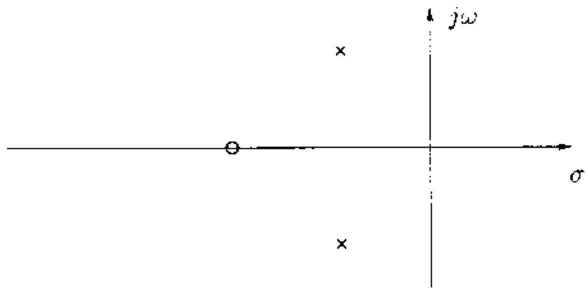
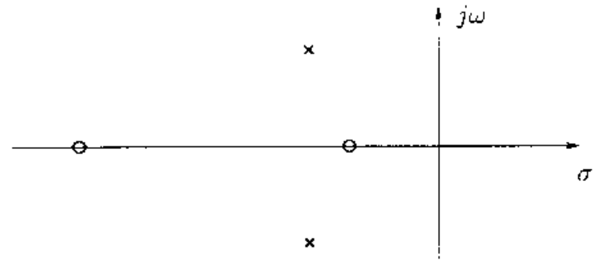
EE 231 **Exam#3** **May 05, 1992**

75 minutes

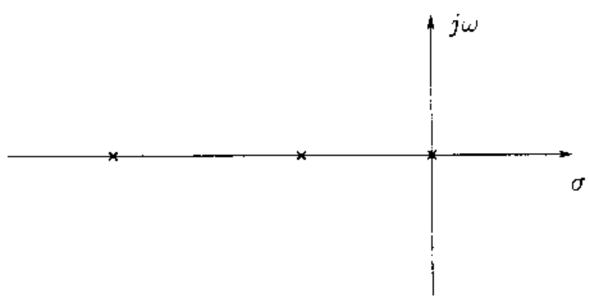
1. For the following open-loop pole/zero locations, sketch *expected* root-locus diagrams. *Do not* determine any features of the diagram, simply show the expected shapes of all the root-locus branches. (20pts)



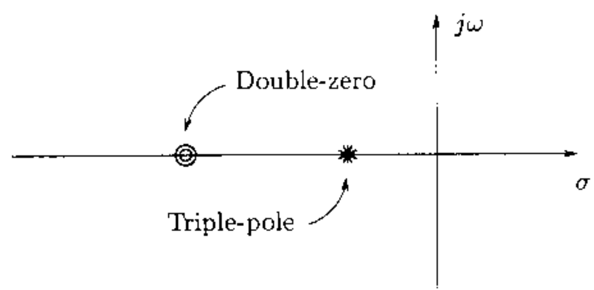
(a)



(b)



(c)



(d)

2. Consider a unity-feedback control system with the open-loop transfer function

$$G(s) = K\frac{s^2 - 2s + 5}{(s+1)(s+2)(s^2+8s+17)}.$$

   (a) Construct the root-locus diagram. Determine all the important features like asymptotes, imaginary-axis crossings, angle of arrivals and departures; however *do not* determine the break-away and/or break-in points explicitly. Obtain only the equation whose solutions would give those points, i.e. *do not solve that equation.* (25pts)

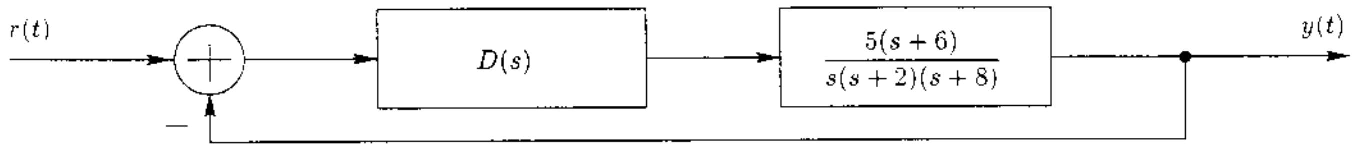
   (b) Determine all the values of K such that the closed-loop system is asymptotically stable. (05pts)

3. Consider a unity-feedback system under a proportional control with the open-loop transfer function

$$D(s)G(s) = K\frac{2(s+4)(s+50)}{(s^2+9)(s+60)}.$$

   Using graphical root-locus methods, design for the proportional control constant K, such that the closed-loop system behaves underdamped, and its 2% settling time is 0.5 second. (25pts)

