1. Consider the following feedback control systems.

\[ u(t) \xrightarrow{+} D(s) \xrightarrow{\frac{s^2 + 4s + 8}{(s-3)(s-1)}} y(t) \]

Note: This problem might require hand construction of some diagrams. In this case, the inaccuracies in the closed-loop pole locations are acceptable.

(a) Design a proportional (P) controller, such that the normalized maximum overshoot is \( M_p \approx 0.25 \). (10pts)

(b) Design an integral (I) controller, such that the steady-state error for the unit ramp input is at most 0.1. (5pts)

(c) Design the simplest controller, such that the 2% settling time is at most 4 seconds. After the design, determine the normalized maximum overshoot. (10pts)

2. For the following open-loop pole/zero locations, sketch expected root-locus diagrams. Do not determine any features of the diagram, except the asymptote angles. Simply show the expected shapes of all the root-locus branches. Make reasonable assumptions for all the undetermined features. (20pts)

\[ (a) \quad (b) \quad (c) \quad (d) \]
3. Consider a unity-feedback control system with the open-loop transfer function

\[ G(s) = K \frac{1}{s(s + 2)(s^2 + 6s + 13)} = K \frac{s^2 + 3s + 1}{s^4 + 8s^3 + 25s^2 + 26s} \]

(a) Construct the root-locus diagram. Determine all the important features like asymptotes, imaginary-axis crossings, angle of arrivals and departures; however do not determine the break-away and/or break-in points explicitly. In other words, obtain the equation whose solutions would give those points, but do not solve that equation. (30pts)

(b) Determine all the values of \( K \) such that the closed-loop system is asymptotically stable. (5pts)

4. For the following system, design a first order compensator \( D(s) \), without increasing the order of the system, such that the steady-state error is zero for a step input, and the 5% settling time is approximately 0.5 second. (20pts)
Proportional Control \( \Rightarrow D(s) = K \)

\[ G_H = K \frac{s^2 + 4s + 8}{(s-3)(s+1)} \]

\[ s = -2 \pm j2 \]

\[ W_p \approx 0.25 \]

\[ \theta = \frac{\pi}{1 - 4^2} = 0.25 \]

\[ \theta \approx 0.25 \]

\[ \theta = \frac{1 + 0.25}{\sqrt{1 + 0.25} + 0.25^2} \approx 0.4 \]

\[ \cos^{-1} = \cos^{-1} 0.4 = 66.19^\circ \]

Need to find out the imaginary and opening and the angle of final and the break-away point.

(For the purposes of the exam, it was O.K. to sketch the root-locus without finding the above points.)
Imaginary axis crossing $\Rightarrow$ Routh Table

\[
1 + G(s) = 0 \\
1 + K \frac{s^2 + 4s + 8}{(s-3)(s-1)} = 0 \\
(s^2 - 4s + 3) + K(s^2 + 4s + 8) = 0 \\
(K+1)s^2 + (4K-4)s + (8K+3) = 0
\]

\[
\begin{array}{c|cc}
s^2 & K+1 & 8K+3 \\
s & 4K-4 & 4K-4 = 0, K = 1 \\
1 & 8K+3 &
\end{array}
\]

\[
2s^2 + 1 = 0 \\
s = \pm j23452
\]

Angle of arrival $\Rightarrow$ Angular condition at $s_0 = -2+j2$

\[-\Delta (s_0-1) - \Delta (s_0-3) + \Delta (s_0+2+j2) + \theta_{in} = 180^0 + k360^0 \]

\[-\tan^{-1} \frac{2-0}{-2-1} - \tan^{-1} \frac{2-0}{-2-3} + 90^0 + \theta_{in} = 180^0 + k360^0 \]

\[-146.31^0 - 158.20^0 + 90^0 + \theta_{in} = 180^0 + k360^0 \]

\[\theta_{in} = 34.51^0 \]
Break-away point \( \Rightarrow \frac{dK}{ds} = 0 \)

\[
1 + q^2 = 0 \\
1 + k \frac{s^2 + 4s + 8}{(s-3)(s-1)} = 0 \\
- k = \frac{s^2 - 4s + 3}{s^2 + 4s + 8}
\]

\[
\frac{dK}{ds} = \frac{(2s - 4)(s^2 + 4s + 8) - (s^2 - 4s + 3)(2s + 4)}{(s^2 + 4s + 8)^2}
\]

\[
\frac{dK}{ds} = 0, \quad 8s^2 + 10s - 44 = 0 \\
3 = -5 \pm \sqrt{377} = 1.8021, -3.0121
\]

