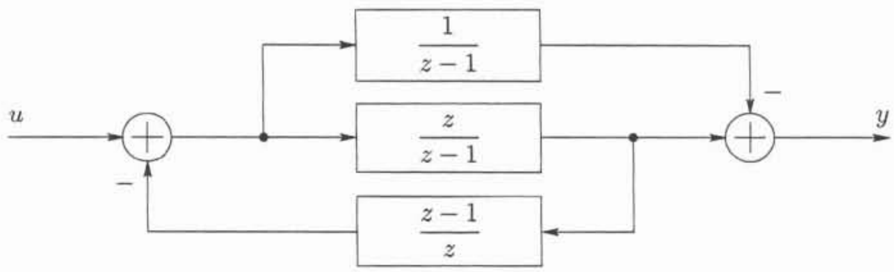
EE 331 **Exam#2** **May 04, 2004**

75 minutes

1. The block diagram of a control system is given in the following figure.



   (a) Obtain a state-space representation of the system without any block-diagram reduction. (15pts)

   (b) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$. (10pts)

2. A linear, discrete-time control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} c & 0 \end{bmatrix} \boldsymbol{x}(k),$$

   where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively, and a, b, and c are real constants.

   (a) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system has a reachability index of 1. (05pts)

   (b) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system has a reachability index of 2. (05pts)

   (c) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system is observable. (05pts)

3. A discrete-time linear control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.05 & 0.6 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k),$$

   where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

   (a) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$. (15pts)

(b) Design a full state-feedback controller; such that the maximum percent overshoot is between 1.5% and 4%, and the 5% settling-time is reached in 3 sampling periods. (15pts)

(c) Assuming that only the output is available, implement the controller of the previous part. (10pts)

4. Consider a system described by the difference equation

$$x(k+1) = 2x(k) + u(k),$$

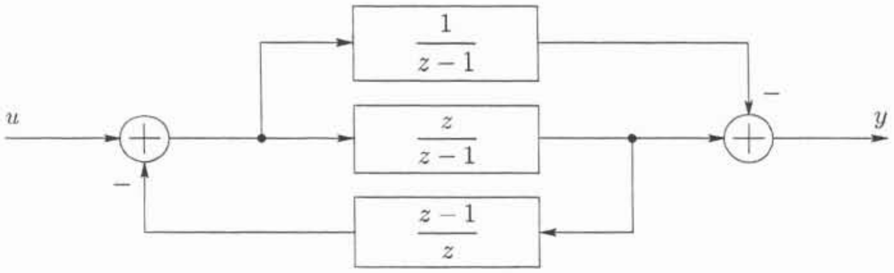
where x and u are the state and the input variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = 10x^2(4) + \sum_{k=0}^{3} \frac{1}{2}\left(x^2(k) + 5u^2(k)\right),$$

when $x(0) = -1$. (20pts)

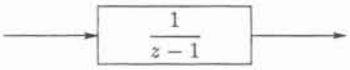
 Exam#2 Solutions

1. The block diagram of a control system is given in the following figure.

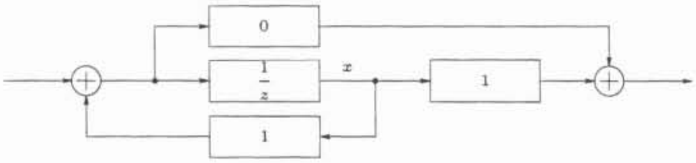


(a) Obtain a state-space representation of the system without any block-diagram reduction.

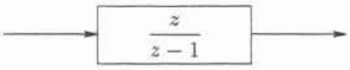
Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.



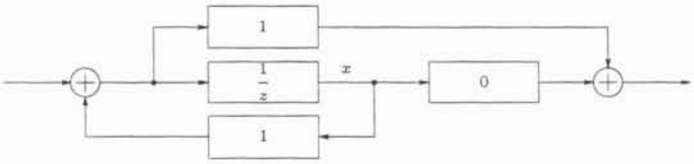
(a) The first feedforward gain block.



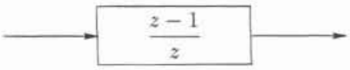
(b) Controller realization form.



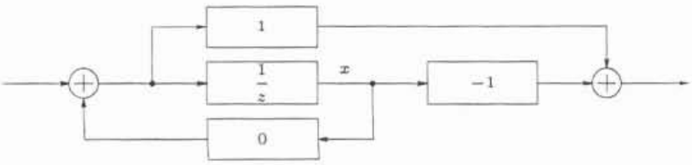
(a) The second feedforward gain block.



