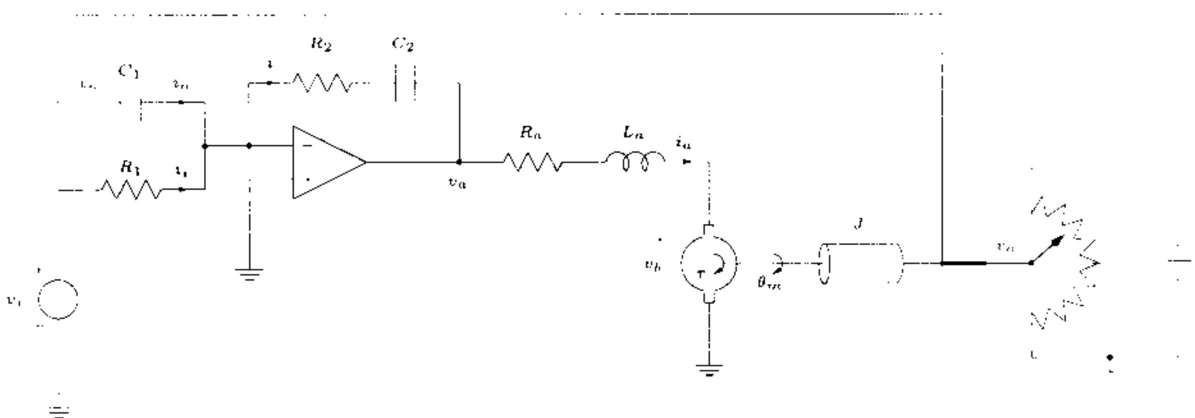
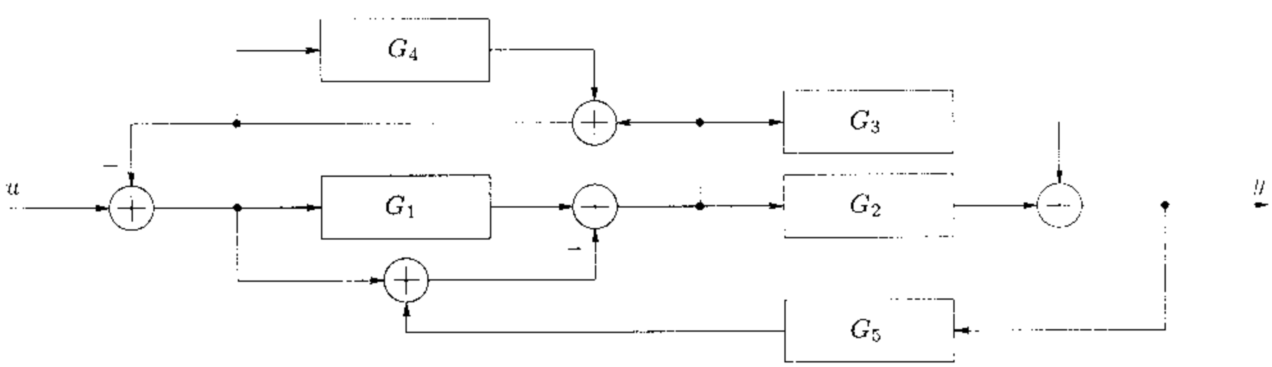
EE 231 | Exam#1 | Nov. 10, 1999

50 minutes

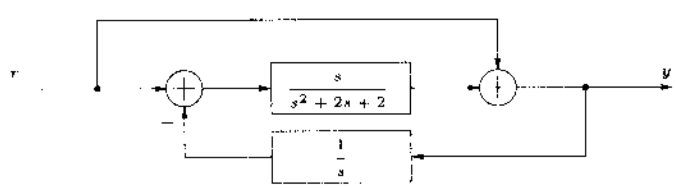
1. In the following system, an armature controlled motor is used to control the location of a rotational object. At the output a voltage that is proportional to the motor angle is generated by the variable resistor, such that $v_o = K_o\theta_m$. Obtain the detailed block diagram of the system, where v_i is the input and θ_m is the output, and show the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, and θ_m on the block diagram. (25pts)



2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (20pts)



3. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (20pts)