Desired location

\( s_d \approx -1 + j2.3 \)

approx. read from Nyquist graph

To find \( k \) from \( s_d \), use the magnitude condition

\[
|G(s)| = 1, \quad \left| k \frac{s^2 + 4s + 8}{(s-3)(s-1)} \right| = 1 \Rightarrow k = 3.05
\]

\( s = s_d \)
by Integral Control \( \Rightarrow \frac{L(s)}{D(s)} = \frac{K}{s} \)

\[ G(s) = K \frac{s^2 + 4s + 8}{s(s-3)(s+1)} \]

\[ \text{different mass configuration, break-away point, angle of arrival, but the shape is similar.} \]

\[ \varepsilon_{ss} = \frac{1}{K_v} \quad \text{where } K_v = \lim_{s \to s_{\infty}} sG(s) \quad \text{for unit ramp} \]

\[ K_v = \lim_{s \to s_{\infty}} sK \frac{s^2 + 4s + 8}{s(s-3)(s+1)} = K \frac{8}{(-3x-1)} = \frac{8}{3}K \]

\[ \varepsilon_{ss} = \frac{3}{8K} \]

\[ \varepsilon_{ss} \leq 0.1 \Rightarrow \frac{3}{8K} \leq 0.1 \Rightarrow K \geq 3.75 \]

Need to check stability \( \Rightarrow \) use Routh-Hurwitz criterion

\[ 1 + \theta_H = 0 \]

\[ 1 + K \frac{s^2 + 4s + 8}{s(s-3)(s+1)} = 0 \]

\[ s^3 + (k-4)s^2 + (4k+3)s + 8K = 0 \]
\[
\begin{array}{c|cc}
S^3 & 1 & 4K+3 \\
S^2 & K-4 & 8K \\
S & (4K+3) - \frac{8K}{K-4} \\
1 & 8K \\
\end{array}
\]

Stability \iff K-4 > 0 \ , \ K > 4

\[
4K+3 - \frac{8K}{K-4} > 0 , \quad (K-4)(4K+3) - 8K > 0 \\
4K^2 - 21K - 12 > 0
\]

\[
\therefore K = -0.52 \\
\quad K = 5.77
\]

\[
\begin{array}{c|c|c}
eg n > 0 & \neg n < 0 & \neg n > 0 \\
\hline
-0.52 & \text{STAB} & \text{D unstab} \\
\end{array}
\]

So \ K < -0.52 \\
\text{or} \ K > 5.77

\[
8K > 0 , \quad K > 0
\]

\[
\Rightarrow K > 5.77
\]

So let \ K = 6 \ \text{or} \ \text{Den} = \frac{6}{S}
c) \[ t_{2\%} s \leq 4 \Rightarrow \frac{4}{s_0} \leq 4, \quad s_0 \geq 1 \] for a 2nd order system

\[ G(s) = \frac{s^2 + 4s + 8}{(s-3)(s-1)} \]

Desired location \( s_d = -1 + j2.4 \) approx.

\[ W_p = e^{-\frac{s_0\pi}{\omega_d \pi}} = e^{-\frac{1}{2.4\pi}} \approx 0.27 \text{ or } 27\% \]

\( K \) from the magnitude condition

\[ |G(s)| = 1 \]

\[ s = s_d \]

\[ |K \frac{s^2 + 4s + 8}{(s-3)(s-1)}| = 1 \Rightarrow K = 2.99 \]

\[ s = -1 + j2.4 \]

i.e. \[ t_{2\%} s \leq 4 \Rightarrow K > 2.99 \], let \( K = 3.5 \)
\[ T_0 = \frac{-180^\circ + 2360^\circ}{3 - 0} = \pm 60^\circ, 180^\circ \]

\[ T_\alpha = \frac{180^\circ + k360^\circ}{3 - 1} = \pm 90^\circ \]

\[ T_\alpha = \frac{180^\circ + 360^\circ}{3 - 2} = 180^\circ \]

\[ \text{No asymptotes} \]
\( G(s) = K \frac{1}{s(s+2)(s^2+6s+13)} = K \frac{1}{s^4 + 8s^3 + 25s^2 + 26s} \)

\( s = -3 \pm j^2 \)

\( \text{Root locus plot} \)

Need to determine:
- Breakaway pt
- Asymptotes
- Points on s-plane
- Angle of departure

\( \text{Breakaway pt} \Rightarrow \frac{dK}{ds} = 0 \)

\[ 1 + K \frac{1}{s^4 + 8s^3 + 25s^2 + 26s} = 0 \]

\[ -K = s^4 + 8s^3 + 25s^2 + 26s \]

\[ \frac{dK}{ds} = 4s^3 + 24s^2 + 50s + 26 \]

\[ \frac{dK}{ds} = 0 \Rightarrow 4s^3 + 24s^2 + 50s + 26 = 0 \]

