EE 231 | Exam#2 | May 01, 2008

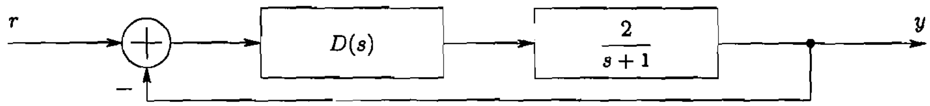
75 minutes

1. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{(s+2)(s+5)}{s^2}.$$

Determine the range of the constant K, such that the 5% settling-time is less than 3 seconds. (25pts)

2. Consider the following feedback control system.



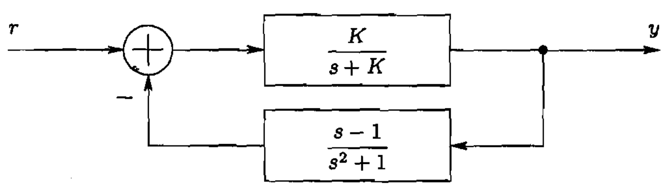
Determine the simplest controller $D(s)$ amongst P, I, or PI controllers, such that the maximum percent-overshoot is less than 10%, the 5% settling-time is less than 1 s, and the steady-state error is zero for a step input. (25pts)

3. Consider the negative-feedback control-system with the following open-loop transfer-function. Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angles of departure and/or arrival.

$$G(s) = K\frac{s^2+4s+5}{s(s+10)(s^2+2s+2)}.$$

(30pts)

4. Sketch the root-locus diagram for the following control system. (20pts)



EE 231 | Exam#2 Solutions | May 01, 2008

1. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{(s+2)(s+5)}{s^2}.$$

Determine the range of the constant K, such that the 5% settling-time is less than 3 seconds.

Solution: Since the 5% settling time $t_{5\%s} = (3/\sigma_o)$; we have

$$t_{5\%s} = \frac{3}{\sigma_o} < 3,$$

$\sigma_o > 1$, or the poles need to be on the left-hand-side of the $\Re[s] = -\sigma_o = \sigma = -1$ vertical line. One way to determine the conditions for the poles to be on the left-hand-side of a vertical line is to use the Routh-Hurwitz's Table after shifting the vertical line from the $\sigma = 0$ line to the desired line.

The closed-loop poles are determined from the factors of the characteristic polynomial or the denominator of the closed-loop transfer function. In our case, the characteristic equation is

$$1 + G(s) = 1 + K\frac{(s+2)(s+5)}{s^2} = \frac{s^2 + K(s+2)(s+5)}{s^2} = 0,$$

and the characteristic polynomial becomes

$$q_c(s) = s^2 + K(s+2)(s+5).$$

If we use the Routh-Hurwitz's Table on this polynomial, we would determine the conditions for the poles to be on the left-hand-side of the $\sigma = 0$ vertical line. To determine the conditions for the left-hand-side of the $\sigma = -1$ line, we need to shift the characteristic polynomial, such that

$$\begin{aligned} q_c(s-1) &= (s-1)^2 + K\big((s-1)+2\big)\big((s-1)+5\big) \\ &= (K+1)s^2 + (5K-2)s + (4K+1). \end{aligned}$$

With the new polynomial, the Routh-Hurwitz's Table becomes as given below.

s^2	$K+1$	$4K+1$
s	$5K-2$	
1	$4K+1$	

Applying the Routh-Hurwitz's criterion on the new polynomial gives the conditions for the left-hand-side of the $\sigma = -1$ line. The s^2-term gives

$$K + 1 > 0, \text{ or } K > -1.$$

The s-term gives

$$5K - 2 > 0, \text{ or } K > 0.4.$$

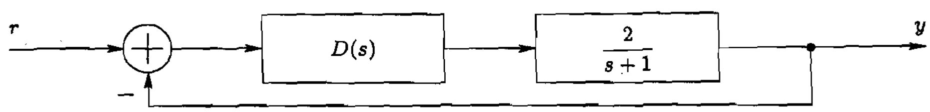
The 1-term gives

$$4K + 1 > 0, \text{ or } K > -0.25.$$

Therefore, from the intersection of all the regions, we get

$$K > 0.4.$$

2. Consider the following feedback control system.



Determine the simplest controller $D(s)$ amongst P, I, or PI controllers, such that the maximum percent-overshoot is less than 10%, the 5% settling-time is less than 1 s, and the steady-state error is zero for a step input.

