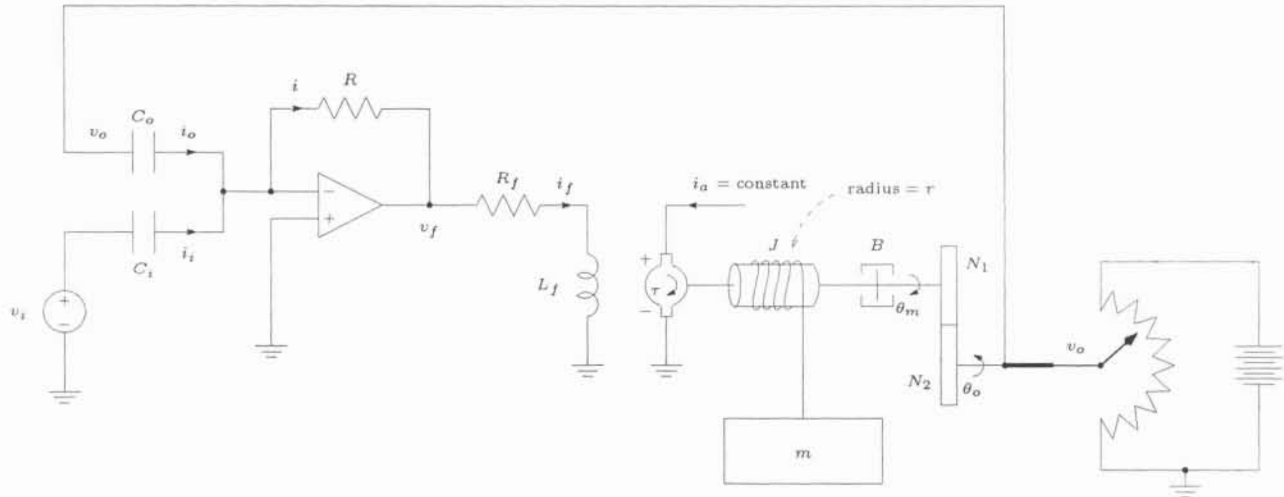
EE 231 | Exam#1 | Oct. 19, 2006

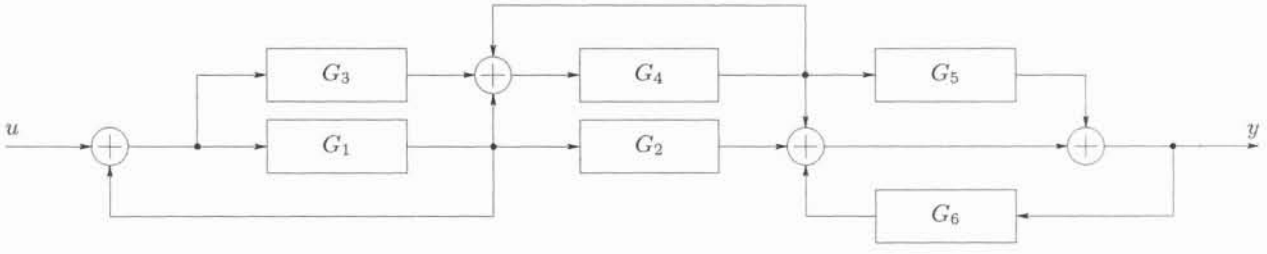
75 minutes

1. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, θ_m, and θ_o on the block diagram. (25pts)

2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly. (25pts)



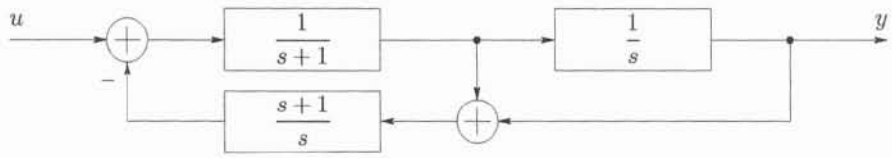
3. A nonlinear system with input u and output y has the following nonlinear dynamics.

$$\dot{y}(t) = -a\left(y^2(t) + 1\right) + u(t).$$

(a) Obtain a linear approximation of the system dynamics about $y = 1$. (05pts)