4. Consider a control system described in state space, such that

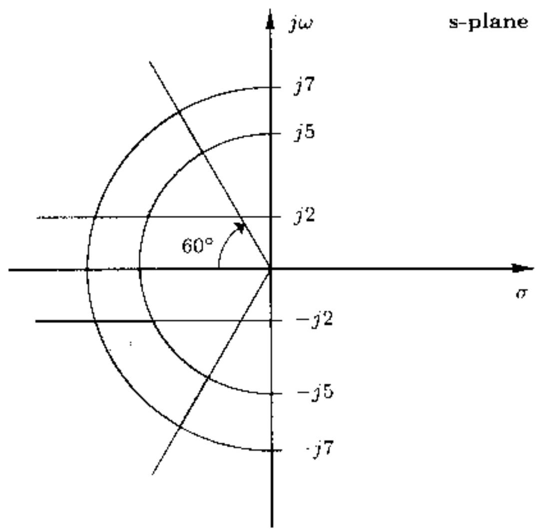
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) + u(t).$$

(a) Determine the transfer function, $Y(s)/U(s)$, of the system. (10pts)

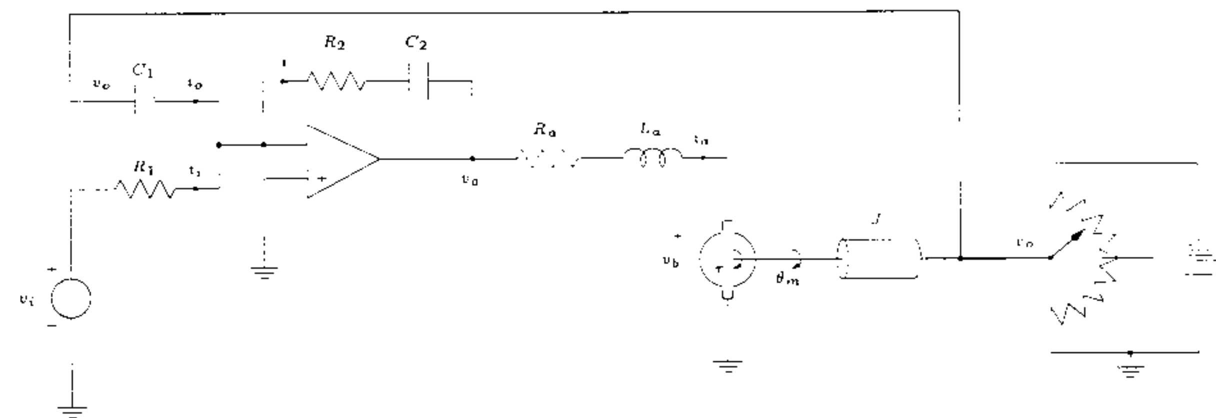
(b) Determine $y(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and the input is the unit step, i.e. $u(t) = \mathbb{1}(t)$. (15pts)

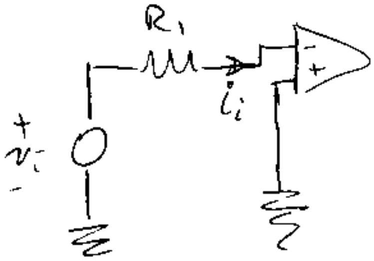
5. Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. (10pts)



Solutions

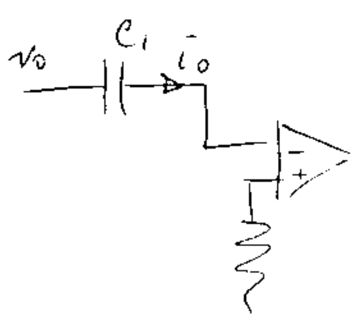
1. In the following system, an armature controlled motor is used to control the location of a rotational object. At the output a voltage that is proportional to the motor angle is generated by the variable resistor, such that $v_o = K_o\theta_m$. Obtain the detailed block diagram of the system, where v_i is the input and θ_m is the output, and show the variables v_i, i_i, v_o, i_o, i, v_a, i_a, v_b, τ, and θ_m on the block diagram.





$$i_i = \frac{1}{R_1} V_i$$

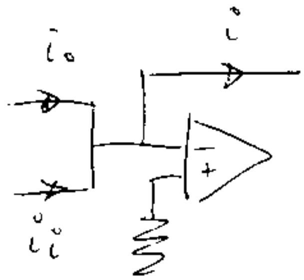
$$v_i \longrightarrow \boxed{1/R_1} \longrightarrow i_i$$