\( \text{Soln.} \quad s = -0.7652 \)

\[ s = -2.6174 \pm j1.2820 \]

Not required
→ Asymptotes

\[ \tau_a = \frac{2\pi r_1 - 2\pi r_2}{u - v_i} \]

\[ \Phi_a = \frac{180^\circ + k360^\circ}{u - v_i} \]

\[ \tau_a = \frac{(0 + (-2)) + (-3 + j2) + (-3 - j2)}{4 - 0} = -2 \]

\[ \Phi_a = \frac{180^\circ + k360^\circ}{4 - 0} = \pm 45^\circ, \pm 135^\circ \]

→ Imag. axis crossing → Routh–Hurwitz criterion

\[ 1 + gH = 0 \]

\[ 1 + \frac{1}{s^4 + 8s^3 + 25s^2 + 26s} = 0 \]

\[ s^4 + 8s^3 + 25s^2 + 26s + K = 0 \]

\[
\begin{array}{c|ccc}
  s^4 & 1 & 25 & K \\
  s^3 & 8 & 26 & \\
  s^2 & 25 - \frac{26}{8} = 21.25 & 1K & \rightarrow 21.75s^2 + K = 0 \\
  s & 26 - \frac{8K}{21.75} & \rightarrow 26 - \frac{8K}{21.75} = 0 \Rightarrow K = 70.69 \\
  1 & K & \\
\end{array}
\]

\[ 21.75s^2 + 70.69 = 0 \]

\[ s = \pm j 1.8 \]
Angle of departure \( \theta_p = 3 + j2 \):

\[
\Delta s_p = \Delta (s_p + 2) - \Delta (s_p + 3 + j2) - \Theta_{out} = 180^\circ + 360^\circ
\]

\[
\tan^{-1} \frac{2-0}{-3-0} - \tan^{-1} \frac{2-0}{-3-(-2)} = 90^\circ - \Theta_{out} = 180^\circ + k360^\circ
\]

\[
-146.31^\circ - 116.57^\circ - 90^\circ - \Theta_{out} = 190^\circ + k360^\circ
\]

\[\Theta_{out} = -172.87^\circ\]
b) For stability, the Routh-Hurwitz table should have positive first column.

\[ s \to 26 - \frac{8k}{21.75} > 0 \Rightarrow K < 70.69 \]

\[ 1 \to K \Rightarrow K > 0 \]

so \[ 0 < K < 70.69 \]
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\[ \rightarrow D(s) \rightarrow \left[ \frac{s+2}{s(s+1)} \right] \]

\( e_{ss} = 0 \) for step input \( \Rightarrow G_H = \frac{1}{s} \gamma_i(s) \)

we have already have this term in \( G_H \) so there is no need for \( D(s) \) to supply it.

\[ t_{5\%} \approx 0.5 \Rightarrow \frac{3}{\tau_0} = 0.5 \Rightarrow \tau_0 = 6 \]

For a 2nd order system

When \( D(s) = K \)

\[ \begin{align*}
\tau_0 &= 6 \\
\circ \text{circle with center} &= -2 \\
\text{radius} &= \sqrt{(-2 - 0)(-2 + 1)} \\
&= 1.4142
\end{align*} \]

Since \( t_{5\%} \approx 0.5 \), one pole should have \( \tau_0 \approx 6 \) and the other one \( \tau_0 \approx 6 \). This condition cannot be satisfied from the above root-locus diagram since one poles approaches to \(-2\).
Cancelling one pole and placing it any stable location does not help since the root locus becomes "this pole will always be too slow."

So cancel the zero, and place it such that the circle intersect with the \( \sigma_0 = 6 \). One such choice is a zero at \(-4\), or

\[ D(s) = K \frac{s + 4}{s + 2} \]

\[ \sigma_d = -6 + j2.8284 \]

\[ \sqrt{(3.4641)^2 - 2^2} = 2.8284 \]

\[ K \text{ from the magnitude condition} \]

\[ |G + H| = 1 \Rightarrow \left| K \frac{s + 4}{s(s + 2)} \right| = 1 \]

\[ s = \sigma_d \]

\[ \sigma_d = 6 \]

\[ \text{circle with center} = -4 \]

\[ \text{real} = \sqrt{(-4-0)(-4+4)} = 3.4641 \]

\[ \Rightarrow K = 9.38 \quad \text{or} \quad D(s) = 9.38 \frac{s + 4}{s + 2} \]