

**Solution:** The transfer matrix of a control system described in the state-state representation

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t) + Du(t),
\]

is

\[
F(s) = C(sI - A)^{-1}B + D,
\]

where

\[
A = \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix},
\]

\[
C = \begin{bmatrix} -5 & 3 \end{bmatrix}, \quad D = 1,
\]
and $I$ is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1$$

$$= \begin{bmatrix} -5 & 3 \end{bmatrix} \left( \begin{bmatrix} s & 2 \\ -8 & s+3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1$$

$$= \frac{1}{(s-5)(s+3) - (-8)(2)} \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} s+3 & -2 \\ 8 & s-5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1$$

$$= \frac{1}{(s-1)^2} (2s - 2) + 1$$

$$= \frac{2}{s-1} + 1.$$

Therefore, the transfer matrix is

$$F(s) = \frac{s+1}{s-1}.$$