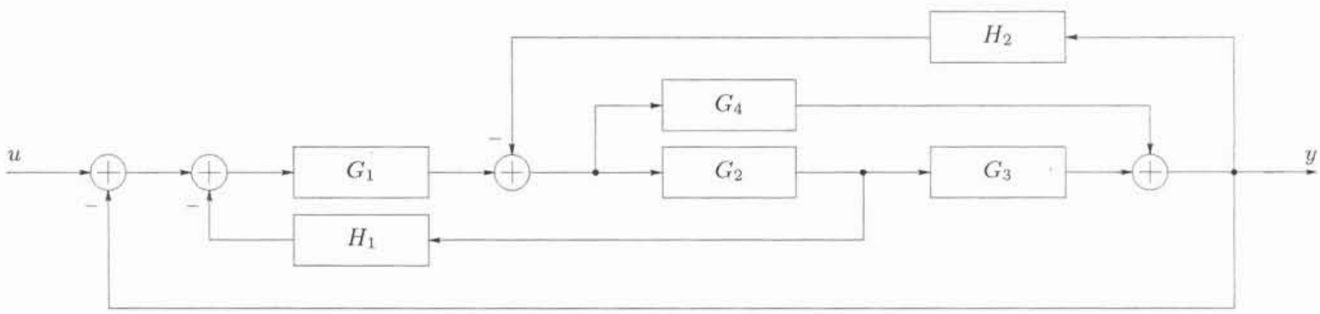
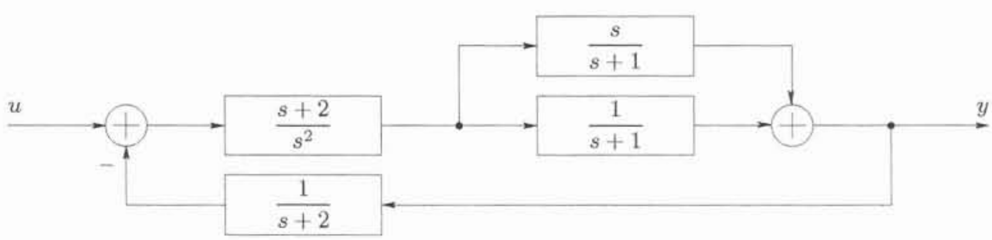
EE 231 — Exam#1 — Oct. 21, 2009

50 minutes

1. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (35pts)

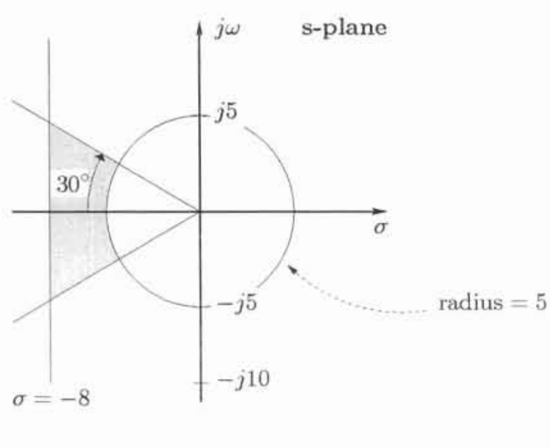


2. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (35pts)

3. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.

