1. Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only $\zeta$ and $\omega_n$ of a second-order system described by $Y(s)/U(s) = \omega_n^2 / (s^2 + 2\zeta\omega_ns + \omega_n^2)$. (20pts)

![Diagram of the s-plane with shaded region](image)

2. Consider the following control system.

![Control system diagram](image)

Assume that $R_1 = 1\, \Omega$, $R_2 = 2\, \Omega$, $R_3 = 4\, \Omega$, $C = 1\, \text{F}$, and $L = 2\, \text{H}$. Only the capacitor and the inductor are sensitive to temperature changes; such that the sensitivity of the capacitance with respect to temperature $S_C^T = 5$, and the sensitivity of the inductance with respect to temperature $S_L^T = 4$. Determine the sensitivity of the transfer function $V_o(s)/V_i(s)$ with respect to the temperature. (20pts)

3. Design the simplest controller $D(s)$, such that the steady-state error for a unit-step reference input is zero, and the maximum percent overshoot is about 10%. Show all your work clearly. (15pts)
4. A PID controller is to be designed for the following control system.

(a) Determine the requirements for the PID constants $K_P$, $K_I$, and $K_D$, such that the closed-loop system is asymptotically stable. 

(b) Determine whether or not the values $K_P = 1$, $K_I = 0$, and $K_D = 1$ result in an asymptotically-stable system. 

(15pts) 

(05pts) 

5. Consider a negative unity-feedback control system with the open-loop transfer function

$$G(s) = K \frac{s + 1}{(s - 1)(s^2 + 4s + 16)} = K \frac{s + 1}{s^4 + 3s^3 + 12s^2 - 16s}.$$ 

(a) Determine the values of $K$ such that the closed-loop system is asymptotically stable. 

(b) Determine the value (or values) of $K$ and the natural frequency (or frequencies), such that the closed-loop system would have sustained oscillations. 

(15pts) 

(10pts)
1. Obtain the necessary inequalities to describe the strictly complex poles in the shaded region below in terms of only \( \zeta \) and \( \omega_n \) of a second-order system described by \( Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2) \).

Solution: To be able to describe the shaded region, we need to separate it into unions or intersections of simpler regions.

The equi-distance points from the origin designate constant value for \( \omega_n \). As a result, the shown shaded area is represented by

\[ 10 \leq \omega_n \leq 20. \]
A straight line originating from the origin designates a constant $\zeta$ value, where $\cos^{-1}(\zeta)$ is the acute angle between the line and the negative real axis. So for the shaded area shown, we have

$$30^\circ \leq \cos^{-1}(\zeta) \leq 45^\circ,$$

or

$$\cos(30^\circ) \geq \zeta \geq \cos(45^\circ).$$

since $\cos(\theta)$ is a monotonically decreasing function for $0 < \theta < 180^\circ$. So, we have

$$\frac{\sqrt{2}}{2} \leq \zeta \leq \frac{\sqrt{3}}{2}.$$

Therefore, the shaded area given in the problem is the intersection of the individual shaded areas. and it can be represented by

$$10 \leq \omega_n \leq 20,$$

$$\frac{\sqrt{2}}{2} \leq \zeta \leq \frac{\sqrt{3}}{2}.$$
Ideal opamp \[ \frac{v_2'}{v_i'} = - \frac{R_2 + 1/C}{R_1} \]

Voltage division \[ \frac{v_0}{v_2} = \frac{L_s}{R_3 + L_s} \]

\[ \frac{v_0}{v_i} = \frac{v_0}{v_2}, \quad \frac{v_2}{v_i} = - \left( \frac{R_2}{R_1} + \frac{1}{R_1 C, s} \right) \left( \frac{L_s}{L_s + R_3} \right) \]

\[ = - \frac{R_2 C s + 1}{R_1 C s} \cdot \frac{L_s}{L_s + R_3} = - \frac{L}{R_1 C} \frac{R_2 C s + 1}{L_s + R_3} \]

\[ = - \frac{R_2/r_1 s + 1/r_1 C}{s + R_3/L} \]

