1. The following requirements are given for a second-order system that is described by the transfer function 
\[ \frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}. \]

Maximum percent overshoot: \( 5\% \leq M_p \leq 15\% \).
Peak time: \( t_p \leq 1 \text{s} \).
2% settling time: \( t_{2\%s} \leq 2 \text{s} \).

(a) Describe and sketch the s-plane regions of the pole locations satisfying the requirements. (15pts)
(b) Determine the largest possible rise time of a system with the poles satisfying the requirements. (10pts)

2. Consider the following feedback control system.

Design a proportional-integral (PI) controller
\[ D(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} = \frac{K_P s + (K_I/K_P)}{s}; \]
such that the 2\% settling-time is less than 2 seconds, and the steady-state error for the unit-ramp input \( e(\infty) = (1/2) \). (25pts)

3. Consider the following feedback control system with the reference input \( r \) and the disturbance input \( d \).

For the case when
\[ G(s) = \frac{s + 5}{s + 8^1} \]
design a minimal-order controller, such that the output tracks the reference input that has the laplace transform
\[ R(s) = \frac{2(s - 4)}{{(s^2 + 4)(s + 1)}} \]
with zero steady-state error, and a step disturbance is rejected at the output. (25pts)
4. Consider a negative unity-feedback control system with the open-loop transfer function

\[ G(s) = K \frac{(s + 2)^2}{s^3} = K \frac{s^2 + 4s + 4}{s^3}. \]

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, break-away and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures. (25pts)
1. The following requirements are given for a second-order system that is described by the transfer function

\[ \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

Maximum percent overshoot: \( 5% \leq M_p \leq 15\% \).
Peak time: \( t_p \leq 1\text{s}. \)
2\% settling time: \( t_{2\%s} \leq 2\text{s}. \)

(a) Describe and sketch the \( s \)-plane regions of the pole locations satisfying the requirements.

Solution:

<table>
<thead>
<tr>
<th>Given Specifications</th>
<th>System Constraints</th>
<th>Geometrical Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5% \leq M_p \leq 15% ).</td>
<td>( 0.05 \leq e^{-\left(\frac{c}{\sqrt{1-c^2}}\right)^2} \leq 0.15, )  [ \frac{</td>
<td>\ln(0.15)</td>
</tr>
<tr>
<td>( t_p \leq 1\text{s}. )</td>
<td>( \frac{\pi}{\omega_d} \leq 1, )  or  ( \omega_d \geq \pi/1; )  since ( t_p = \pi/\omega_d. )</td>
<td>(</td>
</tr>
<tr>
<td>( t_{2%s} \leq 2\text{s}. )</td>
<td>( \frac{4}{\sigma_o} \leq 2, )  or  ( \sigma_o \geq 2; )  since ( t_{2%s} = 4/\sigma_o. )</td>
<td>( \sigma \leq -2, )  since the poles are at ( s = -\sigma_o \pm j\omega_d )</td>
</tr>
</tbody>
</table>

The shaded region describes the region specified by the given requirements.
(b) Determine the largest possible rise time of a system with the poles satisfying the requirements.

**Solution:** The rise time of the system is given by

\[ t_r = \frac{\pi - \cos^{-1}(\zeta)}{\omega_d}. \]

The largest rise time is when we have the largest \( \pi - \cos^{-1}(\zeta) \) or the smallest \( \cos^{-1}(\zeta) \) and the smallest \( \omega_d \). From the shaded region of the sketch in the previous part, we realize that the smallest \( \omega_d \) is when \( \omega_d = \pi \approx 3.14 \) and the smallest \( \cos^{-1}(\zeta) \) is when \( \cos^{-1}(\zeta) = \cos^{-1}(0.69) = 46.36^\circ = 0.2576\pi \), which is at the intersection of the radial line with the angle of 46.36° with respect to the negative real axis and the horizontal line at \( \omega = 3.14 \). Therefore,

\[ t_{r_{\text{max}}} = \frac{\pi - \cos^{-1}(\zeta_{\text{max}})}{\omega_{d_{\text{min}}}} = \frac{\pi - 0.2576\pi}{\pi}, \]

or the largest possible rise time of the system is 0.74s.