(b) Controller realization form.

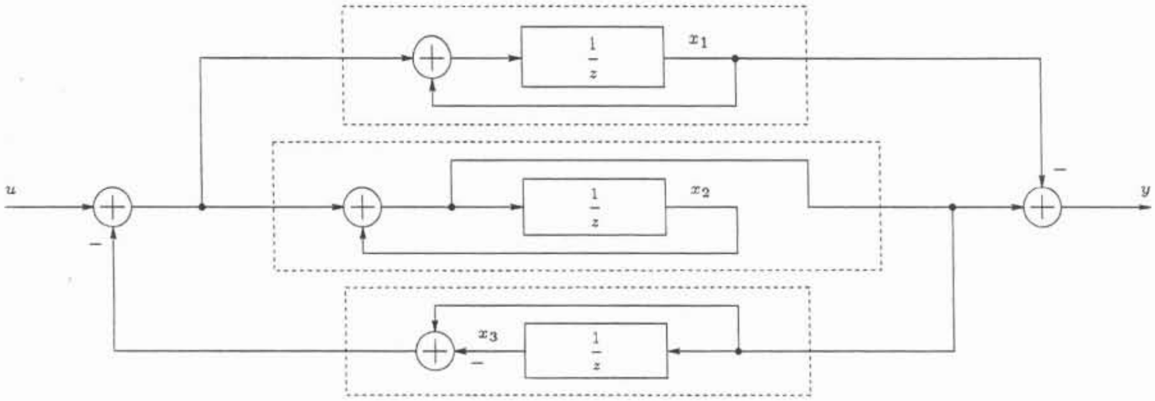


(a) The feedback gain block.



(b) Controller realization form.

The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$x_1(k+1) = x_1(k) + \big(u(k) - \big(x_3(k+1) - x_3(k)\big)\big) = x_1(k) + x_3(k) + u(k) - x_3(k+1),$$
$$x_2(k+1) = x_2(k) + \big(u(k) - \big(x_3(k+1) - x_3(k)\big)\big) = x_2(k) + x_3(k) + u(k) - x_3(k+1),$$
$$x_3(k+1) = x_2(k+1) = x_2(k) + x_3(k) + u(k) - x_3(k+1),$$

and

$$y(k) = x_2(k+1) - x_1(k) = -x_1(k) + x_3(k+1).$$

From the above $x_3(k+1)$ equation, we can solve for $x_3(k+1)$ to get

$$x_3(k+1) = (1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k).$$

Substituting the expression for $x_3(k+1)$ into the above state and the output equations, we get

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_3(k) + u(k) - \big((1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k)\big) \\ &= x_1(k) - (1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k), \\ x_2(k+1) &= x_2(k) + x_3(k) + u(k) - \big((1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k)\big) \\ &= (1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k), \\ x_3(k+1) &= x_2(k+1) \\ &= (1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k), \end{aligned}$$

and

$$\begin{aligned} y(k) &= -x_1(k) + \big((1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k)\big) \\ &= -x_1(k) + (1/2)x_2(k) + (1/2)x_3(k) + (1/2)u(k). \end{aligned}$$

Rewriting the equations in matrix form, we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} -1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1/2 \end{bmatrix} u(k).$$

Note here that the observer realization form results in a very similar realization diagram.

(b) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$.

Solution: We can determine the transfer function using couple of approaches.