Solution: The performance requirements are listed below, where

$$G(s) = \frac{2}{s+1},$$

and

$$D(s) = K_p,$$

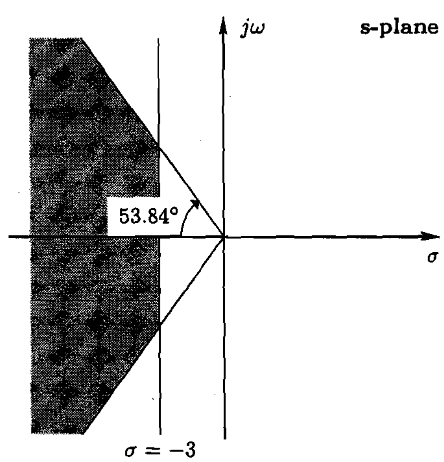
$$D(s) = \frac{K_I}{s},$$

or

$$D(s) = K_P + \frac{K_I}{s} = K_P \frac{s + K_I/K_P}{s}.$$

Given Requirements	General System Restrictions	Specific System Restrictions
The steady-state error is zero for a step input.	$G(s)D(s)$ has a pole at 0.	$D(s) = \frac{K_I}{s}$ or $D(s) = K_P \frac{s + K_I/K_P}{s}$.
The maximum percent overshoot is less than 10%.	$e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} < M_{p_{\text{given}}}$, or $\zeta > \frac{\lvert \ln(M_{p_{\text{given}}}) \rvert}{\sqrt{(\ln(M_{p_{\text{given}}}))^2 + (\pi)^2}}$.	$\zeta > 0.59$.
The 5% settling-time is less than 1 second.	$\frac{3}{\sigma_o} < t_{5\%s_{\text{given}}}$, or $\sigma_o > \frac{3}{t_{5\%s_{\text{given}}}}$.	$\sigma_o > 3$.

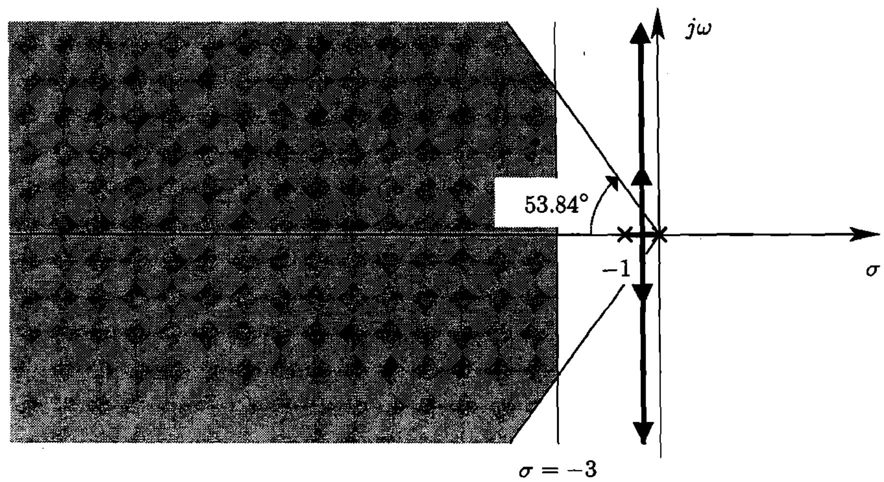
In order words, from the steady-state error requirement, we conclude that the P controller won't work. The s-plane region for the dominant closed-loop poles from the inequalities, $\zeta > 0.59$ or $\alpha < \cos^{-1}(\zeta) = 53.84°$ and $\sigma_o > 3$ or $\sigma < -3$ is given in the following figure.