2. Consider the following feedback control system.

\[
\begin{array}{c}
  r \rightarrow + \rightarrow e \rightarrow D(s) \rightarrow \frac{1}{(s+1)(s+5)} \rightarrow y
\end{array}
\]

Design a proportional-integral (PI) controller

\[ D(s) = K_P + \frac{K_I}{s} = \frac{K_PS + K_I}{s} = K_P \frac{s + (K_I/K_P)}{s}; \]

such that the 2% settling-time is less than 2 seconds, and the steady-state error for the unit-ramp input \( e(\infty) = (1/2) \).
Solution: Since the open-loop gain of the system is

\[ D(s)G(s) = K_P \left( \frac{s + (K_I/K_P)}{s} \right) \left( \frac{1}{(s + 1)(s + 5)} \right), \]

the system is type-1, and we can get a constant steady-state error for a ramp input. The 2\% settling-time requirement gives us the restriction such that

\[ t_{2\%} = \frac{4}{\sigma_o} \leq 2, \]

or \( \sigma_o \geq 2 \). In other words, the real part of the dominant complex closed-loop poles or the dominant real pole needs to be less than \(-2\). In the PI design, we have the choice of a zero and the loop gain. Since we have three open-loop gain poles at \( s = 0, -1, \) and \(-5\), as well as a zero at \( s = -K_I/K_P \); we get the following possible root-locus diagrams.

(a) \(-K_I/K_P = 0\)

(b) \(-1 < -K_I/K_P < 0\)

(c) \(-K_I/K_P = -1\)

(d) \(-5 < -K_I/K_P < -1\)

(e) \(-K_I/K_P = -5\)

(f) \(-K_I/K_P < -5\)

In case (a), the system is no longer type-1. In case (b), the dominant closed-loop real pole is greater than \(-1\). In the cases (d)-(f), the dominant poles all have real parts greater than \(-2\). So the only good option is to cancel the pole at \(-1\) as in the case (c). In other words,

\[ D(s) = K_P \frac{s + (K_I/K_P)}{s} = K_P \frac{s + 1}{s} \]

satisfies the settling-time requirement for the values of \( K_P \) after the root-locus branch crosses \( s = -2 \). When \( s = -2 \), from the magnitude condition we have

\[ \left| D(s)G(s) \right|_{s=-2} = \left| K_P \left( \frac{s + 1}{s} \right) \left( \frac{1}{(s + 1)(s + 5)} \right) \right|_{s=-2} = \left| K_P \frac{1}{s(s + 5)} \right|_{s=-2} = 1, \]

or \( K_P = 6 \). So, as long as we choose \( K_P \geq 6 \), we satisfy the settling-time requirement.

The steady-state error for the unit-ramp input is \( e(\infty) = (1/K_v) \), where

\[ K_v = \lim_{s \to 0} sD(s)G(s) = \lim_{s \to 0} sK_P \frac{1}{s(s + 5)} = \frac{K_P}{5}. \]
Since we need \( e(\infty) = (1/2) \),
\[
e(\infty) = \frac{1}{K_v} = \frac{1}{K_P/5} = \frac{5}{K_P} = \frac{1}{2},
\]
or \( K_P = 10 \) will satisfy the steady-state error requirement as well as the settling-time requirement. Therefore,
\[
D(s) = 10 \frac{s + 1}{s},
\]
or
\[
D(s) = 10 + \frac{10}{s}.
\]

3. Consider the following feedback control system with the reference input \( r \) and the disturbance input \( d \).

For the case when
\[
G(s) = \frac{s + 5}{s + 8},
\]
design a minimal-order controller, such that the output tracks the reference input that has the laplace transform
\[
R(s) = \frac{2(s - 4)}{(s^2 + 4)(s + 1)}
\]
with zero steady-state error, and a step disturbance is rejected at the output.

Solution: In order to have a zero steady-state error for any reference input and to reject a disturbance signal at the output, we need to match the non-asymptotically stable poles of the input and the disturbance in the open-loop gain of the system. In the case of the reference input, we need to have poles at \( s = \pm j2 \); since the pole at \( s = -1 \) of \( R(s) \) is asymptotically stable, and its contribution will disappear on its own at steady state. To reject a step disturbance, we also need to match the disturbance pole at \( s = 0 \), or the system has to be of type-1. With these choices, the open-loop gain
\[
D(s)G(s) = \left( \frac{1}{s(s^2 + 4)} D'(s) \right) \left( \frac{s + 5}{s + 8} \right) = \frac{s + 5}{s(s^2 + 4)(s + 8)} D'(s),
\]
where
\[
D(s) = \frac{1}{s(s^2 + 4)} D'(s)
\]
for some \( D'(s) \). Since there is no other explicit requirement, we only need to ensure stability by a proper and simple choice of \( D'(s) \).