(b) Determine the value of a such that the linearized system about $y = 1$ would have a 2% settling time of 2 s. (05pts)

4. The block diagram of a control system is given below.



Obtain a state-space representation of the system without any block-diagram reduction. (25pts)

5. The state-space equations of a control system are given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

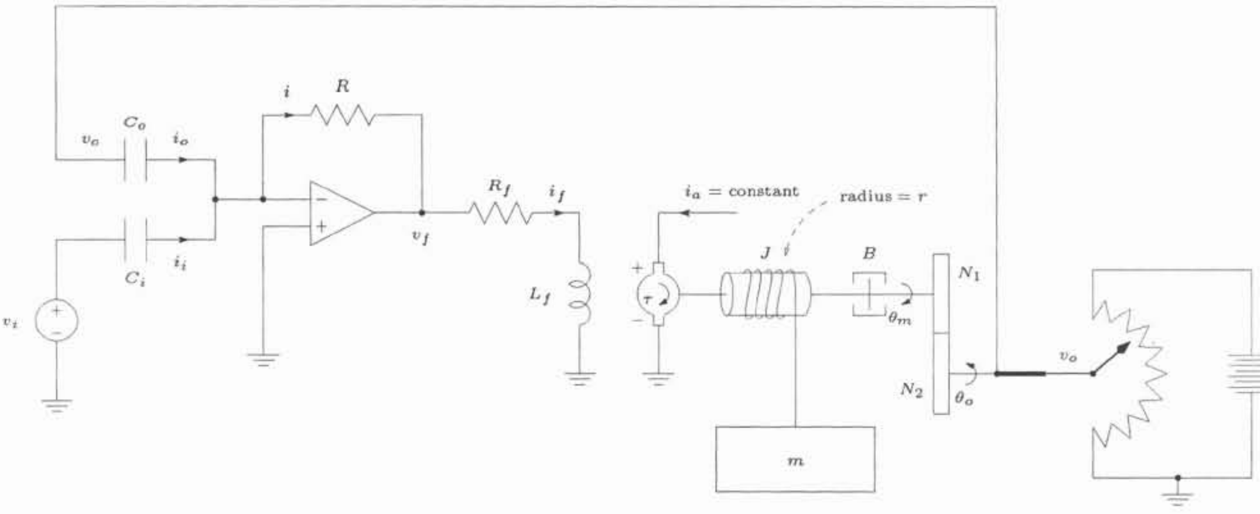
(a) Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$. (10pts)

(b) Determine the transfer function of the system. (05pts)

EE 231 Exam#1 Solutions Oct. 19, 2006

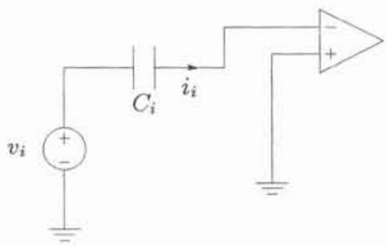
Exam#1 Solutions

1. The angular position of the shaft of a motor is controlled by the system shown below.



The angular position of the motor shaft is detected by a variable resistor which provides a voltage v_o proportional to the angle, such that $v_o = K_o\theta_o$. Draw the most detailed block diagram of the system, where v_i is the input, and θ_o is the output. Show all the variables v_i, i_i, v_o, i_o, i, v_f, i_f, τ, θ_m, and θ_o on the block diagram.

Solution: To determine the block diagram of the system, we first separate it into simpler components.



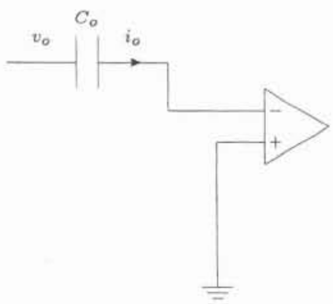
Because the input variable is v_i, we write i_i in terms v_i, such that

$$i_i(t) = C_i\frac{\mathrm{d}v_i(t)}{\mathrm{d}t},$$

or

$$I_i(s) = C_i s V_i(s),$$

since the operational amplifier is assumed to be ideal.