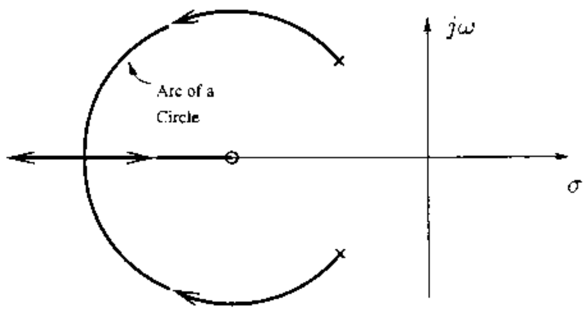
4. A compensator $D(s)$ needs to be designed for the following system, such that the steady-state error is zero for a step input, the complex closed-loop poles are at $s = -5 \pm j4$, and the order of the closed-loop system stays the same. (25pts)



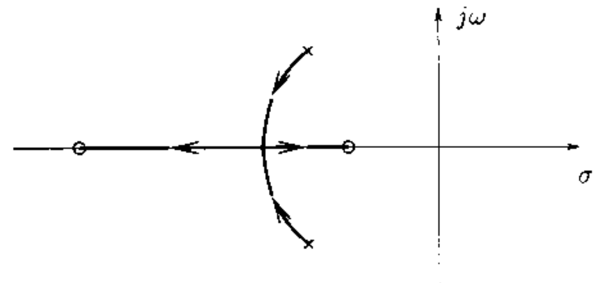
EE 231 | Exam#3 | May 08, 1992

Solutions

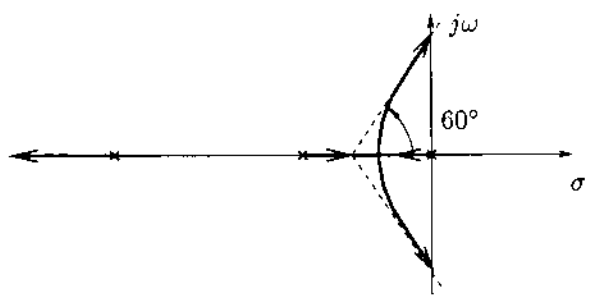
1. From the rules of obtaining root-locus diagrams, we can sketch the following without computing any of the important features.



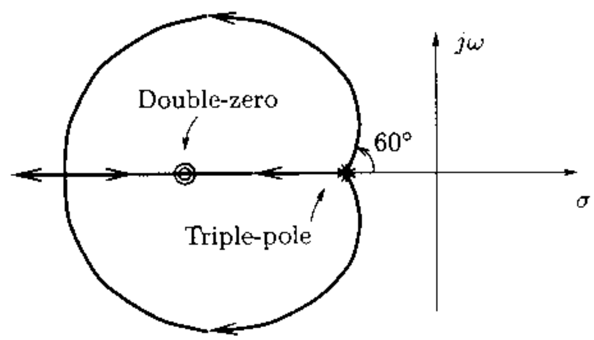
(a)



(b)



(c)



(d)

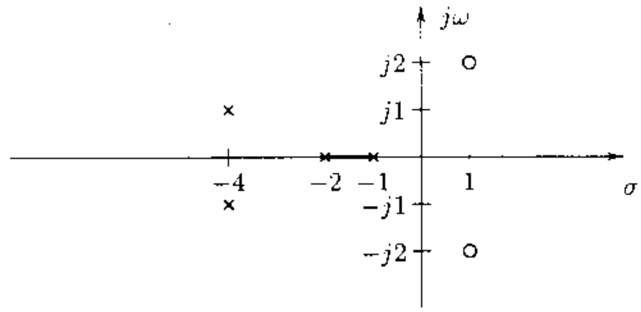
2. The open-loop transfer function of the unity-feedback control system is

$$
\begin{aligned}
G(s) &= K\frac{s^2-2s+5}{(s+1)(s+2)(s^2+8s+17)} \\
&= K\frac{(s-(1+j2))(s-(1-j2))}{(s+1)(s+2)(s-(-4+j))(s-(-4-j))}.
\end{aligned}
$$