Using the expression $Y(z)/U(z) = C(zI - A)^{-1}B + D$

In this approach, we may use the expression for the transfer function, and it involves the determination of $(zI - A)^{-1}$, where I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(zI - A)$ is to use row operations on the augmented matrix $\begin{bmatrix} (zI - A) & I \end{bmatrix}$ to generate $\begin{bmatrix} I & (zI - A)^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} z-1 & 1/2 & -1/2 & 1 & 0 & 0 \\ 0 & z-1/2 & -1/2 & 0 & 1 & 0 \\ 0 & -1/2 & z-1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2(z-1) & 1 & -1 & 2 & 0 & 0 \\ 0 & 2z-1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 2z-1 & 0 & 0 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2(z-1) & 1 & -1 & 2 & 0 & 0 \\ 0 & 2z-1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 4z(z-1)/(2z-1) & 0 & 2/(2z-1) & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2(z-1) & 1 & -1 & 2 & 0 & 0 \\ 0 & 2z-1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2z(z-1) & 0 & 1 & 2z-1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2(z-1) & 0 & -2(z-1)/(2z-1) & 2 & -2/(2z-1) & 0 \\ 0 & 2z-1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2z(z-1) & 0 & 1 & 2z-1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2(z-1)(2z-1) & 0 & -2(z-1) & 2(2z-1) & -2 & 0 \\ 0 & 2z-1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2z(z-1) & 0 & 1 & 2z-1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/(z-1) & -1/(2z(z-1)) & 1/(2z(z-1)) \\ 0 & 1 & 0 & 0 & (2z-1)/(2z(z-1)) & 1/((2z(z-1)) \\ 0 & 0 & 1 & 0 & 1/(2z(z-1)) & (2z-1)/(2z(z-1)) \end{array}\right].$$

Therefore,

$$C(zI - A)^{-1}B + D = \frac{1}{2z(z-1)} \begin{bmatrix} -1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2z & -1 & 1 \\ 0 & 2z-1 & 1 \\ 0 & 1 & 2z-1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 1/2$$

$$= \frac{1}{2z(z-1)} \begin{bmatrix} -1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} z \\ z \\ z \end{bmatrix} + 1/2 = 1/2.$$

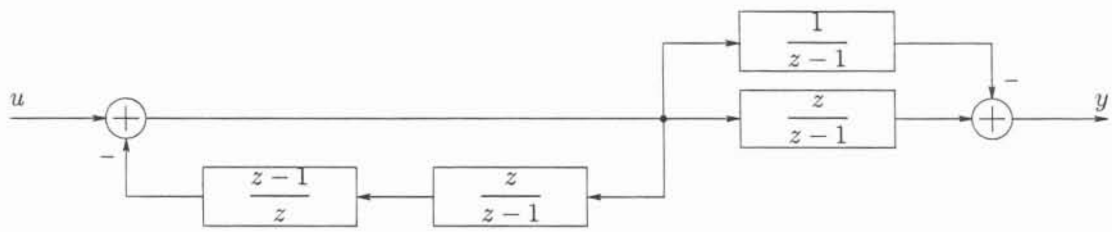
In other words, the transfer function

$$\frac{Y(z)}{U(z)} = \frac{1}{2}.$$

Block-diagram reduction

In this approach, we can use the block diagram reduction methods to determine the transfer function. Considering the simplicity of the block-diagram reduction and the complexity of the inversion process in the previous method, the block-diagram reduction method, in this case, should be the preferred method.

After one block-diagram reduction step, we get the following diagram.



Writing the transfer function from the block diagram, we get

$$\frac{Y(z)}{U(z)} = \left(\frac{1}{1+\big(z/(z-1)\big)\big((z-1)/z\big)}\right)\left(\frac{z}{z-1}-\frac{1}{z-1}\right) = \frac{1}{1+1}\left(\frac{z-1}{z-1}\right) = \frac{1}{2},$$

as before.

2. A linear, discrete-time control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} c & 0 \end{bmatrix} \boldsymbol{x}(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively, and a, b, and c are real constants.

(a) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system has a reachability index of 1.

Solution: To determine reachability index of the system, we need to check on the columns of the controllability matrix. The controllability matrix

$$\mathcal{C}(A,B) = \left[\ B \mid AB\ \right] = \left[\begin{array}{cc|cc} 0 & b & a & 0 \\ a & 0 & 0 & 0.25b \end{array}\right],$$

where A and B are the state and the input matrices of the system, respectively; and since

$$AB = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0.25b \end{bmatrix}.$$

In order for an nth order system to have the reachability index equal to l, the columns associated with the B, AB, $\ldots$, $A^{l-1}B$ terms in the controllability matrix should provide

n linearly independent columns, but the columns associated with the B, AB, $\ldots$, $A^{l-2}B$ terms could not.

As a result, to have the the reachability index equal to 1, B should provide 2 linearly independent columns. Therefore, the reachability index is 1, when $a \neq 0$ and $b \neq 0$.

(b) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system has a reachability index of 2.