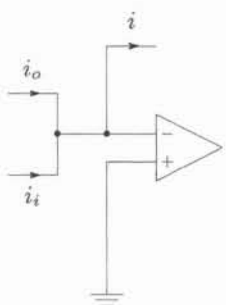


Similarly, we have

$$i_o(t) = C_o \frac{\mathrm{d}v_o(t)}{\mathrm{d}t},$$

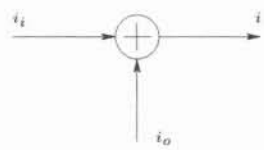
or

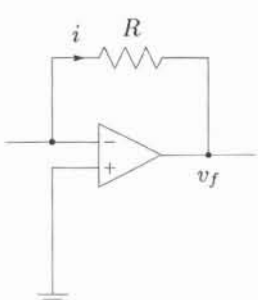
$$I_o(s) = C_o s V_o(s).$$



For an ideal operational amplifier,

$$i(t) = i_i(t) + i_o(t).$$



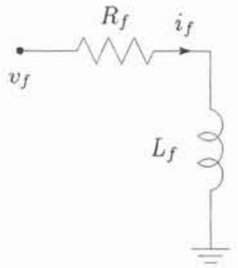


Again for an ideal operational amplifier,

$$v_f(t) = -Ri(t),$$

or

$$V_f(s) = -RI(s).$$

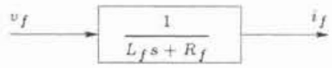


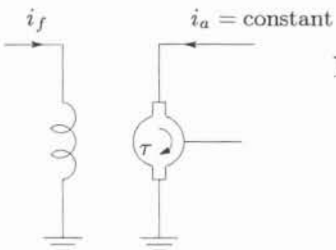
The field current of the motor can be obtained from the Kirchhoff's Voltage Law, where

$$L_f \frac{\mathrm{d}i_f(t)}{\mathrm{d}t} + R_f i_f(t) = v_f(t),$$

or

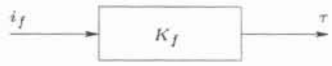
$$I_f(s) = \frac{1}{L_f s + R_f} V_f(s).$$

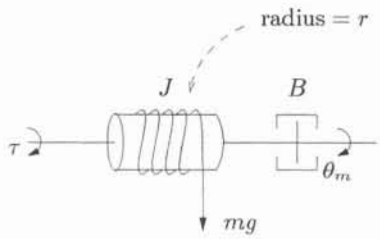




From the field controlled motor,

$$\tau(t) = K_f i_f(t).$$



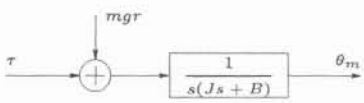


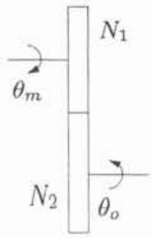
The torque equation is

$$J \frac{\mathrm{d}^2\theta_m(t)}{\mathrm{d}t^2} = \tau(t) - B\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + (mg)r,$$

or

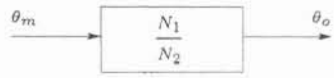
$$\Theta_m(s) = \frac{1}{s(Js+B)}\big(T(s) + mgr\big).$$





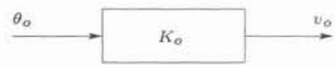
The gear relates the two angle values, such that

