EE 231 **Exam#2** May 04, 2001

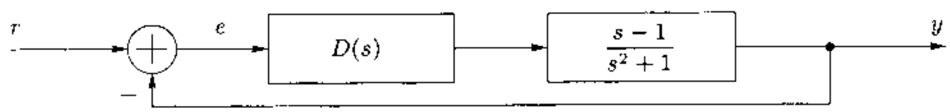
50 minutes

1. Describe and sketch the s-plane region specified by the following requirements for a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

Maximum percent overshoot	$M_p \leq 20\%$.
Peak time	$t_p \leq 2\,\text{s}$.
5% settling time	$t_{5\%s} \leq 4\,\text{s}$.

Also, determine whether any of the specifications is unnecessary or not. (15pts)

2. For the following feedback control system, determine the steady-state error $e(\infty)$ for the unit-step input.



(a) Assume $D(s) = 0.5$. (10pts)

(b) Assume $D(s) = 1$. (10pts)

(c) Assume $D(s) = 1/s$. (10pts)

3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s(s+1)}{s^4 + 4s^3 + s^2 + 5s + 4}.$$

Determine the value(s) of K such that the closed-loop system is marginally stable. (25pts)

4. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{1}{s(s^2+1)(s+1)} = K\frac{1}{s^4 + s^3 + s^2 + s}.$$

(a) Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures. (25pts)

(b) Determine all the values of K such that the closed-loop system is asymptotically stable. (05pts)

1. Describe and sketch the s-plane region specified by the following requirements for a second-order system described by $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

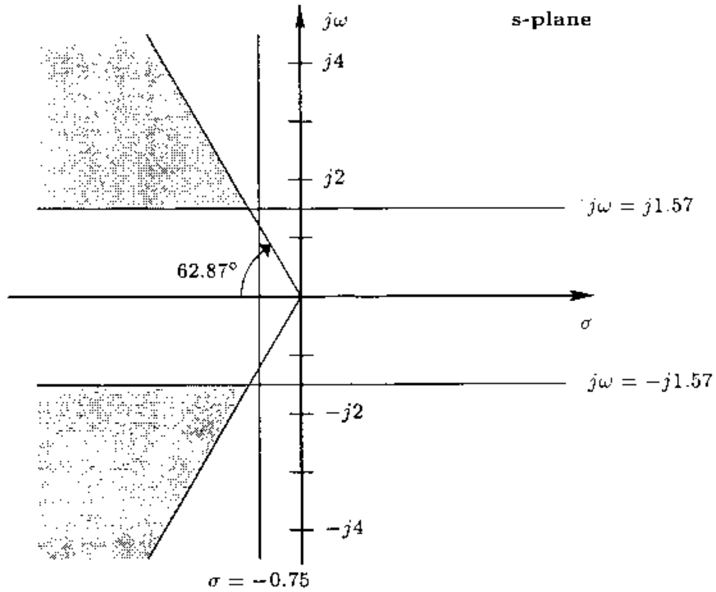
Maximum percent overshoot	$M_p \leq 20\%$.
Peak time	$t_p \leq 2\,\text{s}$.
5% settling time	$t_{5\%s} \leq 4\,\text{s}$.

Also, determine whether any of the specifications is unnecessary or not.

Solution:

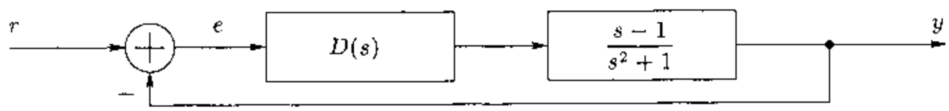
Given Specifications	System Constraints	Geometrical Representations
$M_p \leq 20\%$.	$M_p = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} \leq 0.2,$ or $\zeta \geq \dfrac{\lvert \ln(0.2) \rvert}{\sqrt{\left(\ln(0.2)\right)^2 + (\pi)^2}} \approx 0.46.$	$\alpha = \cos^{-1}(\zeta)$ $\leq \cos^{-1}(0.46) \approx 62.87°,$ where α is the angle measured from the negative real axis.
$t_p \leq 2\,\text{s}$.	$t_p = \dfrac{\pi}{\omega_d} \leq 2,$ or $\omega_d \geq \pi/2 \approx 1.57.$	$\omega \geq 1.57,$ since the poles are at $s = -\sigma_o \pm j\omega_d$
$t_{5\%s} \leq 4\,\text{s}$.	$\dfrac{3}{\sigma_o} \leq 4,$ or $\sigma_o \geq 3/4 = 0.75.$	$\sigma \leq -0.75,$ since the poles are at $s = -\sigma_o \pm j\omega_d$

The shaded region describes the region specified by the given requirements.