(a) First, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Breakaway point
- Asymptotes
- Angle of Departure
- Angle of Arrival
- Imaginary-axis crossings



Breakaway Point: $\dfrac{dK}{ds} = 0$

From the characteristic equation.

$$1 + G(s) = 0.$$

$$1 + K\frac{s^2 - 2s + 5}{(s+1)(s+2)(s^2+8s+17)} = 0.$$

and

$$-K = \frac{(s+1)(s+2)(s^2+8s+17)}{s^2 - 2s + 5}.$$

Therefore,

$$-\frac{\mathrm{d}K}{\mathrm{d}s} = \frac{\begin{array}{r}[(s+2)(s^2+8s+17) + (s+1)(s^2+8s+17) + (s+1)(s+2)(2s+8)](s^2-2s+5) \\ -(s+1)(s+2)(s^2+8s+17)(2s-2)\end{array}}{(s^2-2s+5)^2},$$

and since $\mathrm{d}K/\mathrm{d}s = 0$, the solution of

$$\begin{array}{r}[(s+2)(s^2+8s+17) + (s+1)(s^2+8s+17) + (s+1)(s+2)(2s+8)](s^2-2s+5) \\ -(s+1)(s+2)(s^2+8s+17)(2s-2) = 0.\end{array}$$

which is between -2 and -1, is the breakaway point.

Asymptotes, Real-Axis Crossing: $\sigma_a = \dfrac{\sum p_i - \sum z_i}{n - m}$

So, the real-axis crossing of the asymptotes is at

$$\begin{aligned}\sigma_a &= \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{n} z_i}{n - m} \\ &= \frac{(-1) + (-2) + (-4+j) + (-4-j) - (1+j2) - (1-j2)}{4-2} \\ &= \frac{-13}{2} = -6.5.\end{aligned}$$

Asymptotes, Real-Axis Angles: $\theta_a = \dfrac{\pm(2k+1)\pi}{n-m}$

So, the angles the asymptotes make with the real axis are determined from

$$\begin{aligned}\theta_a &= \frac{\pm(2k+1)180°}{n-m}\\ &= \frac{\pm(2k+1)180°}{4-2} = \pm 90°.\end{aligned}$$

Angle of Departure: $\sum \angle(\cdot) = \pm(2k+1)\pi$

The angles of departures from complex open-loop poles are determined from the angular conditions about the open-loop poles. Therefore, the angular condition about $s = -4 + j$ is

$$-\theta_{\text{dep}} - \tan^{-1}\frac{1-(-1)}{-4-(-4)} - \tan^{-1}\frac{1-0}{-4-(-2)} - \tan^{-1}\frac{1-0}{-4-(-1)} + \tan^{-1}\frac{1-2}{-4-1} + \tan^{-1}\frac{1-(-2)}{-4-1} = \pm(2k+1)180°.$$

$$-\theta_{\text{dep}} - 90° - 153.43° - 161.57° + 191.31° + 149.04° = \pm(2k+1)180°,$$

or

$$\theta_{\text{dep}} = 115.35°.$$

Angle of Arrival: $\sum \angle(\cdot) = \pm(2k+1)\pi$

The angles of arrivals to complex open-loop zeros are also determined from the angular conditions about the open-loop zeros. Therefore, the angular condition about $s = 1 + j2$ is

$$\theta_{\text{arr}} + \tan^{-1}\frac{2-(-2)}{1-1} - \tan^{-1}\frac{2-0}{1-(-1)} - \tan^{-1}\frac{2-0}{1-(-2)} - \tan^{-1}\frac{2-1}{1-(-4)} - \tan^{-1}\frac{2-(-1)}{1-(-4)} = \pm(2k+1)180°.$$

$$\theta_{\text{arr}} + 90° - 45° - 33.69° - 11.31° - 30.96° = \pm(2k+1)180°,$$

or

$$\theta_{\text{arr}} = -149.04°.$$

Imaginary-Axis Crossing: Routh-Hurwitz Table

The imaginary-axis crossings are determined from the Routh-Hurwitz table, so consider the characteristic equation,