$$\theta_o(t) = \frac{N_1}{N_2}\theta_m(t).$$

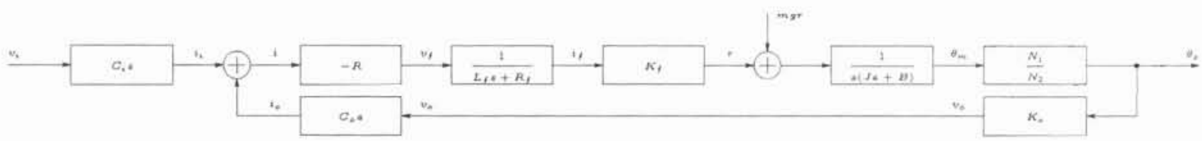


And, finally the given relationship

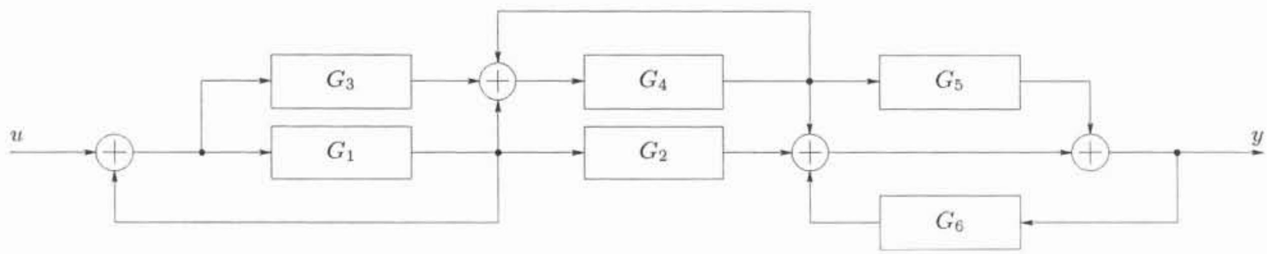
$$v_o(t) = K_o\theta_o(t).$$



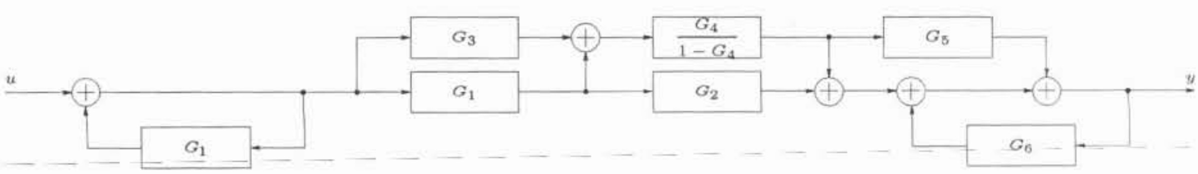
When we connect all the individual blocks together, we get the following block diagram.

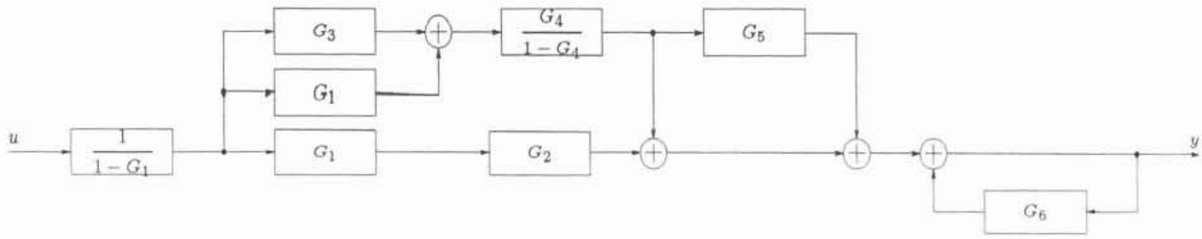


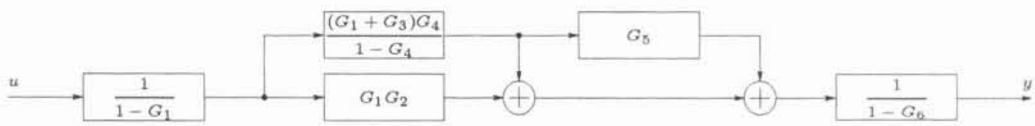
2. For the block diagram given below, determine the transfer function *either* by block-diagram reduction *or* by Mason's formula. Show your work clearly.

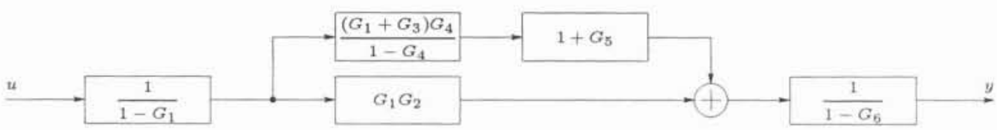


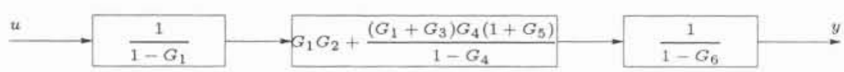
Solution: If we choose to use the block-diagram reduction, best approach is to reduce the block diagram step by step, until we obtain the transfer function.