The restriction $\sigma \le -0.75$ could be unnecessary, if the straight line that is making 62.87° with the negative real axis intersects the $j\omega = j1.57$ line to the left of the $\sigma = -0.75$ line. That intersection is at $-1.57/\tan(62.87°) = -0.80$. So, the specification that resulted in the restriction $\sigma \le -0.75$ is unnecessary. In other words, the specification $t_{5\%s} \le 4\,\text{s}$ is unnecessary.

2. For the following feedback control system, determine the steady-state error $e(\infty)$ for the unit-step input.



(a) Assume $D(s) = 0.5$.

Solution: For $D(s) = 0.5$, the system is type 0; since there is no pole at $s = 0$ in the open-loop gain $D(s)G(s)$. As a result, the steady-state error for a unit-step input is given by

$$e(\infty) = \frac{1}{1+K_p},$$

where $K_p = \lim_{s\to 0}\big(D(s)G(s)\big)$ provided that the closed-loop system is stable. Therefore,

$$K_p = \lim_{s\to 0}\left(0.5\left(\frac{s-1}{s^2+1}\right)\right) = -0.5,$$

and

$$e(\infty) = \frac{1}{1+(-0.5)} = 2$$

provided that the closed-loop system is stable. Stability of the closed-loop system may be checked from the characteristic equation. Since the characteristic equation is

$$1 + D(s)G(s) = 0,$$

$$1 + 0.5\left(\frac{s-1}{s^2+1}\right) = 0,$$

$$s^2 + 0.5s + 0.5 = 0;$$

the poles of the closed-loop system are at $s = -0.25 \pm j0.6614$. Therefore, the system is stable, and the steady-state error $e(\infty) = 2$.

(b) Assume $D(s) = 1$.

Solution: Similarly, for $D(s) = 1$,

$$K_p = \lim_{s\to 0}\left(\frac{s-1}{s^2+1}\right) = -1,$$

and

$$e(\infty) = \frac{1}{1+(-1)}.$$

In this case, we get $|e(\infty)| = \infty$. Indeed, when we check for the stability of the closed-loop system from the characteristic equation, we get

$$1 + D(s)G(s) = 0,$$

$$1 + \left(\frac{s-1}{s^2+1}\right) = 0,$$

and

$$s(s+1) = 0.$$

The poles of the closed-loop system are at $s = 0$ and $s = -1$. The system seems to be marginally stable; but since one of the poles is at $s = 0$, and the unit-step input provides another pole at $s = 0$, the repeated pole on the imaginary axis results in an unbounded output whereas the input stays at unity.

(c) Assume $D(s) = 1/s$.

Solution: For $D(s) = 1/s$, the system is type 1, and the steady-state error for a step input is 0 provided that the closed-loop system is stable. Again, checking the characteristic equation, we get

$$1 + D(s)G(s) = 0,$$

$$1 + \left(\frac{1}{s}\right)\left(\frac{s-1}{s^2+1}\right) = 0,$$

and

$$s^3 + 2s - 1 = 0.$$

The poles of the closed-loop system are at $s = 0.4534$ and $s = -0.2267 \pm j1.4677$. Since one of the poles has a positive real part, the output will be unbounded as the input stays at unity. Therefore, $|e(\infty)| = \infty$.

3. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s(s+1)}{s^4+4s^3+s^2+5s+4}.$$

Determine the value(s) of K such that the closed-loop system is marginally stable.