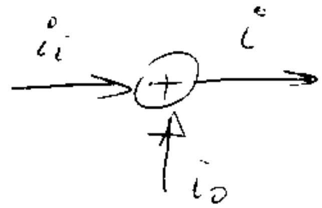
$$i_o = \frac{1}{1/sC_1} V_o$$

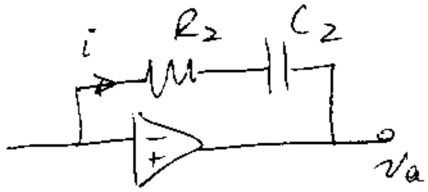
$$= sC_1 V_o$$

$$v_o \longrightarrow \boxed{sC_1} \longrightarrow i_o$$

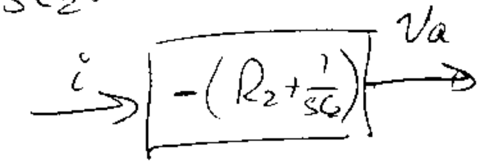


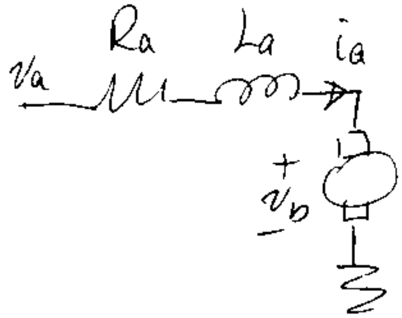
$i = i_0 + i_1$



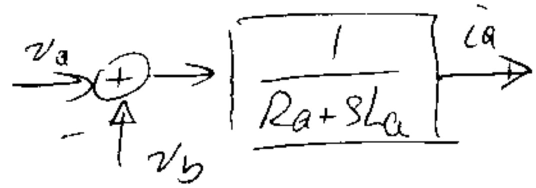


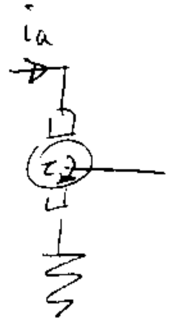
$$V_a = -\left(R_2 + \frac{1}{sC_2}\right) I$$