Solution: To determine reachability index of the system, we need to check on the columns of the controllability matrix. The controllability matrix

$$\mathcal{C}(A,B) = \left[\begin{array}{c|c} B & AB \end{array}\right] = \left[\begin{array}{cc|cc} 0 & b & a & 0 \\ a & 0 & 0 & 0.25b \end{array}\right],$$

where A and B are the state and the input matrices of the system, respectively; and since

$$AB = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0.25b \end{bmatrix}.$$

In order for an nth order system to have the reachability index equal to l, the columns associated with the B, AB, $\ldots$, $A^{l-1}B$ terms in the controllability matrix should provide n linearly independent columns, but the columns associated with the B, AB, $\ldots$, $A^{l-2}B$ terms could not.

As a result, to have the the reachability index equal to 2, $\mathcal{C}(A,B)$ should provide 2 linearly independent columns, but B should not. In order B not to provide 2 linearly independent columns, either $a = 0$ or $b = 0$.

If $a = 0$, then $b \neq 0$; since $\mathcal{C}(A,B)$ should provide 2 linearly independent columns. Similarly, if $b = 0$, then $a \neq 0$.

Therefore, the reachability index is 2; when either $a = 0$ and $b \neq 0$, or $a \neq 0$ and $b = 0$.

(c) Determine the necessary and sufficient conditions in terms of a, b, and c, such that the system is observable.

Solution: To determine the observability of the system, the rank of the observability matrix needs to be full. The observability matrix

$$\mathcal{O}(C,A) = \left[\begin{array}{c} C \\ \hline CA \end{array}\right] = \left[\begin{array}{cc} c & 0 \\ \hline 0 & c \end{array}\right],$$

where A and C are the state and the output matrices of the system, respectively; and since

$$CA = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \end{bmatrix}.$$

In order for the rank of $\mathcal{O}(C,A)$ to be full, the determinant of $\mathcal{O}(C,A)$ should not be zero, since the observability matrix is a 2×2 square-matrix. So,

$$\det\big(\mathcal{O}(C,A)\big) = \det \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \neq 0,$$

or $c^2 \neq 0$. Therefore, the system is observable, when $c \neq 0$.

3. A discrete-time linear control system is described by

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.05 & 0.6 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k),$$

where u, $\boldsymbol{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine the transfer function $Y(z)/U(z)$ of the system, where $U = \mathcal{Z}[\,u\,]$ and $Y = \mathcal{Z}[\,y\,]$.

Solution: The transfer matrix of a control system described in the state-state representation

$$\dot{\boldsymbol{x}}(k+1) = A\boldsymbol{x}(k) + Bu(k),$$
$$y(k) = C\boldsymbol{x}(k) + Du(k),$$

is

$$F(z) = C(zI - A)^{-1}B + D,$$

where I is the appropriately dimensioned identity matrix. So,

$$F(z) = \begin{bmatrix} 1 & -1 \end{bmatrix} \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.05 & 0.6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z & -1 \\ 0.05 & z-0.6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{1}{z(z-0.6)+0.05} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z-0.6 & 1 \\ -0.05 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{1}{z^2 - 0.6z + 0.05} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{-z+1}{z^2 - 0.6z + 0.05} + 1 = \frac{z^2 - 1.6z + 1.05}{z^2 - 0.6z + 0.05}.$$

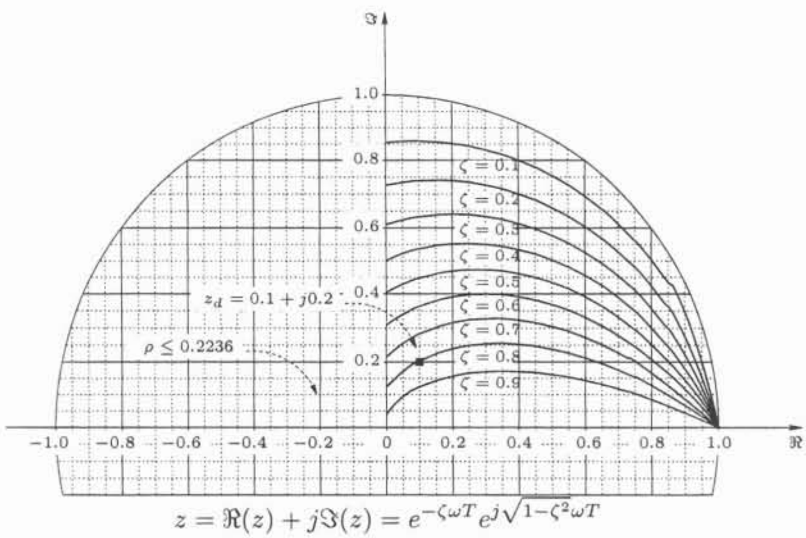
In other words, the transfer function is $F(z) = (z^2 - 1.6z + 1.05)/(z^2 - 0.6z + 0.05)$.