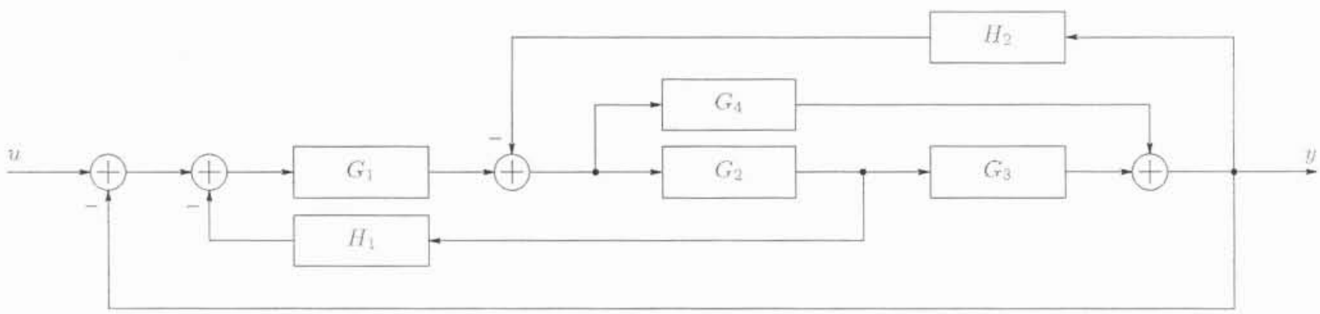


Determine the largest possible maximum percent-overshoot, the smallest possible peak time, and the smallest possible 2% settling-time of the system. Determine also a set of poles that has all these properties. (30pts)