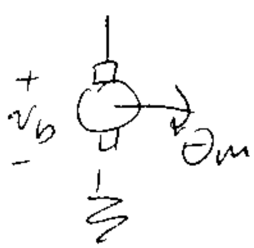


$$I_a = \frac{1}{R_a + sL_a}\left(V_a - V_b\right)$$



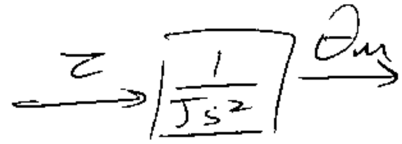


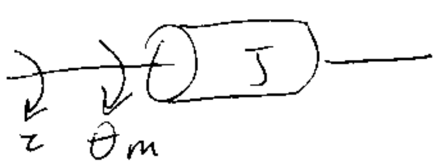
$\tau = K_a i_a$



$V_b = K_b s \Theta$

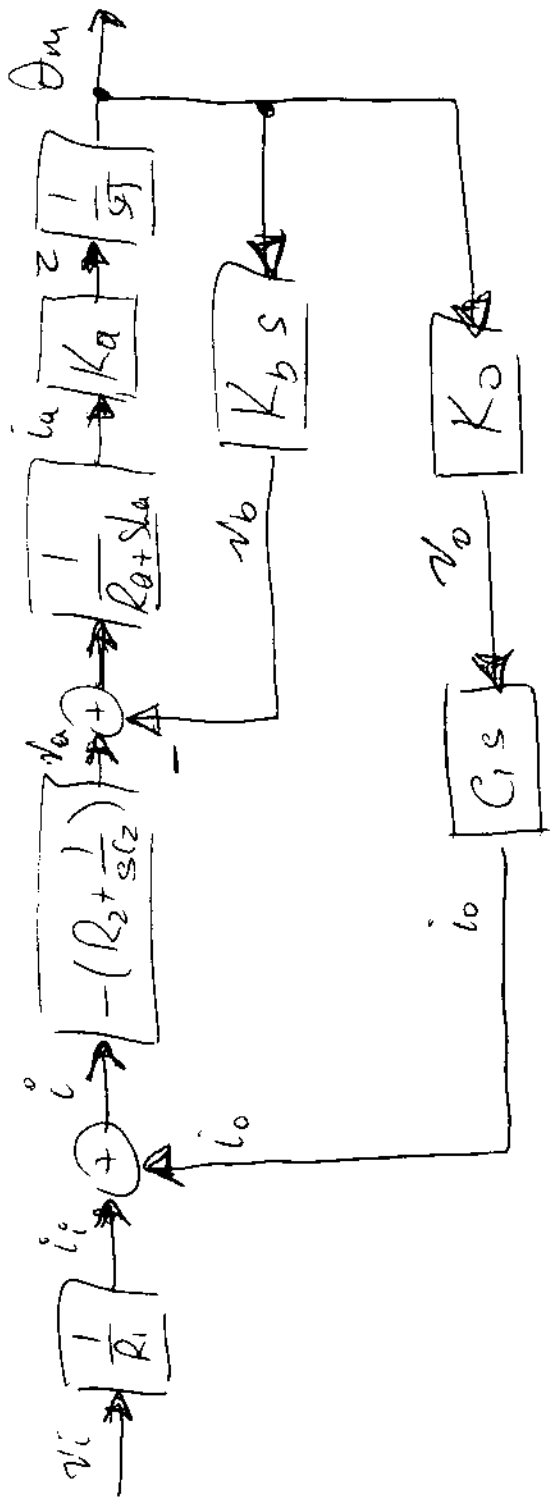
$J\ddot{\theta}_m = \tau$



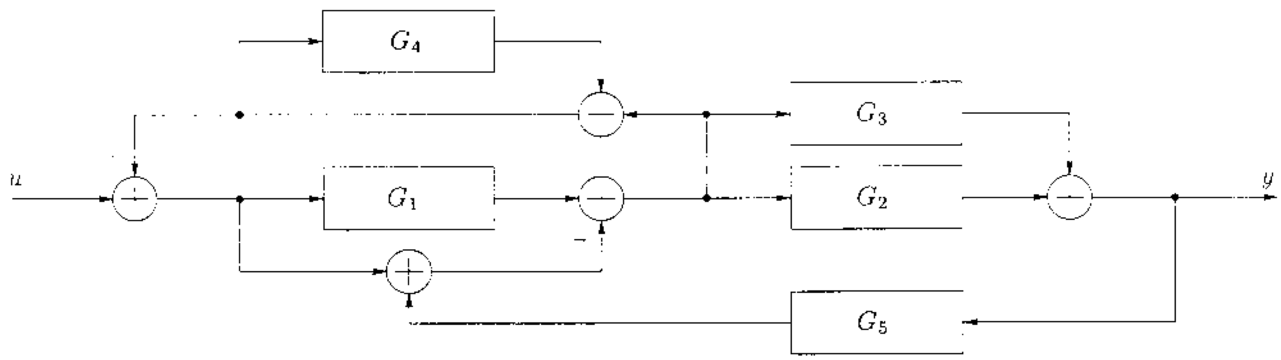


$$\Theta_m = \frac{1}{Js^2} T$$

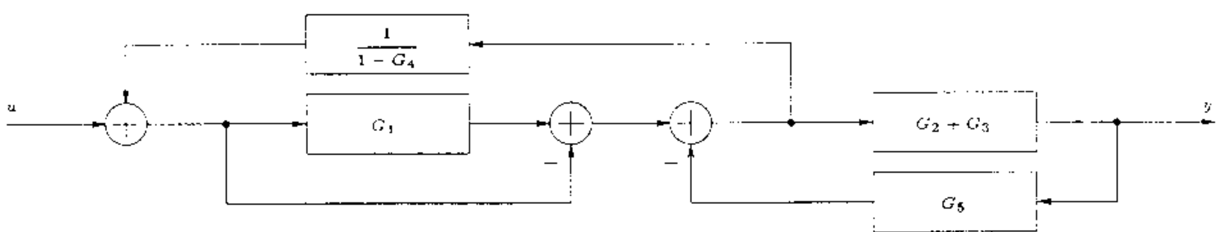
Given $v_0 = K_0 \theta_m$ $\quad \theta_m \rightarrow \boxed{K_0} \rightarrow v_0$