$$1 + K\frac{s^2-2s+5}{(s+1)(s+2)(s^2+8s+17)} = 0.$$

$$(s+1)(s+2)(s^2+8s+17) + K(s^2-2s+5) = 0,$$

$$(s^2+3s+2)(s^2+8s+17) + K(s^2-2s+5) = 0,$$

$$(s^4+11s^3+43s^2+67s+34) + K(s^2-2s+5) = 0,$$

or

$$s^4 + 11s^3 + (43+K)s^2 + (67-2K)s + (34+5K) = 0.$$

Therefore,

s^4	1	$43+K$	$34+5K$
s^3	11	$67-2K$	
s^2	$\dfrac{406+13K}{11}$	$34+5K$	
s	$\dfrac{(406+13K)(67-2K)/11-11(34+5K)}{(406+13K)/11}$		
1	$34+5K$		

In this case, the only way that there is an imaginary-axis crossing (other than at $s=0$) is when the s-row is all zero, or when

$$\frac{(406+13K)(67-2K)/11-11(34+5K)}{(406+13K)/11}=0,$$

$$\begin{aligned}
\frac{(406+13K)(67-2K)}{11}-11(34+5K)&=0,\\
(406+13K)(67-2K)-121(34+5K)&=0,\\
27,202+59K-26K^2-4,114-605K&=0,\\
23,088-546K-26K^2&=0,\\
-(13K^2+273K-11,544)&=0.
\end{aligned}$$

The solution to the above equation is

$$\begin{aligned}
K_{1,2}&=\frac{-273\pm\sqrt{(273)^2-4(13)(-11,544)}}{2(13)}\\
&=\frac{-273\pm 821.472}{26},\\
K_1&=21.095,\\
K_2&=-42.095.
\end{aligned}$$

Since, we are interested in positive K, we substitute $K=21.095$ in the polynomial from the s^2-row, i.e.

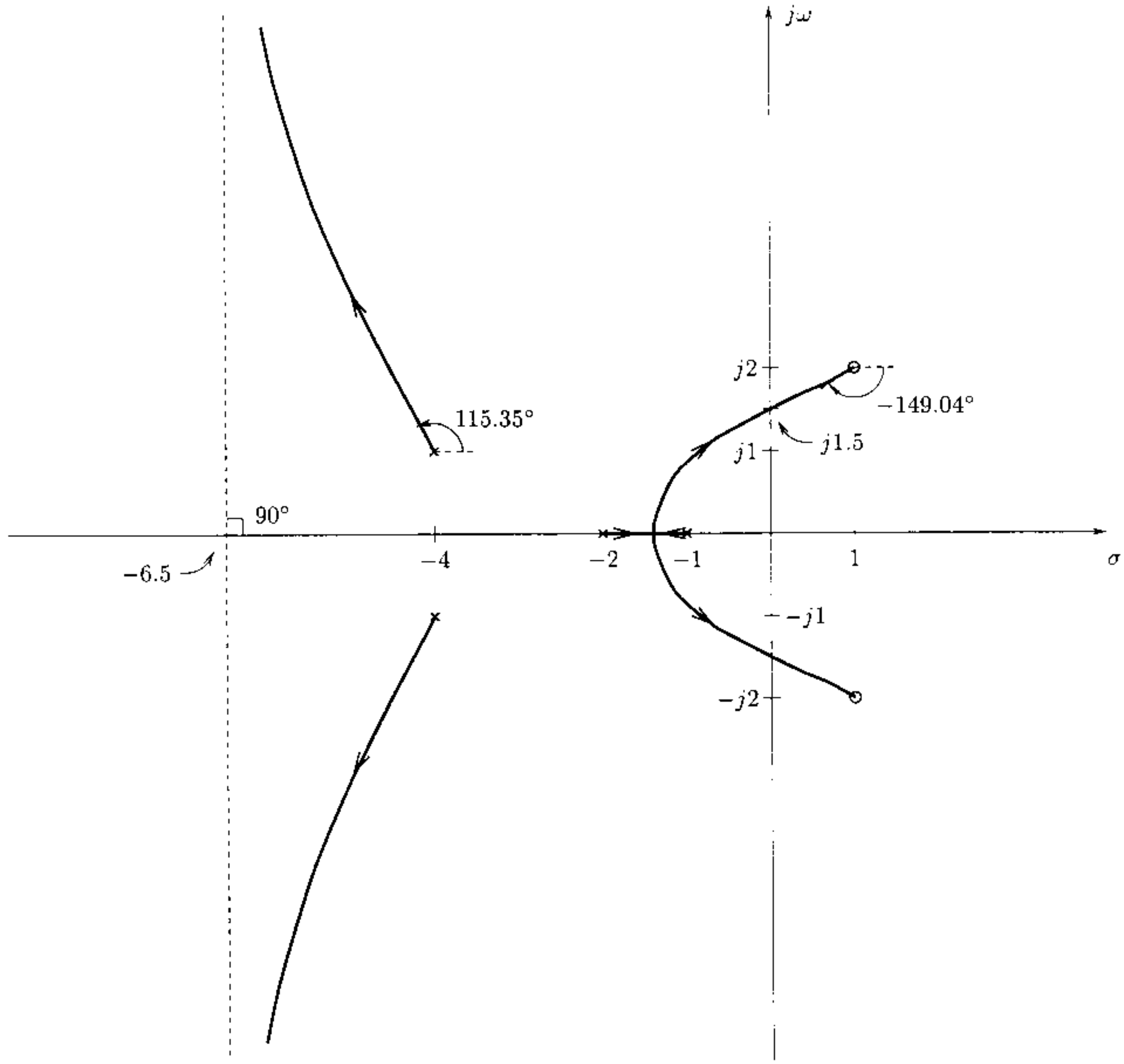
$$\left(\frac{406+13K}{11}\right)s^2+(34+5K)=0,$$