The next simpler controller is the I controller. However, when we have

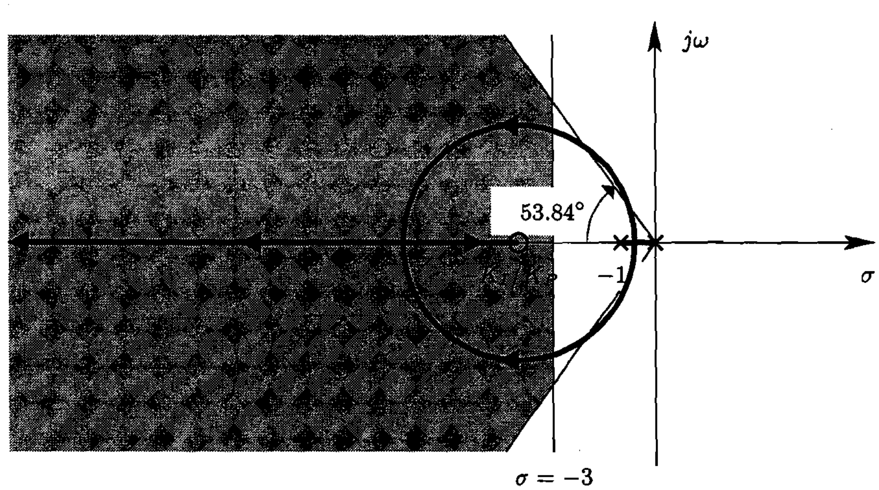
$$D(s)G(s) = K_I \frac{2}{s(s+1)},$$

the root-locus diagram doesn't go through the desired region as we can see from the following sketch.



With the PI controller, we have an extra zero to pull the root-locus branches towards the desired region, where

$$D(s)G(s) = K_P \frac{2(s + K_I/K_P)}{s(s+1)}.$$



Since a lot of choices for the zero would work, one possible choice is $K_I/K_P = 4$. For that choice, the radius is $r = \sqrt{(-4-0)(-4-(-1))} = \sqrt{12}$, and the intersection of the root-locus branch and the $\{\sigma{=}-3\}$ line is at $s = -3 \pm j\sqrt{11}$. The gain at $s = -3 \pm j\sqrt{11}$ can be determined from the magnitude condition, such that

$$\left| D(s)G(s) \right|_{s=-3+j\sqrt{11}} = \left| K_P \frac{2(s+4)}{s(s+1)} \right|_{s=-3+j\sqrt{11}} = 1,$$

or $K_P = 2.5$. Therefore, any $K_P > 2.5$ satisfies the requirements. By the way, we can also check the $\{\alpha < 53.84°\}$ requirement, where $\alpha = \tan^{-1}(\sqrt{11}/3) = 47.87°$.

One possible choice is $K_P = 3 > 2.5$ and $K_I = 4K_P = 12$. Therefore, the simplest controller is a PI controller, and one such controller is

$$D(s) = 3 + \frac{12}{s}.$$

The actual condition satisfying the $\{\sigma < -3\}$ requirement is $K_I > 3(K_P - 1)$.

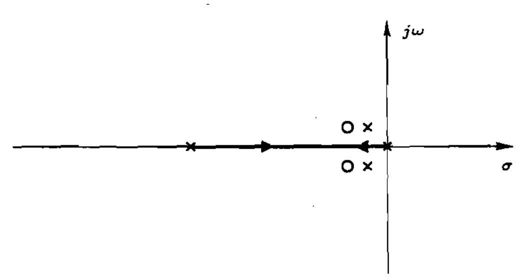
3. Consider the negative-feedback control-system with the following open-loop transfer-function. Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angles of departure and/or arrival.

$$G(s) = K\frac{s^2 + 4s + 5}{s(s+10)(s^2 + 2s + 2)}.$$

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes,
- Break-away point, and
- Angles of departure and arrival.



Asymptotes

Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n - m}$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n - m} = \frac{((-10) + (-1 + j) + (-1 - j) + (0)) - ((-2 + j) + (-2 - j))}{4 - 2} = -4.$$

Real-Axis Angles: $\theta_a = \pm(2k+1)\pi/(n - m)$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n - m} = \frac{\pm(2k+1)\pi}{4 - 2} = \pm\frac{\pi}{2}.$$

Break-Away Point: $\mathrm{d}K/\mathrm{d}s = 0$

From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{s^2+4s+5}{s(s+10)(s^2+2s+2)} = 0,$$

and

$$-K = \frac{s(s+10)(s^2+2s+2)}{s^2+4s+5}.$$

Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = \frac{2(s^5+12s^4+58s^3+124s^2+110s+50)}{(s^2+4s+5)^2}.$$

and for $\mathrm{d}K/\mathrm{d}s = 0$, the equation

$$s^5+12s^4+58s^3+124s^2+110s+50 = 0$$

gives $s = -2.9753$ and two sets of complex poles. So, the only break-away point is at $s = -2.9753$.