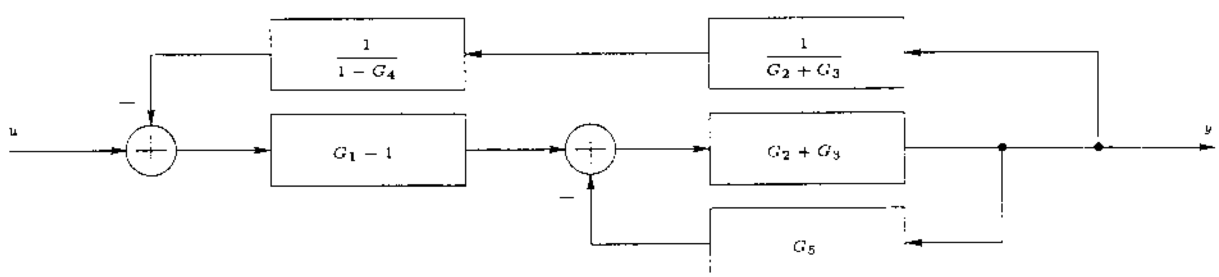


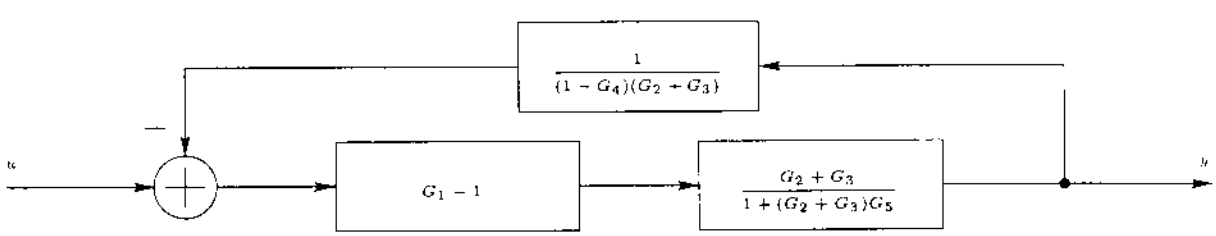
2. For the block diagram given below, determine the transfer function *either* by block diagram reduction, *or* by Mason's formula. Show your work clearly.

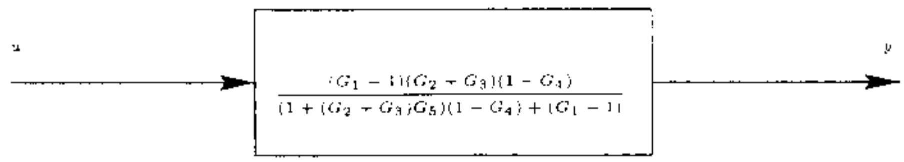


Solution: If we choose to use the block diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.









If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.

In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 1_3 G_2 1_4 1_5 = G_1 G_2,$$

$$F_2 = 1_1 1_2 1_6 (-1_7) 1_3 G_2 1_4 1_5 = -G_2,$$

$$F_3 = 1_1 1_2 G_1 1_3 1_8 G_3 1_4 1_5 = G_1 G_3,$$

and

$$F_4 = 1_1 1_2 1_6 (-1_7) 1_3 1_8 G_3 1_4 1_5 = -G_3.$$

The loop gains are

$$L_1 = 1_3 G_2 1_4 G_5 (-1_7) = -G_2 G_5,$$

$$L_2 = 1_3 1_8 G_3 1_4 G_5 (-1_7) = -G_3 G_5,$$

$$L_3 = 1_2 G_1 1_3 1_8 1_9 1_{10} (-1_{11}) = -G_1,$$

$$L_4 = 1_2 1_6 (-1_7) 1_3 1_8 1_9 1_{10} (-1_{11}) = 1,$$

and

$$L_5 = G_4 1_{10} = G_4.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3	L_4	L_5
L_1	✔	✔	✔	✔	✘
L_2		✔	✔	✔	✘
L_3			✔	✔	✔
L_4				✔	✔
L_5					✔

Loops on Forward Paths

	L_1	L_2	L_3	L_4	L_5
F_1	✔	✔	✔	✔	✘
F_2	✔	✔	✔	✔	✘
F_3	✔	✔	✔	✔	✘
F_4	✔	✔	✔	✔	✘

Therefore,

$$
\begin{aligned}
\Delta &= 1 - (L_1 + \ldots + L_5) + (L_1 L_5 + L_2 L_5) \\
&= 1 - ((-G_2G_5) + (-G_3G_5) + (-G_1) + (1) + (G_4)) + ((-G_2G_5)(G_4) + (-G_3G_5)(G_4)) \\
&= G_2G_5 + G_3G_5 + G_1 - G_4 - G_2G_4G_5 - G_3G_4G_5.
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_1 &= \Delta|_{L_1=\cdots=L_4=0} = 1 - L_5 = 1 - G_4, \\
\Delta_2 &= \Delta|_{L_1=\cdots=L_4=0} = 1 - L_5 = 1 - G_4, \\
\Delta_3 &= \Delta|_{L_1=\cdots=L_4=0} = 1 - L_5 = 1 - G_4, \\
\Delta_4 &= \Delta|_{L_1=\cdots=L_4=0} = 1 - L_5 = 1 - G_4,
\end{aligned}
$$