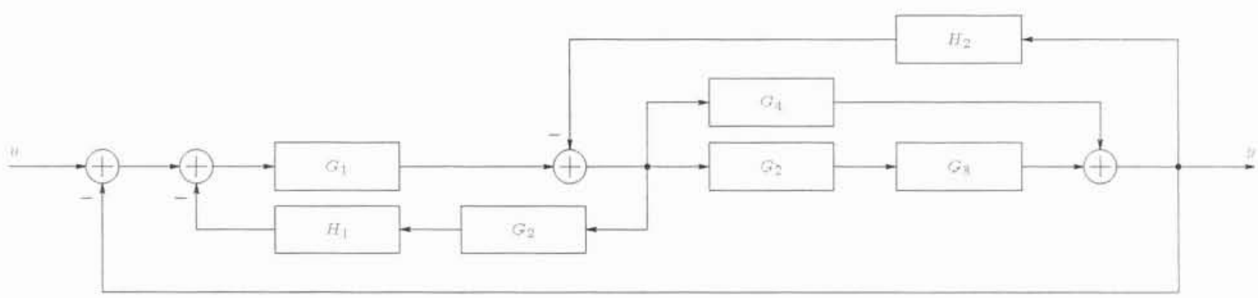
EE 231 Exam#1 Oct. 21, 2009

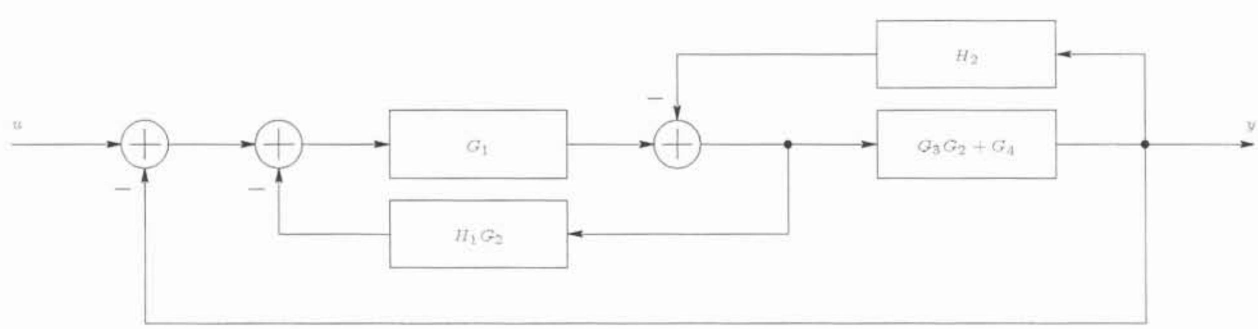
Solutions

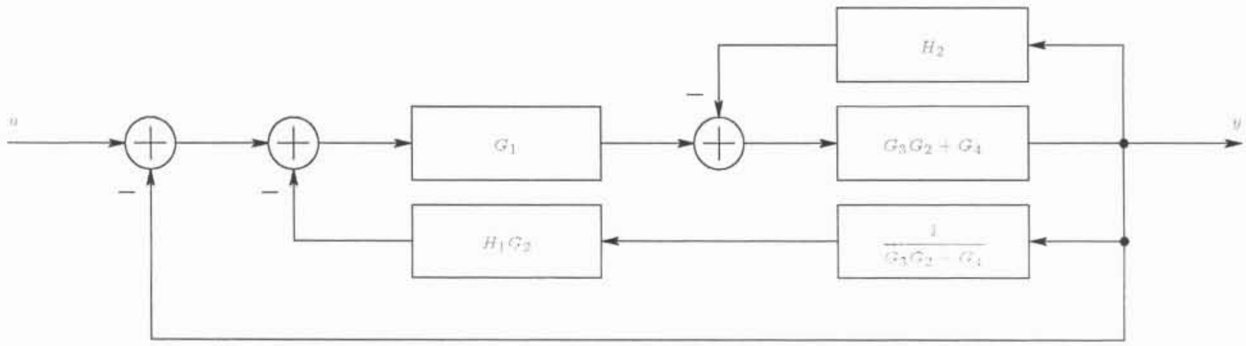
1. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

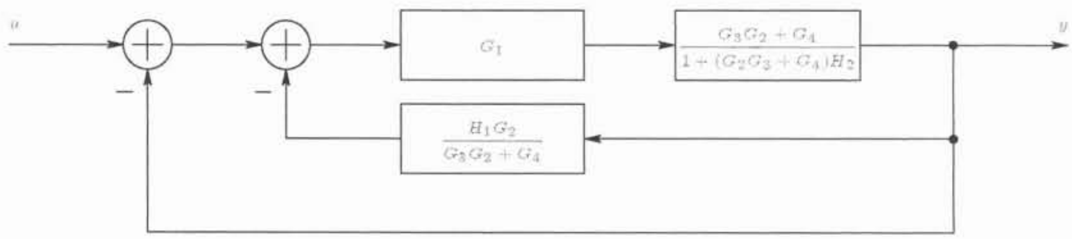


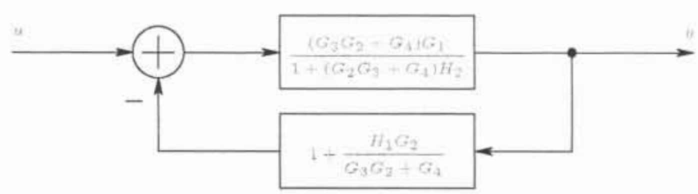
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

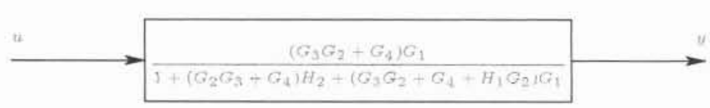




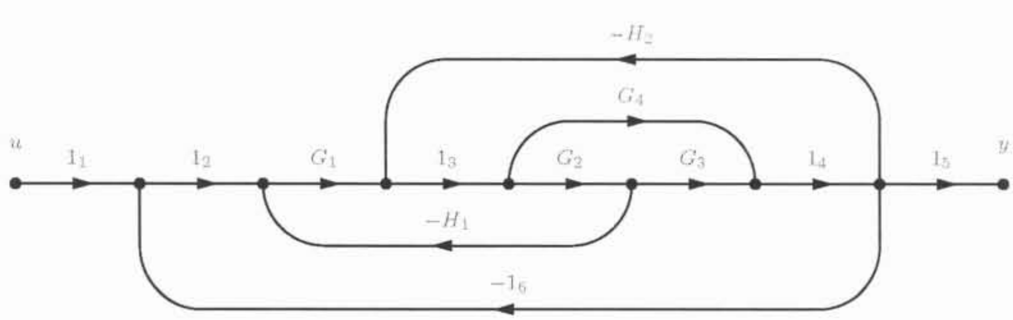








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 G_3 1_4 1_5 = G_1 G_2 G_3,$$
$$F_2 = 1_1 1_2 G_1 1_3 G_4 1_4 1_5 = G_1 G_4.$$