Solution: The stability of the closed-loop system can be determined using the Routh-Hurwitz's stability criterion on the characteristic polynomial. From the characteristic equation, $1+G(s)=0$,

$$1+K\frac{s(s+1)}{s^4+4s^3+s^2+5s+4}=0,$$

or

$$s^4+4s^3+(K+1)s^2+(K+5)s+4=0.$$

The Routh-Hurwitz table for the system becomes as given below.

s^4	1	$K+1$	4
s^3	4	$K+5$	
s^2	$-\frac{(1)(K+5)-(4)(K+1)}{4}=(3K-1)/4$	4	
s	$-\frac{(4)(4)-((3K-1)/4)(K+5)}{(3K-1)/4}=\frac{(3K-1)(K+5)-64}{3K-1}$		
1	4		

For marginal stability, we need to choose K, such that there are distinct poles on the imaginary axis and no pole on the right-half plane. The candidates for such a choice are obtained by generating a row of zeros on the Routh-Hurwitz table. Observing from the table, the only such row is the s row. From the only element on the s row, we let

$$\frac{(3K-1)(K+5)-64}{3K-1}=0,$$

or

$$3K^2+14K-69=0.$$

The solution of the above equation gives $K=3$ and $K=-23/3$.

Next, we need to obtain the factor of the original polynomial from the previous row, and verify that we get poles on the imaginary axis. From the upper or the s^2 row,

$$\Big(\big((3K-1)/4\big)s^2+4\Big)_{K=(-23/3),3}=0.$$

Note here that the above equation gives some of the poles of the closed-loop system *only* for the values of K that make the s row all zero.

For $K=-23/3$, we get $-6s^2+4=0$, or $s=\pm\sqrt{2/3}$. In this case, we don't have imaginary axis crossings but a stable and an unstable pole combination.

For $K = 3$, we get $2s^2 + 4 = 0$, or $s = \pm j\sqrt{2}$. So, when $K = 3$, we have imaginary axis crossings at $s = \pm j\sqrt{2}$. And, from the first elements of the remaining rows of the Routh-Hurwitz table, we conclude that the rest of the poles are in the left-half plane.

Therefore, the only value of K to generate a marginally stable closed-loop system is when $K = 3$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

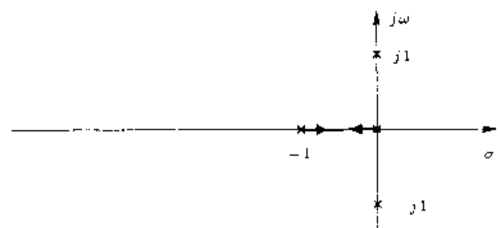
$$G(s) = K\frac{1}{s(s^2+1)(s+1)} = K\frac{1}{s^4+s^3+s^2+s}.$$

(a) Construct the root-locus diagram. Determine all the important features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, angle of arrivals and/or departures.

Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Asymptotes,
- Breakaway point,
- Imaginary-axis crossings, and
- Angle of departures.



Asymptotes

Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n-m}$

The real-axis crossing of the asymptotes is at

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{n-m} = \frac{((-1)+(0)+(j1)+(-j1))}{4-0} = \frac{-1}{4} = -0.25.$$

Real-Axis Angles: $\theta_a = \dfrac{\pm(2k+1)\pi}{n-m}$

The angles that the asymptotes make with the real axis are determined from

$$\theta_a = \frac{\pm(2k+1)\pi}{n-m} = \frac{\pm(2k+1)\pi}{4-0} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}.$$

Breakaway Point: $\dfrac{\mathrm{d}K}{\mathrm{d}s} = 0$

From the characteristic equation,

$$1 + G(s) = 0,$$

$$1 + K\frac{1}{s^4+s^3+s^2+s} = 0,$$

and

$$-K = s^4 + s^3 + s^2 + s.$$

Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = 4s^3 + 3s^2 + 2s + 1.$$

and for $\mathrm{d}K/\mathrm{d}s = 0$, the equation

$$4s^3 + 3s^2 + 2s + 1 = 0$$

gives $s = -0.6058$ and $s = -0.0721 \pm j0.6383$. So, the break-away point is $s = -0.6058$, since it is between -1 and 0..

Imaginary-Axis Crossings: Routh-Hurwitz Table

The imaginary axis crossings can be determined from the Routh-Hurwitz table. We have determined the characteristic equation above as

$$s^4 + s^3 + s^2 + s + K = 0.$$

The Routh-Hurwitz table for this characteristic equation is given below.

s^4	1	1	K
s^3	1	1	
s^2	0	K	
s			
1			

The imaginary-axis crossings will correspond to the values of K that would make a row of all zeros on the table. The first such candidate is the s^2 row. The s-row is all zero, when $K = 0$. For this value of K, we get a factor of the characteristic polynomial from the upper or the s^3-row. So,

$$\left(s^3 + s\right)_{K=0} = 0,$$

or $s = 0$ and $s = \pm j1$. Indeed, as expected both of these crossings correspond to the open-loop poles on the imaginary axis.