The simplest choice is \( D'(s) = K \) for a constant \( K \). We may use a number of methods to check the stability of the system for this choice, but a rough sketch of the root-locus, as shown below, is simple enough to see the location of the closed-loop poles.
As we observe from the root-locus diagram, there is no value of $K$ that would result in a stable closed-loop system; mainly because the asymptote angles are $\theta_a = \pm 60^\circ, 180^\circ$, and there are poles on the imaginary axis.

In order to have the asymptote intersection and the angles stay inside the left-half plane, we need to have zeros in $D'(s)$. Since we are placing three poles, we may have up to three zeros in $D'(s)$. With only one zero, we will be able to have the asymptote angle as $\theta_a = \pm 90^\circ$. We need to make sure that the asymptote intersection is on the left-half plane. For

\[ D'(s) = K(s - a), \]

or

\[ D(s)G(s) = K \frac{(s - a)(s + 5)}{s(s^2 + 4)(s + 8)}, \]

the asymptote intersection

\[ \sigma_a = \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m} = \frac{((-8) + (0) + (j2) + (-j2)) - ((a) + (-5))}{4 - 1} = \frac{-a - 3}{3}, \]

where $\sum p_i$ and $\sum z_i$ are the sums of the pole and zero locations, respectively. As long as $a > -3$, we get $\sigma_a < 0$, and the complex poles will go towards the asymptotically stable region. However, if $a > 0$; this time the pole at $s = 0$ will go towards the zero at $s = -a$, and the system will still be unstable. So, we need to choose $-3 < a < 0$ and $K \gg 0$. (For small $K > 0$, there might be a region of the root-locus branch that is still in the unstable region.)

Therefore, one possible simplest controller is

\[ D(s) = \frac{K(s - a)}{s(s^2 + 4)}, \]

where $-3 < a < 0$ and $K \gg 0$.

4. Consider a negative unity-feedback control system with the open-loop transfer function

\[ G(s) = K \frac{(s + 2)^2}{s^3} = K \frac{s^2 + 4s + 4}{s^3}. \]

Construct the root-locus diagram. Determine all the important necessary features like asymptotes, breakaway and/or break-in points, imaginary-axis crossings, and angle of arrivals and departures.
Solution: First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:
- Break-in points, and
- Imaginary-axis crossings.

There is only one asymptote, since two out of the three poles will go towards the double zero. The initial break-away from the triple poles at \( s = 0 \) will be at \( \pm60° \) and \( 180° \), since they will leave the real axis equally apart and one has to leave at \( 180° \).

Break-in Point: \( \frac{dK}{ds} = 0 \)

From the characteristic equation,
\[
1 + G(s) = 0,
\]
\[
1 + K \frac{(s + 2)^2}{s^3} = 0,
\]
and
\[
-K = \frac{s^3}{(s + 2)^2}.
\]
Therefore,
\[
\frac{dK}{ds} = \frac{3s^2(s + 2)^2 - s^3(2(s + 2))}{(s + 2)^4};
\]
and for \( \frac{dK}{ds} = 0 \), the equation
\[
3s^2(s + 2)^2 - s^3(2(s + 2)) = s^2(s + 2)(3(s + 2) - 2s) = s^2(s + 2)(s + 6) = 0
\]
gives \( s = -6, s = -2, s = 0, \) and \( s = 0 \). Since the double \( s = 0 \) location is the initial break-away, and the \( s = -2 \) location is due to the double zero, the break-in point is at \( s = -6 \).

Imaginary-Axis Crossings: Routh-Hurwitz Table

The imaginary axis crossings can be determined from the Routh-Hurwitz table. The characteristic equation from
\[
1 + G(s) = 1 + K \frac{s^2 + 4s + 4}{s^3} = \frac{s^3 + K(s^2 + 4s + 4)}{s^3} = 0
\]
is
\[
s^3 + Ks^2 + 4Ks + 4K = 0.
\]
The Routh-Hurwitz table for this characteristic equation is given below.
The imaginary-axis crossings will correspond to the values of $K$ that would make a row of all zeros on the table. When $K = 0$, the $s^2$-row and the 1-row become all zeros, because there are multiple imaginary-axis crossings at the start of the root-locus diagram. The only other candidate is the $s$-row. The $s$-row is all zero, when $K = 1$. For this value of $K$, we get a factor of the characteristic polynomial from the upper or the $s^2$-row. So,

$$\left(Ks^2 + 4K\right)_{K=1} = s^2 + 4 = 0,$$

or $s = \pm j2$. Therefore, the imaginary-axis crossings are at $s = \pm j2$.

With the features determined, we can now sketch the root-locus diagram.