The loop gains are

$$L_1 = 1_2 G_1 1_3 G_2 G_3 1_4 (-1_6) = -G_1 G_2 G_3,$$
$$L_2 = 1_2 G_1 1_3 G_4 1_4 (-1_6) = -G_1 G_4,$$
$$L_3 = G_1 1_3 G_2 (-H_1) = -G_1 G_2 H_1,$$
$$L_4 = 1_3 G_2 G_3 1_4 (-H_2) = -G_2 G_3 H_2,$$
$$L_5 = 1_3 G_4 1_4 (-H_2) = -G_4 H_2.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4	L_5
L_1	✓	✓	✓	✓	✓
L_2		✓	✓	✓	✓
L_3			✓	✓	✓
L_4				✓	✓
L_5					✓

Loops on Forward Paths

	L_1	L_2	L_3	L_4	L_5
F_1	✓	✓	✓	✓	✓
F_2	✓	✓	✓	✓	✓

Therefore,

$$\begin{aligned}\Delta &= 1 - \big(L_1 + L_2 + L_3 + L_4 + L_5\big) \\ &= 1 - \big((-G_1G_2G_3) + (-G_1G_4) + (-G_1G_2H_1) + (-G_2G_3H_2) + (-G_4H_2)\big) \\ &= 1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2,\end{aligned}$$

and

$$\Delta_1 = \Delta\big|_{L_1=L_2=L_3=L_4=L_5=0} = 1,$$
$$\Delta_2 = \Delta\big|_{L_1=L_2=L_3=L_4=L_5=0} = 1.$$

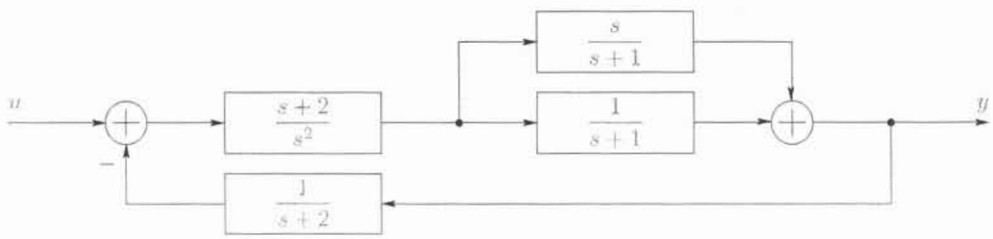
So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{2} F_i\Delta_i = \frac{(G_1G_2G_3)(1) + (G_1G_4)(1)}{1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_1G_4 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}.$$

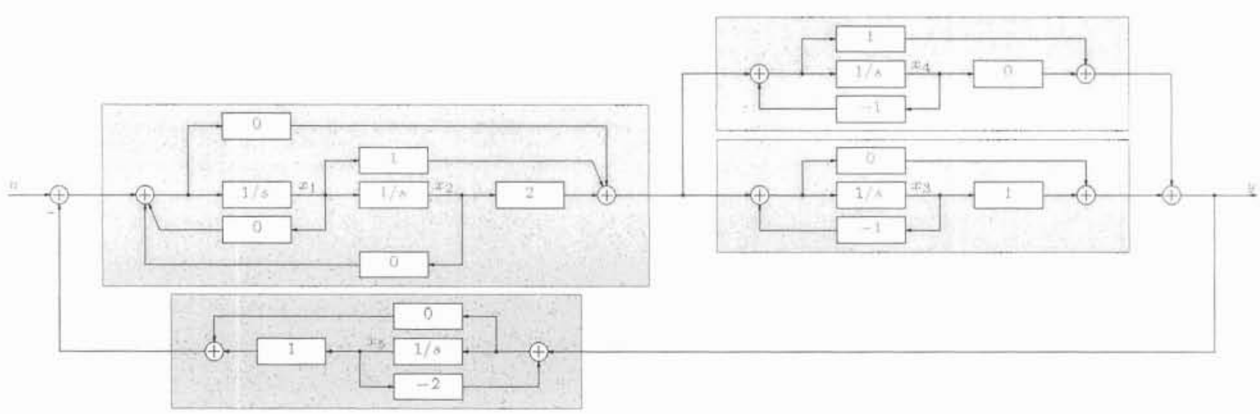
2. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