Angles of Departure: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of departure from complex open-loop poles are determined from the angular conditions about the open-loop poles. Therefore, the angular condition about $s = -1 + j1$ is

$$-\measuredangle\big(s-(-10)\big) + \measuredangle\big(s-(-2+j1)\big) + \measuredangle\big(s-(-2-j1)\big) \\ - \measuredangle\big(s-(-1+j1)\big) - \measuredangle\big(s-(-1-j1)\big) - \measuredangle\big(s-(0)\big) = 180^\circ + k360^\circ,$$

$$-\tan^{-1}\left(\frac{(1)-(0)}{(-1)-(-10)}\right) + \tan^{-1}\left(\frac{(1)-(1)}{(-1)-(-2)}\right) + \tan^{-1}\left(\frac{(1)-(-1)}{(-1)-(-2)}\right) \\ -\theta_{\text{dep}} - \tan^{-1}\left(\frac{(1)-(-1)}{(-1)-(-1)}\right) - \tan^{-1}\left(\frac{(1)-(0)}{(-1)-(0)}\right) = 180^\circ + k360^\circ,$$

or

$$-6.34^\circ + 0^\circ + 63.435^\circ - \theta_{\text{dep}} - 90^\circ - 135^\circ = 180^\circ + k360^\circ.$$

As a result,

$$\theta_{\text{dep}} = 12.095^\circ.$$

Angles of Arrival: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of arrival to complex open-loop zeros are determined from the angular conditions about the open-loop zeros. Therefore, the angular condition about $s = -2 + j1$ is

$$-\measuredangle\big(s-(-10)\big) + \measuredangle\big(s-(-2+j1)\big) + \measuredangle\big(s-(-2-j1)\big) \\ - \measuredangle\big(s-(-1+j1)\big) - \measuredangle\big(s-(-1-j1)\big) - \measuredangle\big(s-(0)\big) = 180^\circ + k360^\circ,$$

$$-\tan^{-1}\left(\frac{(1)-(0)}{(-2)-(-10)}\right) + \theta_{\text{arr}} + \tan^{-1}\left(\frac{(1)-(-1)}{(-2)-(-2)}\right) \\ -\tan^{-1}\left(\frac{(1)-(1)}{(-2)-(-1)}\right) - \tan^{-1}\left(\frac{(1)-(-1)}{(-2)-(-1)}\right) - \tan^{-1}\left(\frac{(1)-(0)}{(-2)-(0)}\right) = 180^\circ + k360^\circ,$$

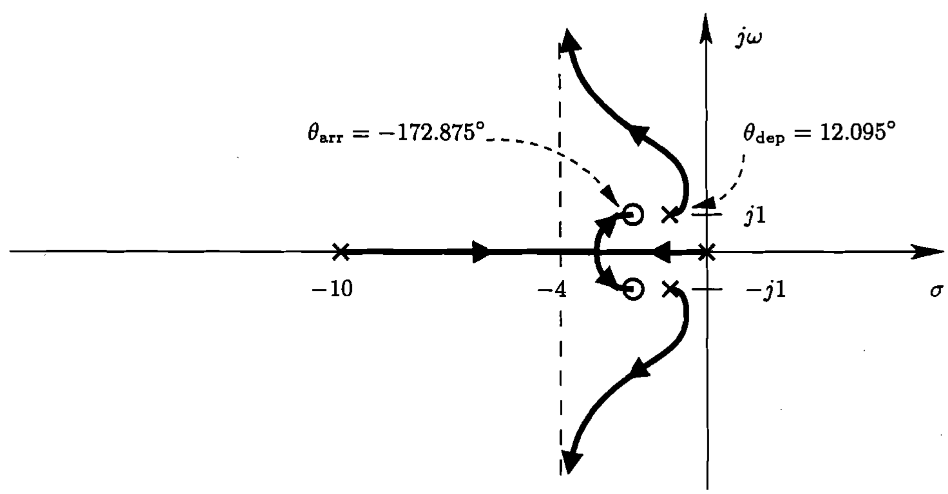
or

$$-7.125° + \theta_{\text{arr}} + 90° - 180° - 116.565° - 153.435° = 180° + k360°.$$

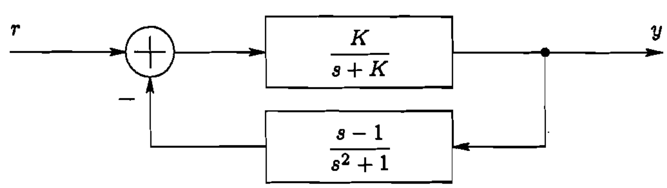
As a result,

$$\theta_{\text{arr}} = -172.875°.$$

With the features determined, we can now sketch the root-locus diagram.