$$61.84s^2+139.48=0.$$

In other words, the imaginary-axis crossings are at

$$s=\pm j1.5.$$

With the features determined, we can now draw the root-locus diagram.



(b) To determine all the values of K for asymptotical stability, we use the Routh-Hurwitz table obtained before. From the table

1. $\dfrac{406 + 13K}{11} > 0,$

   $406 + 13K > 0,$

   $-32.23 < K.$

2. $-(13K^2 + 273K - 11{,}544) > 0,$ since $406 + 13K > 0,$

   $-13(K + 42.095)(K - 21.095) > 0,$

   $-42.095 < K < 21.095,$ since the inequality is satisfied for $K = 0.$

3. $34 + 5K > 0,$

   $-6.8 < K.$

Therefore, the stability requirements can be compactly written as

$$-42.095 < -32.23 < -6.8 < K < 21.095,$$

or

$$-6.8 < K < 21.095.$$

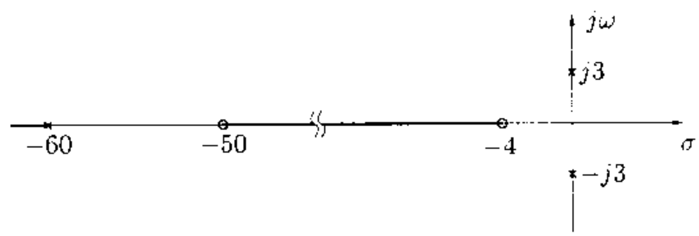
3. The open-loop transfer function of the unity-feedback control system is

$$D(s)G(s) = K\frac{2(s+4)(s+50)}{(s^2+9)(s+60)}.$$

Again, we sketch the pole-zero locations and the real-axis portion of the root-locus diagram. Then, we decide the important features to be determined.

Need to determine:

- Breakaway point
- Angle of Departure



However in this case, we really don't have to determine the breakaway point and the angle of departure, since we can neglect the the zero at -50 and the pole at -60 in drawing the root-locus diagram. This is a good approximation for the system, since the pole-zero pair is so far away from the desired location that the sum of their angular contributions is almost zero.

If the pole-zero pair is neglected, the root-locus diagram will be as shown below, where the arc will be from a circle with its center at the zero, i.e. at $s = -4$, and with the radius $\sqrt{(-4-j3)(-4+j3)} = 5$. This arc puts the breakaway point at -9 (actual analysis would have given -8.8).