We may use a number of possible forms to realize the blocks. If we use the controller canonical form, we get the following diagram.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = (0)x_1 + (0)x_2 + \Big(u - \big((1)x_5 + (0)\dot{x}_5\big)\Big) = -x_5 + u,$$

$$\dot{x}_2 = x_1,$$

$$\dot{x}_3 = (-1)x_3 + \Big((2)x_2 + (1)x_1 + (0)\dot{x}_1\Big) = x_1 + 2x_2 - x_3,$$

$$\dot{x}_4 = (-1)x_4 + \Big((2)x_2 + (1)x_1 + (0)\dot{x}_1\Big) = x_1 + 2x_2 - x_4,$$

$$\dot{x}_5 = (-2)x_5 + y,$$

and

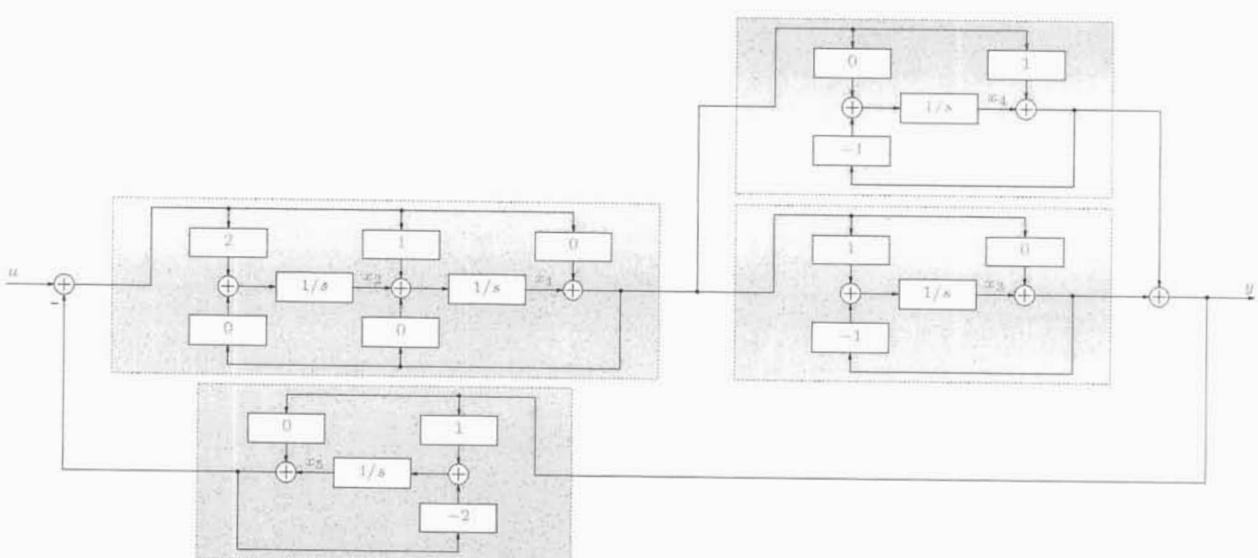
$$y = \Big((1)x_3 + (0)\dot{x}_3\Big) + \Big((0)x_4 + (1)\dot{x}_4\Big) = x_3 + \dot{x}_4 = x_1 + 2x_2 + x_3 - x_4.$$

After substituting for y in the $\dot{x}_5$ equation, we obtain the state-space representation based on the controller canonical form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 1 & 2 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 2 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}.$$

If we use the observer realization form for each of the blocks, then we obtain a different state-space representation.



Similarly, we obtain

$$\dot{x}_1 = (0)y_1 + x_2 + (1)\Big(u - \big(x_5 + (0)y\big)\Big) = x_2 - x_5 + u,$$