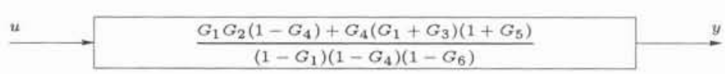




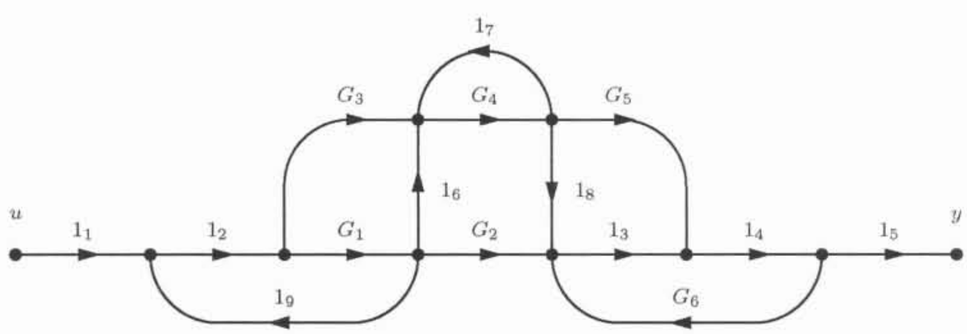








If we choose to use Mason's formula, we need to draw the signal flow graph of the block diagram.



In drawing the signal flow graph, the unity gains are subscribed for easy tracking of the gain expressions. The forward path gains are

$$F_1 = 1_1 1_2 G_1 G_2 1_3 1_4 1_5 = G_1 G_2,$$
$$F_2 = 1_1 1_2 G_3 G_4 1_8 1_3 1_4 1_5 = G_3 G_4,$$
$$F_3 = 1_1 1_2 G_3 G_4 G_5 1_4 1_5 = G_3 G_4 G_5,$$
$$F_4 = 1_1 1_2 G_1 1_6 G_4 1_8 1_3 1_4 1_5 = G_1 G_4,$$
$$F_5 = 1_1 1_2 G_1 1_6 G_4 G_5 1_4 1_5 = G_1 G_4 G_5.$$

The loop gains are

$$L_1 = 1_2 G_1 1_9 = G_1,$$
$$L_2 = G_4 1_7 = G_4,$$
$$L_3 = 1_3 1_4 G_6 = G_6.$$

From the forward path and the loop gains, we determine the touching loops and the forward paths.

Touching Loops

	L_1	L_2	L_3
L_1	✔	✘	✘
L_2		✔	✘
L_3			✔

Loops on Forward Paths

	L_1	L_2	L_3
F_1	✔	✘	✔
F_2	✔	✔	✔
F_3	✔	✔	✔
F_4	✔	✔	✔
F_5	✔	✔	✔

Therefore,

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2 + L_3) + (L_1L_2 + L_1L_3 + L_2L_3) - (L_1L_2L_3)\\ &= 1 - ((G_1) + (G_4) + (G_6)) + ((G_1)(G_4) + (G_1)(G_6) + (G_4)(G_6)) - ((G_1)(G_4)(G_6))\\ &= 1 - G_1 - G_4 - G_6 + G_1G_4 + G_1G_6 + G_4G_6 - G_1G_4G_6,\end{aligned}$$

and

$$\Delta_1 = \Delta|_{L_1=L_3=0} = 1 - L_2 = 1 - G_4,$$
$$\Delta_2 = \Delta|_{L_1=L_2=L_3=0} = 1,$$
$$\Delta_3 = \Delta|_{L_1=L_2=L_3=0} = 1,$$
$$\Delta_4 = \Delta|_{L_1=L_2=L_3=0} = 1,$$
$$\Delta_5 = \Delta|_{L_1=L_2=L_3=0} = 1.$$

So,

$$\frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_{i=1}^{5} F_i\Delta_i = \frac{(G_1G_2)(1-G_4) + (G_3G_4)(1) + (G_3G_4G_5)(1) + (G_1G_4)(1) + (G_1G_4G_5)(1)}{1 - G_1 - G_4 - G_6 + G_1G_4 + G_1G_6 + G_4G_6 - G_1G_4G_6},$$

or

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2(1-G_4) + G_3G_4 + G_3G_4G_5 + G_1G_4 + G_1G_4G_5}{1 - G_1 - G_4 - G_6 + G_1G_4 + G_1G_6 + G_4G_6 - G_1G_4G_6}.$$

3. A nonlinear system with input u and output y has the following nonlinear dynamics.

$$\dot{y}(t) = -a\left(y^2(t) + 1\right) + u(t).$$

(a) Obtain a linear approximation of the system dynamics about $y = 1$.