Sensitivity \[ S_{\text{vo/vi}} = S_{c} \frac{v_0/v_i}{s} + S_{L} \frac{v_0/v_i}{s} \]
\[
S_{\frac{V_{o}}{V_{i}}} = \frac{c}{\frac{V_{o}}{V_{i}}} = \frac{\partial (\frac{V_{o}}{V_{i}})}{\partial c}
\]

where

\[
\frac{\partial (\frac{V_{o}}{V_{i}})}{\partial c} = - \frac{(- \frac{1}{R_{1}C})}{S + \frac{R_{3}}{L}} = \frac{1}{R_{1}C} \frac{1}{S + \frac{R_{3}}{L}}
\]

\[
=\frac{1}{CR_{2}} \frac{1}{S + 1}
\]

\[
S_{\frac{V_{o}}{V_{i}}} = \frac{L}{\frac{V_{o}}{V_{i}}} = \frac{\partial (\frac{V_{o}}{V_{i}})}{\partial L}
\]

where

\[
\frac{\partial (\frac{V_{o}}{V_{i}})}{\partial L} = - \frac{(- \frac{R_{2}}{R_{1}} S + \frac{1}{R_{1}C}) \left(- \frac{R_{3}}{L^2}\right)}{(S + \frac{R_{3}}{L})^2}
\]

\[
\Rightarrow S_{\frac{V_{o}}{V_{i}}} = \left(\frac{L}{\frac{R_{2}}{R_{1}} S + \frac{1}{R_{1}C}}\right) \left(- \frac{R_{3}}{L^2}\right) \left(\frac{1}{S + \frac{R_{3}}{L}}\right)
\]
\[
\begin{align*}
S_L^{\text{ref}} &= \frac{R_3}{L} \frac{1}{s + \frac{R_3}{L}} = \frac{1}{\frac{1}{R_3} s + 1} \\
\Rightarrow \quad S_T^{\text{v/c}} &= -\frac{1}{C R_2 s + 1} S_T^c + \frac{1}{\frac{1}{R_3} s + 1} S_T^L \\
&= -\frac{1}{2s + 1} (5) + \frac{1}{2/4 s + 1} (4) \\
&= -\frac{5}{2s + 1} + \frac{8}{s + 2} = \frac{11s - 2}{(2s + 1)(s + 2)}
\end{align*}
\]
$W_p = 10\% = 0.1 = e^{-\frac{a}{\sqrt{1-a^2}}} \pi$

$\Rightarrow \ \eta = \frac{\ln(W_p)}{\sqrt{\pi^2 + \ln^2(W_p)}} = 0.59$

$e_{ss} = 0 \ \text{for a step input} \Rightarrow \ D(s) = \frac{1}{s} D'(s)$

Simplest $D'(s) = K$

$\Rightarrow \ 1 + D(s) G(s) = 0$

$1 + \frac{K}{s} \ \frac{1}{s+4} = 0$

$s(s+4) + K = 0$

$s^2 + 4s + K = 0$

For a second-order system with poles at

$s = -\frac{\zeta \omega_n \pm \sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} \omega_n$

the characteristic equation is

$s^2 + 2\zeta \omega_n s + \omega_n^2 = 0$
Comparing the terms in
\[ s^2 + 4s + K = 0 \]
and
\[ s^2 + 2\gamma w_n s + w_n^2 = 0 \]
we get
\[ 2\gamma w_n = 4 \implies w_n = \frac{4}{2\gamma} = \frac{4}{2(0.59)} = 3.38 \]
\[ w_n^2 = K \]
so if \( K = w_n^2 = (3.38)^2 \), then we satisfy all the constraints.