In order to continue with the Routh-Hurwitz table, we substitute the leading zero by ε. Then, the updated table becomes as follows.

s^4	1	1	K
s^3	1	1	
s^2	ε	K	
s	$-\frac{(1)(K) - (\varepsilon)(1)}{\varepsilon} = 1 - \frac{K}{\varepsilon}$		
1	K		

Angle of Departure: $\sum \measuredangle(\cdot) = \pm(2k+1)\pi$

The angles of departures from complex open-loop poles are determined from the angular conditions about the open-loop poles. Therefore, the angular condition about $s = j1$ is

$$-\measuredangle(s-(-1)) - \measuredangle(s-(0)) - \measuredangle(s-(-j1)) - \measuredangle(s-(j1)) = 180^\circ + k360^\circ.$$

$$-\tan^{-1}\left(\frac{(1)-(0)}{(0)-(-1)}\right) - \tan^{-1}\left(\frac{(1)-(0)}{(0)-(0)}\right) - \tan^{-1}\left(\frac{(1)-(-1)}{(0)-(0)}\right) - \theta_{\text{dep}} = 180^\circ + k360^\circ.$$

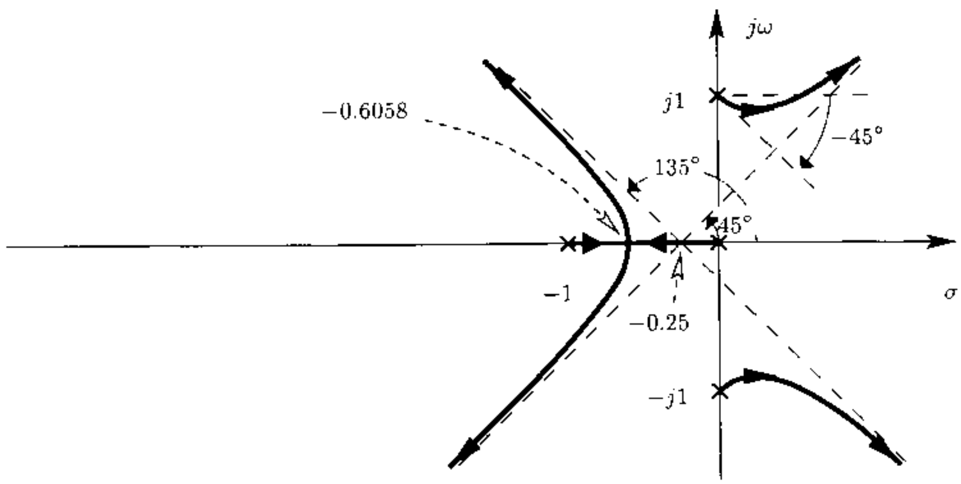
or

$$-45^\circ - 90^\circ - 90^\circ - \theta_{\text{dep}} = 180^\circ + k360^\circ.$$

As a result,

$$\theta_{\text{dep}} = -45^\circ.$$

With the features determined, we can now sketch the root-locus diagram.



(b) Determine all the values of K such that the closed-loop system is asymptotically stable.

Solution: We can determine the conditions for asymptotical stability from the first elements of the rows of the Routh-Hurwitz table.

s^4	1	1	K
s^3	1	1	
s^2	ε	K	
s	$1 - \dfrac{K}{\varepsilon}$		
1	K		

In order for asymptotical stability, we need to have all the closed-loop poles in the left-half plane. The Routh-Hurwitz criterion states that all the solutions to the polynomial that is used to generate the Routh-Hurwitz table are in the left-half plane, if and only if the first elements of the rows of the table are all positive. Since we used the characteristic polynomial to generate the table; if the first elements of the rows of the table are all positive, the closed-loop system is asymptotically stable.

From the first element of the s row, we get $\big(1 - (K/\varepsilon)\big)$, and

$$\lim_{\epsilon \to 0_+} \left(1 - \frac{K}{\varepsilon}\right) = \big(\text{sgn}(-K)\big)\infty.$$

In order for this term to be positive, we need $\text{sgn}(-K) = 1$ or $K < 0$.

From the first element of the 1 row, we get K. So, for a positive first element, we need $K > 0$.

Since the two conditions have an empty intersection, we conclude that there is no value of K that would result in an asymptotically stable system.