$$\dot{x}_2 = (0)y_1 + (2)\Big(u - \big(x_5 + (0)y\big)\Big) = -2x_5 + 2u,$$

$$\dot{x}_3 = (-1)\big(x_3 + (0)y_1\big) + (1)y_1 = -x_3 + y_1,$$

$$\dot{x}_4 = (-1)\big(x_4 + (1)y_1\big) + (0)y_1 = -x_4 - y_1,$$

$$\dot{x}_5 = (-2)\big(x_5 + (0)y\big) + (1)y = -2x_5 + y,$$

and

$$y = \big(x_3 + (0)y_1\big) + \big(x_4 + (1)y_1\big) = x_3 + x_4 + y_1,$$

where

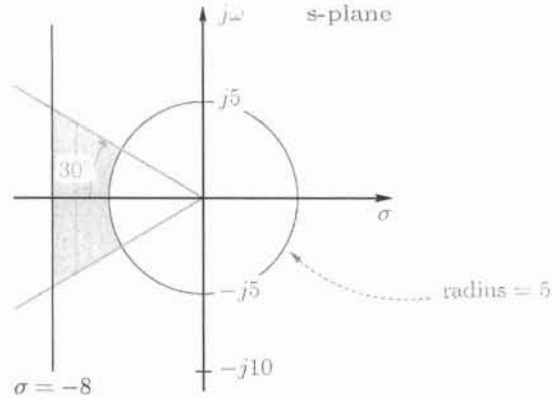
$$y_1 = x_1 + (0)\Big(u - \big(x_5 + (0)y\big)\Big) = x_1.$$

After substituting y_1 and y into the equations, we get the state-space representation based on the observer canonical form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}.$$

3. Consider a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, such that its poles are located in the shaded region below.



Determine the largest possible maximum percent-overshoot, the smallest possible peak time, and the smallest possible 2% settling-time of the system. Determine also a set of poles that has all these properties.

Solution: Maximum overshoot for a second-order system with no zero is given by

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}.$$

The only system parameter that affects the maximum overshoot is ζ. For the maximum M_p, we need to have minimum ζ; since $\zeta = 0$ gives undamped oscillations. In the shaded region, the two boundaries of ζ are when $\zeta = \cos(30°) = \sqrt{3}/2$ and $\zeta = \cos(90°) = 1$. So, the minimum $\zeta = \sqrt{3}/2$, and the corresponding maximum overshoot is

$$M_p = e^{-\frac{\sqrt{3}/2}{\sqrt{1-(\sqrt{3}/2)^2}}\pi} = e^{-\sqrt{3}\pi} \approx 0.0043,$$

or the smallest possible maximum percent-overshoot is 0.43%.

The peak time of a second-order system with no zero is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n}.$$

The single system parameter that affects the settling time is ω_d. For the minimum t_p, we need to have maximum ω_d. In the shaded region, the maximum ω_d is at the intersection of the radial line with the 30° angle and the vertical line at $\sigma = -8$. Since the radial line with the 30° angle gives $\zeta = \cos(30°) = \sqrt{3}/2$, and $\sigma_o = 8$; we get

$$\omega_n = \frac{\sigma_o}{\zeta} = \frac{8}{\sqrt{3}/2} = \frac{16\sqrt{3}}{3},$$

and

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \left(\sqrt{1-(\sqrt{3}/2)^2}\right)\left(16\sqrt{3}/3\right) = \left(1/2\right)\left(16\sqrt{3}/3\right) = 8\sqrt{3}/3.$$

Therefore, the smallest possible peak time is $3\pi/(8\sqrt{3})$ s or approximately 0.68 s.

The 2% settling time of a second-order system with no zero is given by

$$t_{2\%s} = \frac{4}{\sigma_o}.$$

The single system parameter that affects the settling time is σ_o. For the smallest $t_{2\%s}$, we need to have maximum σ_o. In the shaded region, the maximum σ_o is at the vertical line $\sigma = -8$ or $\sigma_o = 8$. Therefore, the smallest possible 2% settling time is $4/8$ s or 0.5 s.

The set of poles that would have the largest maximum percent-overshoot, the smallest peak time, and the smallest 2% settling-time is at the intersection of the radial line with the 30° angle and the vertical line at $\sigma = -8$. Since at the intersection point, $\sigma_o = 8$ and $\omega_d = 8\sqrt{3}/3$, the set of poles is at $s = -\sigma_o \pm j\omega_d = -8 \pm j8\sqrt{3}/3$ or approximately at $s = -8 \pm j4.62$.