\[ K = (3.38)^2 = 11.49 \]

\[ D(r) = \frac{11.49}{s} \]
The characteristic equation is

\[ 1 + D(s) G(s) = 0 \]

\[ \frac{K_p s + K_i}{s} + K_0 s^2 = \frac{1}{s(s+1)(s+2)} = 0 \]

\[ s^2 (s+1)(s+2) + K_p s + K_i + K_0 s^2 = 0 \]

\[ s^4 + 3 s^3 + (K_0 + 2) s^2 + K_p s + K_i = 0 \]

\[ \begin{array}{c|cccc}
    s^4 & 1 & K_0 + 2 & \hline \\
    s^3 & 3 & K_p & \hline \\
    s^2 & 3K_0 + 6 - K_p & 3K_i & \hline \\
    s & (3K_0 + 6 - K_p)K_p - 9K_i & 3K_0 + 6 - K_p & \hline \\
    1 & 3K_i & \hline \\
\end{array} \]
For asymptotical stability

\[ 3K_0 + 6 - K_p > 0 \]

\[ (3 K_0 + 6 - K_p) K_p - 9 K_I > 0 \]

\[ K_I > 0 \]

\[ b \quad \text{For} \quad K_p = 1, \quad K_I = 0, \quad K_0 = 1 \]

\[ 3(1) + 6 - (1) > 0 \]

\[ 8 > 0 \quad \checkmark \]

\[ (3(1) + 6 - (1))(1) - 9(0) > 0 \]

\[ 8 > 0 \quad \checkmark \]

\[ c \quad 0 > 0 \quad \times \quad \text{(Actually, if} \quad K_I = 0 \text{, the order of the system will be reduced.)} \]
For the reduced order system, we get

\[ \frac{1}{s(s+1)(s+2)} = 0 \]

\[ s(s+1)(s+2) + K_0 + K_0 s = 0 \]

\[ s^3 + 3s^2 + (K_0 + 2)s + K_0 = 0 \]

R-H Table

<table>
<thead>
<tr>
<th>( s^3 )</th>
<th>1</th>
<th>( K_0 + 2 )</th>
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<tbody>
<tr>
<td>( s^2 )</td>
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<td>( K_0 )</td>
</tr>
<tr>
<td>( s )</td>
<td>( \frac{3(K_0 + 2) - K_0}{3} )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>( K_0 )</td>
</tr>
</tbody>
</table>
For stability:

(i) $3(K_0+2)-K_0 > 0$
\[ 3((1)+2)-(1) > 0 \checkmark \]

(ii) $K_0 > 0$
\[ (1) > 0 \checkmark \]

So system is asymptotically stable.
The char. eqn. is

\[
1 + K \frac{s+1}{s^4+3s^3+12s^2-16s} = 0
\]

\[
s^4+3s^3+12s^2-16s + Ks + K = 0
\]

**R - H Table**

<table>
<thead>
<tr>
<th>(s^4)</th>
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<th>12</th>
<th>K</th>
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<tr>
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</tr>
<tr>
<td>(s^2)</td>
<td>(\frac{52-16}{3})</td>
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</tr>
<tr>
<td>(s)</td>
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<td>3K</td>
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</tr>
<tr>
<td>1</td>
<td>3K</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conditions

(i) \( K < 52 \)

(ii) \[-K^2 + 59K - 832 > 0\]

\[-(K - 23.32)(K - 35.68) = 0\]

\[
\begin{align*}
23.32 &< K < 35.68
\end{align*}
\]

(iii) \( K > 0 \)

All three conditions are satisfied if \( 23.32 < K < 35.68 \).
(b) Setting a row of zero
(the s-term row)

\[-K^2 + 591K - 832 = 0\]

\[-(K-23.32)(K-35.68) = 0\]

When \[K = 23.32\] \[\Rightarrow (52-K)s^2 + 3K = 0\]
\[\Rightarrow 28.68s^2 + 69.96 = 0\]
\[s = \pm \sqrt{1.5618}\]
\[\Rightarrow \omega = 1.5618\]

When \[K = 35.68\] \[\Rightarrow (52-K)s^2 + 3K = 0\]
\[\Rightarrow 16.32s^2 + 107.04 = 0\]
\[s = \pm 2.5610\]
\[\Rightarrow \omega = 2.5610\]