(b) Design a full state-feedback controller; such that the maximum percent overshoot is between 1.5% and 4%, and the 5% settling-time is reached in 3 sampling periods.

Solution: We determine the restrictions on the location of the desired-pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent-overshoot for the unit-step input	$1.5\% = 0.015 \leq M_p \leq 0.04 = 4\%$.	From the α-M_p curves, $\zeta = 0.8$ provides a range of α values that may satisfy the requirement.
Settling time for the unit-step input	$\rho \leq (0.05)^{1/(k_{5\%s}-1)}$.	For $t_{5\%s} = k_{5\%s}T \leq 3T$, and $k_{5\%s} \leq 3$; $\rho \leq (0.05)^{1/(3-1)} = 0.2236$.

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.1 \pm j0.2$.



Based on our choice of the desired-pole locations, the desired characteristic polynomial is given by

$$q_{c_d}(z) = \big(z - (0.1 + j0.2)\big)\big(z - (0.1 - j0.2)\big) = z^2 - 0.2z + 0.05.$$

We would like to place the closed-loop poles at the desired location via state-feedback control. So assume

$$u(k) = K\boldsymbol{x}(k) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \boldsymbol{x}(k)$$

for some state-feedback matrix K. The characteristic polynomial of the system under state-feedback control can be determined from the denominator of the transfer function, such that

$$q_c(z) = \det\big(zI - (A + BK)\big)$$

$$= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.05 & 0.6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right)$$

$$= z^2 + (-k_2 - 0.6)z + (-k_1 + 0.05).$$

Setting $q_c(z) = q_{c_d}(z)$, we get

$$-k_1 + 0.05 = 0.05,$$

or $k_1 = 0$; and

$$-k_2 - 0.6 = -0.2,$$

or $k_2 = -0.4$. Therefore,

$$u(k) = \begin{bmatrix} 0 & -0.4 \end{bmatrix} \boldsymbol{x}(k).$$

(c) Assuming that only the output is available, implement the controller of the previous part.

Solution: When only the output is available, state-feedback control can still be implemented if an observer is used. Moreover, we know that if a system is observable, we can place the closed-loop poles of the observer at any desired location via error-feedback control. So assume

$$\boldsymbol{e}(k) = L\big(\hat{y}(k) - y(k)\big) = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\big(\hat{y}(k) - y(k)\big)$$

for some observer-error gain matrix L, where $\hat{y}$ is the observer output variable. Assuming that the observer poles are at 0.01 and 0.01, the desired observer-characteristic polynomial

$$q_{o_d}(z) = (z - 0.01)(z - 0.01) = z^2 - 0.02z + 0.0001.$$

The observer-characteristic polynomial q_o under the error-feedback control can be determined from the denominator of the transfer function of the observer, such that

$$q_o(z) = \det\big(zI - (A + LC)\big)$$

$$= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.05 & 0.6 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}\right)\right)$$

$$= z^2 + (-l_1 + l_2 - 0.6)z + (0.55l_1 - l_2 + 0.05).$$

Setting $q_o(z) = q_{o_d}(z)$, we get

$$-l_1 + l_2 - 0.6 = -0.02,$$

and

$$0.55l_1 - l_2 + 0.05 = 0.0001.$$

In matrix form, we get

$$\begin{bmatrix} -1 & 1 \\ 0.55 & -1 \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 0.58 \\ -0.0499 \end{bmatrix},$$

and after solving for $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$, we obtain

$$e(k) = \begin{bmatrix} -1.178 \\ -0.598 \end{bmatrix} \big(\hat{y}(k) - y(k)\big),$$

where e and $\hat{y}$ are the error-feedback control and the observer output variables, respectively.

4. Consider a system described by the difference equation

$$x(k+1) = 2x(k) + u(k),$$

where x and u are the state and the input variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = 10x^2(4) + \sum_{k=0}^{3} \frac{1}{2}\left(x^2(k) + 5u^2(k)\right),$$

when $x(0) = -1$.