4. Sketch the root-locus diagram for the following control system.



Solution: The sketch of the location of the closed-loop poles is the root-locus diagram. However, in this case the open-loop gain of the system is

$$G(s)H(s) = \left(\frac{K}{s+K}\right)\left(\frac{s-1}{s^2+1}\right) = \frac{K(s-1)}{(s+K)(s^2+1)},$$

where the root-locus variable K is not a multiplicative coefficient of the open-loop gain. So, we need to convert the problem into the conventional form while preserving the location of the closed-loop poles the same. The closed-loop poles are obtained from the characteristic equation, where

$$1 + G(s)H(s) = 0,$$

or

$$1 + \frac{K(s-1)}{(s+K)(s^2+1)} = 0,$$

$$\frac{(s+K)(s^2+1) + K(s-1)}{(s+K)(s^2+1)} = 0,$$

$$(s+K)(s^2+1) + K(s-1) = 0,$$

$$s^3 + Ks^2 + Ks + s = 0.$$

We need to regroup the characteristic equation, so that the characteristic equation is in the form

$$1 + K\frac{n(s)}{d(s)} = 0,$$

for some polynomials $n(s)$ and $d(s)$. So,

$$s^3 + Ks^2 + Ks + s = 0,$$

$$(s^3 + s) + K(s^2 + s) = 0,$$

$$\frac{(s^3 + s) + K(s^2 + s)}{(s^3 + s)} = 0,$$

$$1 + K\frac{s^2 + s}{s^3 + s} = 0.$$

Therefore, the new open-loop gain

$$G'(s)H'(s) = K\frac{s^2 + s}{s^3 + s} = K\frac{s(s+1)}{s(s^2+1)} = K\frac{s+1}{s^2+1}$$

generates the same closed-loop poles as the original open-loop gain, but the open-loop gain $G'(s)H'(s)$ of the new system is in the usual form for the generation of the root-locus diagram. In other words, the locations of the closed-loop poles based on the open-loop gains $G(s)H(s)$ and $G'(s)H'(s)$ are identical, however we can use the regular root-locus drawing techniques on the primed system.

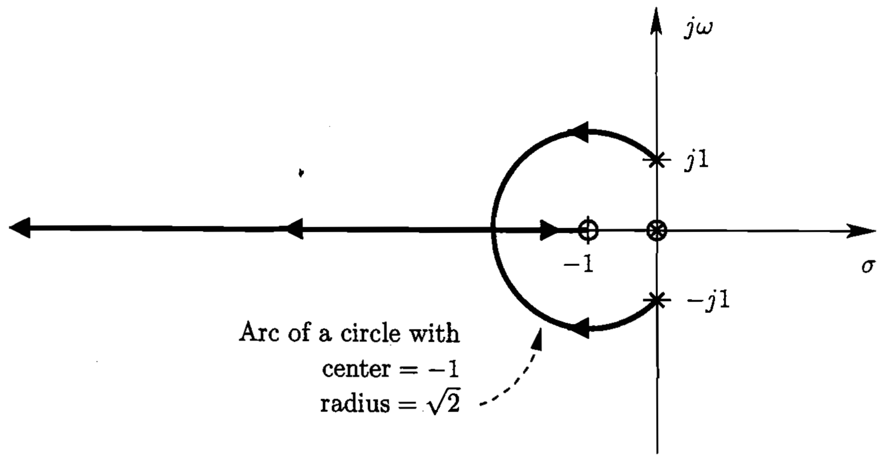
We observe that we have the two-pole one-zero case, where the portion of the root-locus diagram outside of the real axis is on a circle with the center at the zero,

$$\text{center} = z = -1,$$

and the radius that is the geometric mean of the distances of the poles from the zero,

$$\text{radius} = \sqrt{(p_1 - z)(p_2 - z)} = \sqrt{((j) - (-1))((-j) - (-1))} = \sqrt{2}.$$

Therefore,



jω
j1
−1
σ
−j1
Arc of a circle with
center = −1
radius = √2