Solution: The nonlinearity of the system originates from the $-a(y^2+1)$ term in the dynamical equation. Using the taylor series expansion about $y = 1$, we get

$$\begin{aligned} -a\left(y^2+1\right) &= \left[-a\left(y^2+1\right)\right]_{y=1} + \left[\frac{\mathrm{d}}{\mathrm{d}y}\left(-a\left(y^2+1\right)\right)\right]_{y=1}(y-1) + \mathcal{O}(y^2) \\ &= (-2a) + (-2a)(y-1) + \mathcal{O}(y^2) \\ &\approx -2ay. \end{aligned}$$

As a result, the linear approximation of the system dynamics is

$$\dot{y}(t) = -2ay(t) + u(t).$$

(b) Determine the value of a such that the linearized system about $y = 1$ would have a 2% settling time of 2 s.

Solution: The linearized system about $y = 1$ has the dynamical equation

$$\dot{y}(t) = -2ay(t) + u(t).$$

Taking the laplace transform of the differential equation under zero initial condition, we can obtain the transfer function

$$sY(s) = -2aY(s) + U(s),$$

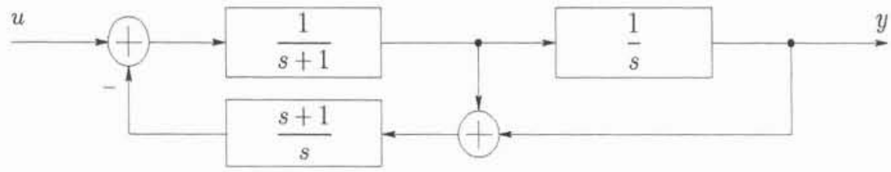
$$\frac{Y(s)}{U(s)} = \frac{1}{s+2a},$$

where $U = \mathfrak{L}[\,u\,]$ and $Y = \mathfrak{L}[\,y\,]$. Since the 2% settling time of a first-order system is $t_{2\%s} = 4T$ for a time constant T, we get

$$t_{2\%s} = 4T = 4\big(1/(2a)\big) = 2,$$

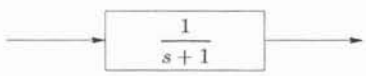
or $a = 1$.

4. The block diagram of a control system is given below.

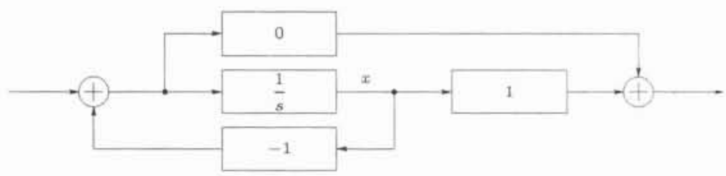


Obtain a state-space representation of the system without any block-diagram reduction.

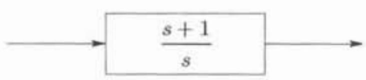
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



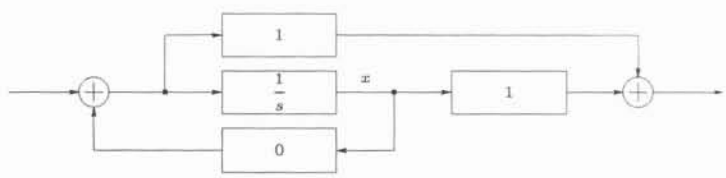
(a) The feedforward gain block.



(b) Controller realization form.