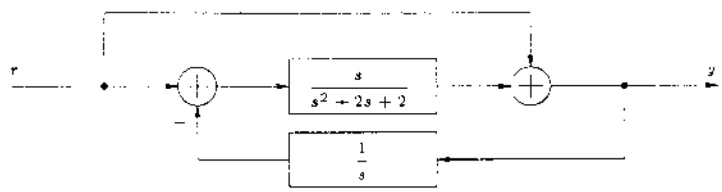
So,

$$
\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{4} F_i\Delta_i = \frac{(G_1G_2)(1-G_4) + (-G_2)(1-G_4) + (G_1G_3)(1-G_4) + (-G_3)(1-G_4)}{G_2G_5 + G_3G_5 + G_1 - G_4 - G_2G_4G_5 - G_3G_4G_5}.
$$

or

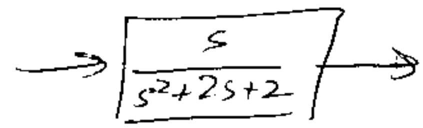
$$
\frac{Y(s)}{U(s)} = \frac{(G_1G_2 - G_2 + G_1G_3 - G_3)(1 - G_4)}{G_2G_5 + G_3G_5 + G_1 - G_4 - G_2G_4G_5 - G_3G_4G_5}.
$$

3. The block diagram of a control system is given below.

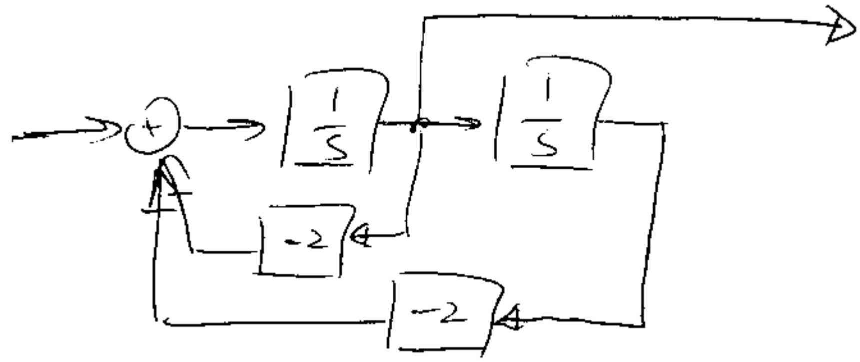


Obtain a state-space representation of the system without any block-diagram reduction.

