1. Consider the mechanical system shown below.

![Mechanical System Diagram]

(a) Obtain the transfer matrix of the system assuming that the external forces $u_1$ and $u_2$ are the inputs, and the displacement $y$ is the output. 

(b) Obtain *either* the force-voltage or the force-current analog of the system.  

2. The angular position of the shaft of a motor is controlled by the system shown below.

![Motor Control System Diagram]

The angular position of the motor shaft is detected by a variable resistor which provides a voltage $v_o$ proportional to the angle, such that $v_o = -K_o \theta_o$. Draw the most detailed block diagram of the system, where $v_i$ is the input, and $\theta_o$ is the output. Show all the variables $v_i$, $v_o$, $v_j$, $v_a$, $v_b$, $i_a$, $\tau$, $\theta_m$, and $\theta_o$ on the block diagram.
3. For the block diagram given below, determine the transfer function either by block-diagram reduction or by Mason's formula. Show your work clearly. (25pts)

4. A control system is represented by

\[
\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),
\]

\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + [1] u(t).
\]

Determine \(y(t)\) for \(t \geq 0\); when \(x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T\), and \(u(t) = 1\) for \(t \geq 0\). (25pts)
1. Consider the mechanical system shown below.

(a) Obtain the transfer matrix of the system assuming that the external forces $u_1$ and $u_2$ are the inputs, and the displacement $y$ is the output.

Solution:

First, we identify the linearly independent displacement locations in the mechanical system and mark them.

Then, we write the differential equations describing the motion from the mechanical system.

$$m_1 \ddot{y} = u_1 - k_1(y - y_2)$$

$$m_2 \ddot{y}_2 = u_2 - k_1(y_2 - y) - k_2 y_2 - b \dot{y}_2.$$

Next, we obtain the transfer function by taking the Laplace transforms of the above equations under zero initial conditions. After some manipulations, we get

$$(m_1 s^2 + k_1)Y(s) - k_1 Y_2(s) = U_1(s),$$

and

$$-k_1 Y(s) + (m_2 s^2 + bs + k_1 + k_2)Y_2(s) = U_2(s),$$
where \( U_1, U_2, Y, \) and \( Y_2 \) are the Laplace transforms of \( u_1, u_2, y, \) and \( y_2, \) respectively. After multiplying the second equation by \( k_1 \) and substituting \( k_1 Y_2(s) \) from the first equation, we get

\[-k_1^2 Y(s) + (m_2 s^2 + bs + k_1 + k_2)\left((m_1 s^2 + k_1) Y(s) - U_1(s)\right) = k_1 U_2(s),\]

or

\[(m_2 s^2 + bs + k_1 + k_2)(m_1 s^2 + k_1) Y(s) = (m_2 s^2 + bs + k_1 + k_2) U_1(s) + k_1 U_2(s).\]

Therefore,

\[Y(s) = \frac{1}{(m_2 s^2 + bs + k_1 + k_2)(m_1 s^2 + k_1) - k_1^2} \begin{bmatrix} m_2 s^2 + bs + k_1 + k_2 & k_1 \\ m_2 s^2 + bs + k_1 + k_2 & 1 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.\]

(b) Obtain either the force-voltage or the force-current analog of the system.

**Solution:** For the force-voltage analog of a mechanical system, there will be a loop charge associated with each displacement variable (or a loop current associated with each velocity variable), and an input force will be associated with a voltage source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of capacitance, the resistance, and the inductance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding loop variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be on the non-common portions of the loops.

The next figure shows the force-voltage analog of the mechanical system, where the loops are identified with the displacements.

For the force-current analog of a mechanical system, there will be a node flux associated with each displacement variable (or a node voltage associated with each velocity variable), and an input force will be associated with a current source. The spring constant, the damping constant, and the mass will be associated with the reciprocal of inductance, the conductance, and the capacitance, respectively. The elements between two displacement variables of the mechanical system will be between the corresponding node variables of the force-voltage analog. The elements that are connected to fixed frames and the elements that are always measured with respect to a fixed frame, such as the mass and the external force, will be connected to the ground.

The next figure shows the force-current analog of the mechanical system, where the nodes are identified with the displacements.
2. The angular position of the shaft of a motor is controlled by the system shown below.

![Block Diagram of Motor Control System]

The angular position of the motor shaft is detected by a variable resistor which provides a voltage $v_o$ proportional to the angle, such that $v_o = -K_o \theta_o$. Draw the most detailed block diagram of the system, where $v_i$ is the input, and $\theta_o$ is the output. Show all the variables $v_i$, $v_o$, $v_j$, $v_a$, $v_b$, $i_a$, $\tau$, $\theta_m$, and $\theta_o$ on the block diagram.

**Solution:** To determine the block diagram of the system, we first separate it into simpler components.

Because the input variable is $v_i$, we would write either $v_j$ or $v_o$ in terms of $v_i$, such that

$$v_j(t) = v_i(t) + v_o(t).$$

Since the operational amplifier is assumed to be ideal, we get

$$\frac{V_o(s)}{R + 1/(Cs)} = \frac{V_j(s)}{1/(Cs)},$$

or

$$V_o(s) = (RCs + 1)V_j(s).$$
The armature current of the motor is

\[ I_a(s) = \frac{1}{L_a s + R_a} (V_a(s) - V_b(s)). \]

From the armature-controlled motor,

\[ \tau(t) = K_a i_a(t). \]

The back-emf voltage of the motor

\[ v_b(t) = K_b \frac{d\theta_m(t)}{dt}, \]

or

\[ V_b(s) = (K_b s) \Theta_m(s). \]
The torque equation for $\theta_m$ is

$$J_m \frac{d^2 \theta_m(t)}{dt^2} = \tau(t) - K_m(\theta_m(t) - \theta_o(t)),$$

or

$$\Theta_m(s) = \frac{1}{J_m s^2 + K_m} \left( T(s) + K_m \Theta_o(s) \right).$$

The torque equation for $\theta_o$ is

$$J_o \frac{d^2 \theta_o(t)}{dt^2} = -K_m(\theta_o(t) - \theta_m(t)),$$

or

$$\Theta_o(s) = \frac{K_m}{J_o s^2 + K_m} \Theta_m(s).$$

And, finally the given relationship

$$v_o(t) = -K_o \theta_o(t).$$

When we connect all the individual blocks together, we get the following block diagram.
3. For the block diagram given below, determine the transfer function either by block-diagram reduction or by Mason's formula. Show your work clearly.

Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.
If we choose to use Mason’s formula, we need to draw the signal flow graph of the block diagram.

In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

\[ F_1 = l_1 l_2 G_1 l_3 G_2 l_4 G_3 l_5 = G_1 G_2 G_3, \]

and

\[ F_2 = l_1 l_2 l_9 l_8 l_3 G_2 l_4 G_3 l_5 = G_2 G_3. \]

The loop gains are

\[ L_1 = l_2 G_1 l_3 l_7 = G_1, \]
\[ L_2 = l_2 l_9 l_8 l_3 l_7 = 1, \]
\[ L_3 = l_3 G_2 l_4 G_4 l_8 = G_2 G_4, \]

and

\[ L_4 = l_4 G_3 l_6 = G_3. \]

From the forward path and the loop gains, we determine the touching loops and the forward paths.

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<th>Touching Loops</th>
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<th>( L_2 )</th>
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</tbody>
</table>
Therefore,
\[
\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_4 + L_2 L_4) \\
= 1 - ((G_1) + (1) + (G_2 G_4 + (G_3)) + ((G_1)(G_3) + (1)(G_3)) \\
= -G_1 - G_2 G_4 - G_1 G_3,
\]
and
\[
\Delta_1 = \Delta|_{L_1=L_2=L_3=L_4=0} = 1, \\
\Delta_2 = \Delta|_{L_1=L_2=L_3=L_4=0} = 1.
\]
So,
\[
\frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_{i=1}^{2} F_i \Delta_i = \frac{(G_1 G_2 G_3)(1) + (G_2 G_3)(1)}{-G_1 - G_2 G_4 + G_1 G_3},
\]
or
\[
\frac{Y(s)}{U(s)} = \frac{G_1 G_2 G_3 + G_2 G_3}{-G_1 - G_2 G_4 + G_1 G_3}.
\]

4. A control system is represented by
\[
\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),
\]
\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \end{bmatrix} u(t).
\]
Determine \( y(t) \) for \( t \geq 0 \); when \( x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \), and \( u(t) = 1 \) for \( t \geq 0 \).

\textbf{Solution:} The general solution to the state-space representation of a system described by
\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]
is obtained from
\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau,
\]
where
\[
e^{At} = \mathcal{L}_s^{-1} [(sI - A)^{-1}](t).
\]
Here, \( I \) is the appropriately dimensioned identity matrix. In our case,
\[
A = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix}.
\]
\( x(0) = [0 \ 0]^T \), and \( u(t) = 1 \) for \( t \geq 0 \). As a result, the initial-condition term in the solution of \( x \) and the first term in the \( y \) equation are identically zero. So,

\[
y(t) = C \int_0^t e^{A(t-\tau)}Bu(\tau) \, d\tau + Du(t)
\]

\[
= \begin{bmatrix} 1 & 0 \end{bmatrix} \left[ \int_0^t \mathcal{L}_s^{-1} \left[ (sI - A)^{-1} \right] (t - \tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1) \, d\tau \right] (1)
\]

\[
= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[ \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} \right)^{-1} \right] (t - \tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \, d\tau + 1
\]

\[
= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[ \left( \begin{bmatrix} s + 1 & 0 \\ -4 & s + 2 \end{bmatrix} \right)^{-1} \right] (t - \tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \, d\tau + 1
\]

\[
= \begin{bmatrix} 1 & 0 \end{bmatrix} \int_0^t \mathcal{L}_s^{-1} \left[ \frac{1}{(s + 1)(s + 2) - (-4)(0)} \begin{bmatrix} s + 2 & 0 \\ 4 & s + 1 \end{bmatrix} \right] (t - \tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \, d\tau + 1
\]

\[
= \int_0^t \begin{bmatrix} 1 & 0 \\ \mathcal{L}_s^{-1} \left[ \frac{1}{s + 1} \right] (t - \tau) & 0 \\ \mathcal{L}_s^{-1} \left[ \frac{4}{(s + 1)(s + 2)} \right] (t - \tau) & \mathcal{L}_s^{-1} \left[ \frac{1}{s + 2} \right] (t - \tau) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \, d\tau + 1
\]

\[
= \int_0^t \mathcal{L}_s^{-1} \left[ \frac{1}{s + 1} \right] (t - \tau) \begin{bmatrix} 1 \\ 0 \\ \mathcal{L}_s^{-1} \left[ \frac{1}{s + 2} \right] (t - \tau) \end{bmatrix} \, d\tau + 1
\]

\[
= \int_0^t e^{-(t-\tau)} \, d\tau + 1
\]

\[
= \left( e^{-(t-\tau)} \right)_{\tau=0}^{\tau=t} + 1
\]

\[
= (1 - e^{-t}) + 1.
\]

Or,

\[
y(t) = 2 - e^{-t} \text{ for } t \geq 0.
\]