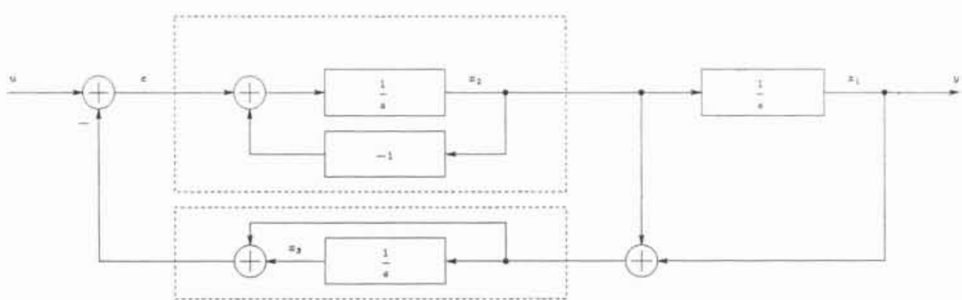


(c) The feedback gain block.



(d) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$\dot{x}_1 = \dot{x}_2,$$
$$\dot{x}_2 = -x_2 + e,$$
$$\dot{x}_3 = x_1 + x_2,$$

and

$$y = x_1.$$

We also have

$$e = u - (x_3 + \dot{x}_3) = -x_1 - x_2 - x_3 + u.$$

After substituting the expression for e into the original set, we obtain the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

In this case, the observer realization form gives the same state-space representation.

5. The state-space equations of a control system are given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine $y(t)$ for $t \geq 0$; when $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $u(t) = 0$ for $t \geq 0$.

Solution: The general solution to the state-space representation of a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

is obtained from

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)} B\boldsymbol{u}(\tau)\, \mathrm{d}\tau,$$

where

$$e^{At} = \mathfrak{L}_s^{-1}\left[\, (sI - A)^{-1} \,\right](t).$$

Here, I is the appropriately dimensioned identity matrix. In our case,

$$A = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix},$$

and $u(t) = 0$ for $t \geq 0$. As a result, the integral term in the solution of $\boldsymbol{x}$ is identically zero. So,

$$
\begin{aligned}
y(t) &= Ce^{At}\boldsymbol{x}(0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}_s^{-1}\left[(sI - A)^{-1} \right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}_s^{-1}\left[\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix} \right)^{-1} \right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}_s^{-1}\left[\left(\begin{bmatrix} s+2 & 2 \\ -1 & s+5 \end{bmatrix} \right)^{-1} \right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathcal{L}_s^{-1}\left[\tfrac{1}{(s+2)(s+5)-(1)(-2)} \begin{bmatrix} s+5 & -2 \\ 1 & s+2 \end{bmatrix} \right](t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{s+5}{s^2+7s+12} \right](t) & \mathcal{L}_s^{-1}\left[\frac{-2}{s^2+7s+12} \right](t) \\ \mathcal{L}_s^{-1}\left[\frac{1}{s^2+7s+12} \right](t) & \mathcal{L}_s^{-1}\left[\frac{s+2}{s^2+7s+12} \right](t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}_s^{-1}\left[\frac{s+5}{s^2+7s+12} \right](t) & \mathcal{L}_s^{-1}\left[\frac{-2}{s^2+7s+12} \right](t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \mathcal{L}_s^{-1}\left[\tfrac{-2}{s^2+7s+12} \right](t) = \mathcal{L}_s^{-1}\left[\tfrac{-2}{s+3} + \tfrac{2}{s+4} \right](t).
\end{aligned}
$$

Or,

$$y(t) = -2e^{-3t} + 2e^{-4t} \text{ for } t \geq 0.$$

(b) Determine the transfer function of the system.

Solution: The transfer function of a control system described in the state-state representation

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \\
\boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t),
\end{aligned}
$$

is

$$F(s) = C(sI - A)^{-1}B + D;$$

where in our case

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

Here, I is the appropriately dimensioned identity matrix.

Therefore,

$$F(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 2 \\ -1 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= \frac{1}{(s+3)(s+4)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & -2 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= \frac{1}{(s+3)(s+4)} \begin{bmatrix} s+5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= \frac{1}{(s+3)(s+4)} \begin{bmatrix} s+3 \end{bmatrix} + 1 = \frac{1}{s+4} + 1.$$

In other words, the transfer function is $F(s) = (s+5)/(s+4)$.