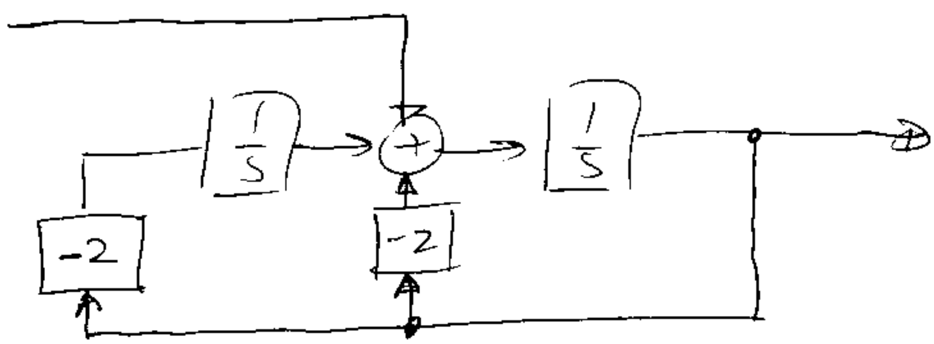
First, we need to realize



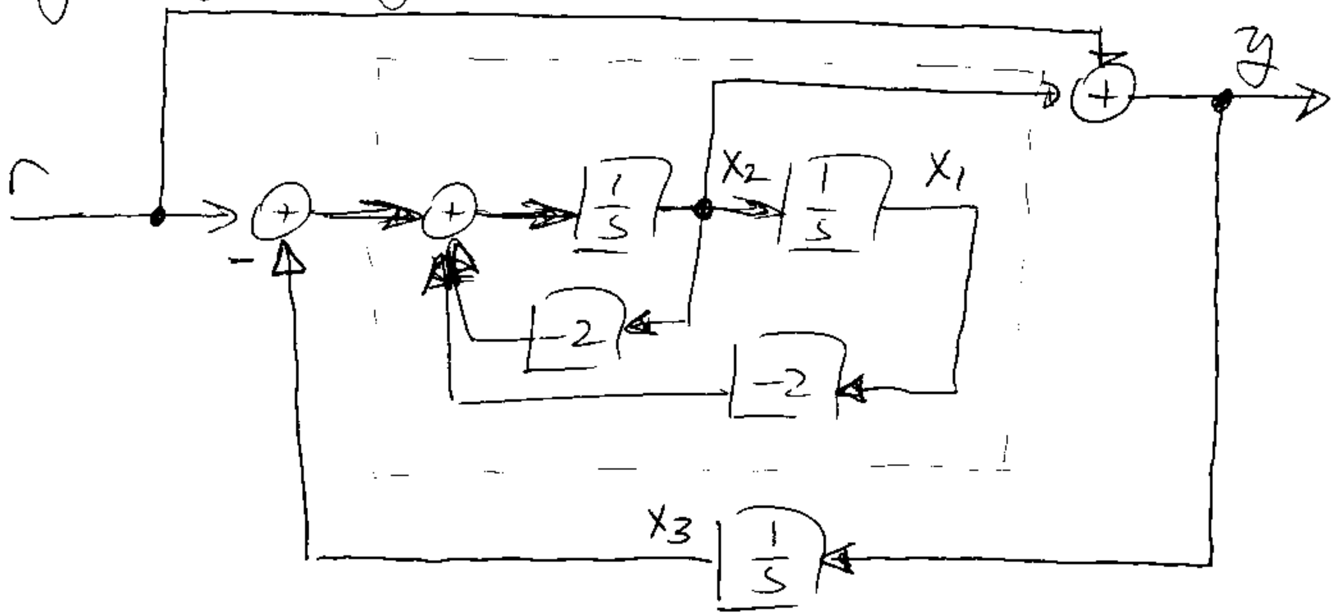
One way is



Another way is



Using the first way, we get



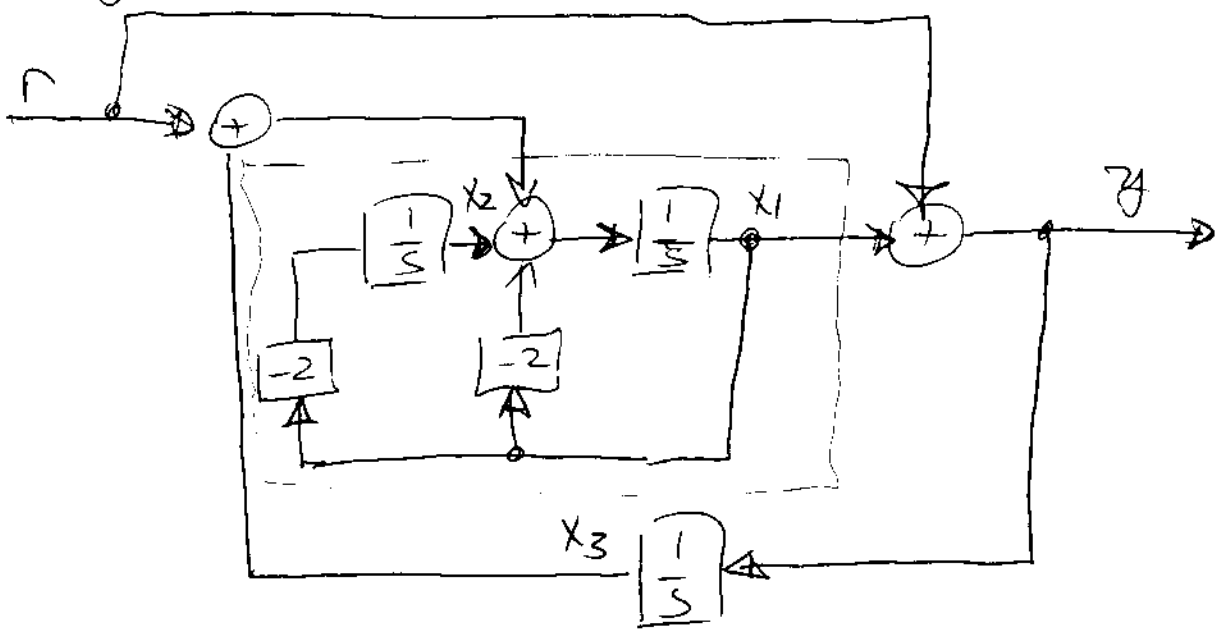
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_2 - 2x_1 + (r - x_3) \qquad y = x_2 + r$$
$$\dot{x}_3 = x_2 + r$$

So

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} r$$

Using the second way, we get



$$\dot{x}_1 = -2x_1 + x_2 + (r - x_3)$$
$$\dot{x}_2 = -2x_1$$
$$\dot{x}_3 = x_1 + r$$

$$y = x_1 + r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1] r$$

4. Consider a control system described in state space, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) + u(t).$$

(a) Determine the transfer function, $Y(s)/U(s)$, of the system.