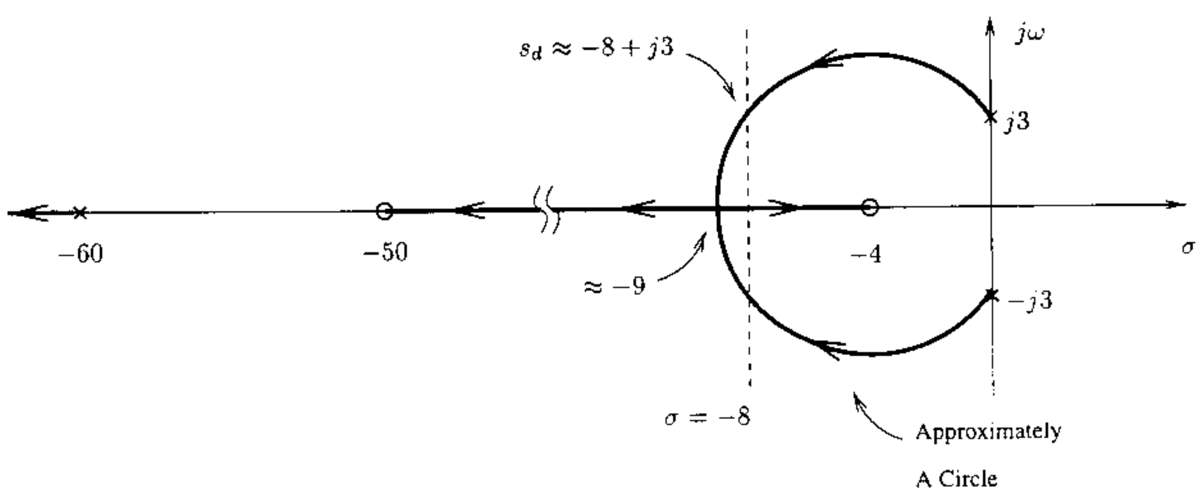
The two percent settling time requirement means

$$\frac{4}{\zeta\omega_n} = 0.5,$$

or

$$\zeta\omega_n = 8,$$

since $t_{s_{2\%}} = 4/(\zeta\omega_n)$. Geometrically, this requirement is represented by a vertical line crossing the real-axis at -8, because $\sigma = -\sigma_o = -\zeta\omega_n$ is the real part of the complex pole pair. Therefore, the desired location is at the intersection of the root-locus arc and the vertical line as shown in the figure.



We read the desired location graphically from the diagram as $s_d \approx -8 + j3$ (the actual location is $s_d = -8 + j2.70$). Then, the controller constant is determined from the magnitude condition at the desired location, i.e.

$$\left| K \frac{2(s+4)(s+50)}{(s^2+9)(s+60)} \right|_{s=s_d=-8+j3} = 1,$$

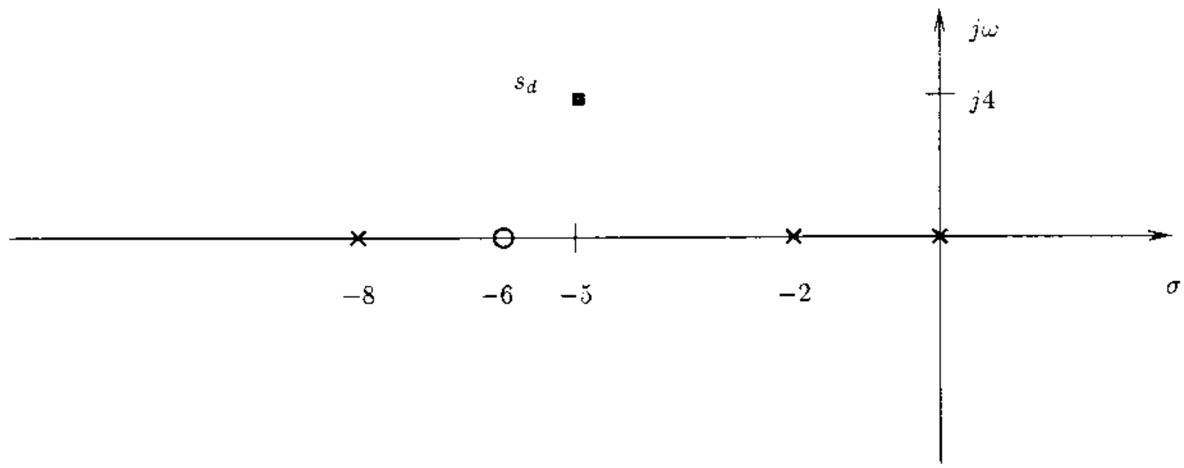
which implies $K = 9.896$ (the actual value is 10.082).