Solution: The Hamiltonian for this cost function and the system is

$$H_k\big(x(k), u(k), \lambda^*(k+1)\big) = \frac{1}{2}\big(x^2(k) + 5u^2(k)\big) + \lambda(k+1)\big(2x(k) + u(k)\big),$$

where λ is the Lagrange multiplier. The optimality conditions in terms of the Hamiltonian are

$$\lambda(k) = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial x(k)} = x(k) + 2\lambda(k+1) \text{ for } 0 \leq k \leq 3,$$

$$0 = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial u(k)} = 5u(k) + \lambda(k+1) \text{ for } 0 \leq k \leq 3$$

$$x(k+1) = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial \lambda(k+1)} = 2x(k) + u(k) \text{ for } 0 \leq k \leq 3.$$

From the above optimality equations, we get

$$\lambda(k+1) = -(1/2)x(k) + (1/2)\lambda(k),$$

and

$$\begin{aligned} x(k+1) &= 2x(k) + u(k) = 2x(k) + \big(-(1/5)\lambda(k+1)\big) \\ &= 2x(k) - (1/5)\big(-(1/2)x(k) + (1/2)\lambda(k)\big) = (21/10)x(k) - (1/10)\lambda(k). \end{aligned}$$

Or, in matrix form

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}.$$

One of the boundary conditions is given as $x(0) = -1$, and the other one needs to be determined from the terminal constraint, such that

$$\left(\frac{\partial g\big(N, \boldsymbol{x}(N)\big)}{\partial \boldsymbol{x}(N)} - \boldsymbol{\lambda}(N)\right)^T = 0,$$

where N is the final time step, and g is the additional terminal cost. Since, in our case, $N = 4$ and $g\big(N, \boldsymbol{x}(N)\big) = 10x^2(4)$, we get

$$\lambda(4) = \frac{\mathrm{d}\big(10x^2(4)\big)}{\mathrm{d}x(4)} = 20x(4).$$

Next, we need to solve the above matrix equation to determine $\lambda(0)$. Since,

$$\begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix}^2 = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} = \begin{bmatrix} (223/50) & -(13/50) \\ -(13/50) & (3/10) \end{bmatrix},$$

$$\begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix}^4 = \begin{bmatrix} (223/50) & -(13/50) \\ -(13/50) & (3/10) \end{bmatrix} \begin{bmatrix} (223/50) & -(13/50) \\ -(13/50) & (3/10) \end{bmatrix}$$

$$= \begin{bmatrix} (25287/1250) & -(1547/1250) \\ -(1547/250) & (107/250) \end{bmatrix},$$

$$\begin{bmatrix} x(4) \\ \lambda(4) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix}^4 \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix},$$

$x(0) = -1$, and $\lambda(4) = 20x(4)$; we get

$$x(4) = -(25287/1250)\big(-1\big) - (1547/1250)\lambda(0),$$

$$20x(4) = -(1547/250)\big(-1\big) + (107/250)\lambda(0),$$

or $\lambda(0) = -(20539/1259) \approx -16.31$.

Since $u(k) = -(1/5)\lambda(k+1)$ for $k = 0, \ldots, 3$ from the optimality condition, we need to determine $\lambda(k)$ for $k = 1, \ldots, 4$.

$$\begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} -0.47 \\ -7.66 \end{bmatrix}.$$

$$\begin{bmatrix} x(2) \\ \lambda(2) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = \begin{bmatrix} -0.22 \\ -3.59 \end{bmatrix}.$$

$$\begin{bmatrix} x(3) \\ \lambda(3) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} x(2) \\ \lambda(2) \end{bmatrix} = \begin{bmatrix} -0.10 \\ -1.68 \end{bmatrix}.$$

$$\begin{bmatrix} x(4) \\ \lambda(4) \end{bmatrix} = \begin{bmatrix} (21/10) & -(1/10) \\ -(1/2) & (1/2) \end{bmatrix} \begin{bmatrix} x(3) \\ \lambda(3) \end{bmatrix} = \begin{bmatrix} -0.04 \\ -0.79 \end{bmatrix}.$$

From $u(k) = -(1/5)\lambda(k+1)$ for $k = 0, \ldots, 3$, we get

$$u(0) = 1.53,\ u(1) = 0.72,\ u(2) = 0.34,\ \text{and } u(3) = 0.16.$$