(b) Determine $y(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and the input is the unit step, i.e. $u(t) = 1(t)$.

a//

$$(sI-A) = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix} ; \quad (sI-A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

$$= [0 \quad 1] \left(\frac{1}{(s+2)(s+1)}\right) \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1]$$

$$= \frac{1}{(s+2)(s+1)} [1 \quad s+2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1] = \frac{s^2+4s+5}{s^2+3s+2}$$

b//

$$e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \begin{bmatrix} e^{-2t} & 0 \\ -e^{-t}+e^{-2t} & e^{-t} \end{bmatrix} ; \quad e^{At}x(0) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

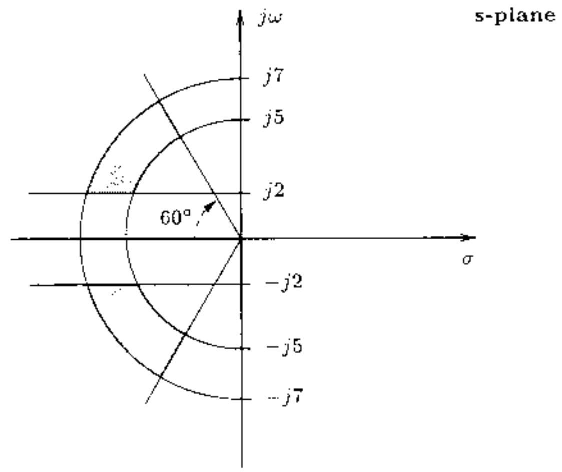
$$\int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ -e^{-(t-\tau)} + e^{-2(t-\tau)} & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

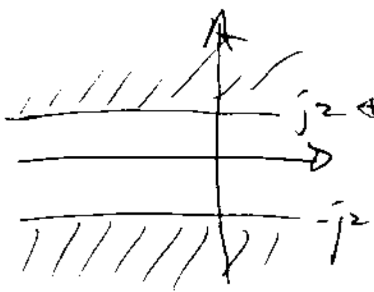
$$= \int_0^t \begin{bmatrix} e^{-2(t-\tau)} \\ e^{-2(t-\tau)} \end{bmatrix} d\tau = \left(-\tfrac{1}{2} + \tfrac{1}{2}e^{-2t}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

So $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}e^{-2t} \\ -\frac{1}{2} + e^{-t} + \frac{1}{2}e^{-2t} \end{bmatrix},$$

and $$y(t) = [0 \quad 1] x(t) + u(t) = \frac{1}{2}\left(1 + 2e^{-t} + e^{-2t}\right), \quad t \geq 0$$

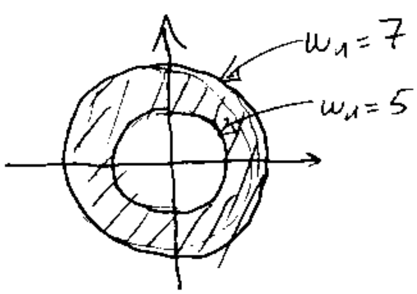
5. Obtain the necessary inequalities to describe the poles in the shaded region below in terms of only ζ and ω_n of a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 - 2\zeta\omega_n s + \omega_n^2)$.



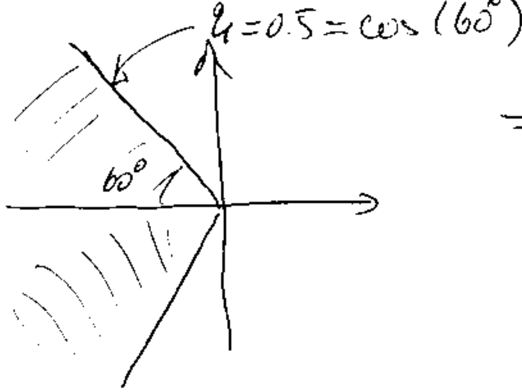


$\Rightarrow$ $\omega_d \ge 2$

or $\boxed{\sqrt{1-\zeta^2}\,\omega_n \ge 2}$



$\Rightarrow$ $\boxed{5 \le \omega_n \le 7}$



$\Rightarrow$ $\boxed{\zeta \ge 0.5}$

since $\cos^{-1}(\zeta) \le 60°$

$\zeta \ge \cos(60°)$