4. The open-loop transfer function of the system is

$$D(s)G(s) = D(s)\frac{5(s+6)}{s(s+2)(s+8)}.$$

In order to have zero steady-state error for a step input, the system has to be TYPE 1. Since in this case, the system already has one pole at zero, there is no need for $D(s)$ to supply anything.

To ensure that the closed-loop poles are at $s_d = -5 \pm j4$, we need to check the deficiency angle at the desired location.



The angular condition at the desired location gives

$$\phi - \tan^{-1}\frac{4-0}{-5-0} - \tan^{-1}\frac{4-0}{-5-(-2)} - \tan^{-1}\frac{4-0}{-5-(-8)} + \tan^{-1}\frac{4-0}{-5-(-6)} = \pm(2k+1)180^\circ.$$

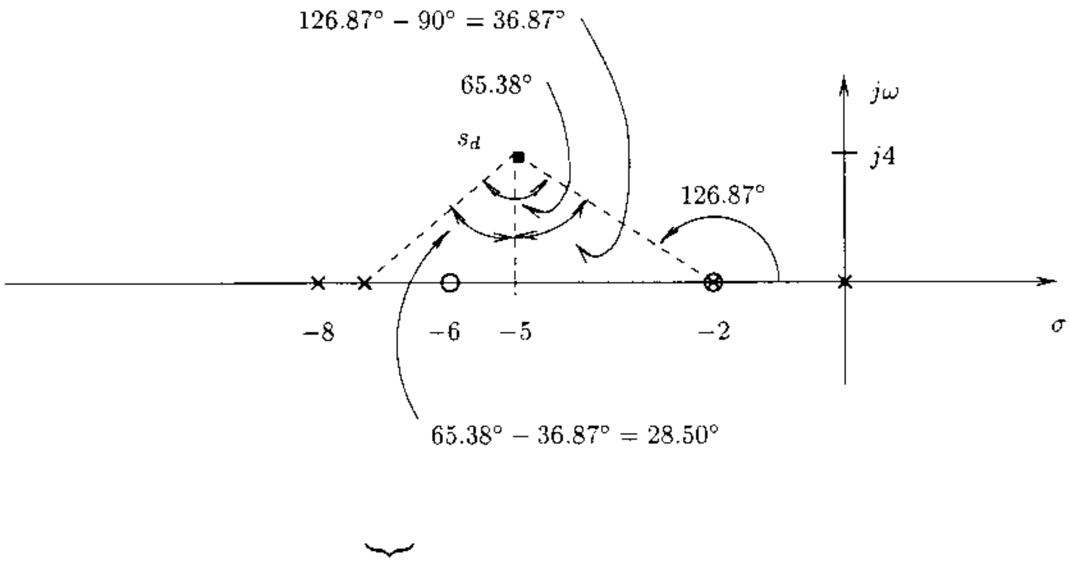
where ϕ is the deficiency angle, and

$$\phi - 141.34^\circ - 126.87^\circ - 53.13^\circ + 75.96^\circ = \pm(2k+1)180^\circ,$$

or

$$\phi = 65.38^\circ > 0.$$

Therefore, we need a lead compensator. In order not to increase the order of the system, we need to cancel a system pole. However, we cannot cancel the pole at zero, since it would drop the TYPE of the system. Cancelling the pole at -8 puts the compensator pole very far away which is undesirable and might even be impossible. So, we cancel the pole at $s = -2$ and determine the location of the compensator pole from the figure.



$$4\tan 28.50^\circ = 2.17 \Longrightarrow \text{pole} = -5 - 2.17 = -7.17$$

Hence,

$$D(s) = K\frac{s+2}{s+7.17},$$

and we determine K from the magnitude condition,

$$\left| K\frac{s+2}{s+7.17}\,\frac{5(s+6)}{s(s+2)(s+8)} \right|_{s=-5+j4} = 1,$$

which implies $K = 7.067$. Therefore,

$$D(s) = 7.067\frac{s+2}{s+7.17}.$$