EE 431 **Exam#1** **Oct. 23, 2003**

75 minutes

1. Assume

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Determine a set of vector(s) that span the subspace that is orthogonal to the subspace spanned by the vectors **a** and **b**. (15pts)

2. A control system is described by

$$Y(s) = \begin{bmatrix} \dfrac{s+1}{s^2+s+1} & \dfrac{2s}{s^2+s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},$$

where $U = \mathcal{L}\left[u\right] = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right]^T$ and $Y = \mathcal{L}\left[y\right]$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than two state variables. (25pts)

3. An autonomous control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 5 & -3 & -1 \\ 8 & -6 & 0 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable.

(a) Determine the transformation that would put the system into a diagonal or Jordan form. (25pts)

(b) Determine $\mathbf{x}(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \left[\begin{array}{ccc} 1 & 0 & 1 \end{array}\right]^T$. (25pts)

4. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system. (10pts)

EE 431 **Exam#1** **Oct. 23, 2003**

Solutions

1. Assume

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Determine a set of vector(s) that span the subspace that is orthogonal to the subspace spanned by the vectors **a** and **b**.

Solution: Since, the vectors **a** and **b** are from a 3-dimensional Euclidean vector space, there can only be three linearly-independent vectors. Since **a** and **b** are linearly independent, there is only one orthogonal vector to **a** and **b**. To find this vector, we may start with the usual basis vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

and remove the components of **a** and **b** from each of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$; until we get a non-zero vector. Once we get a non-zero vector, we don't need to proceed any further. But first, we need to orthogonalize the vectors **a** and **b**. From the Gram-Schmidt orthogonalization procedure, we get

$$\mathbf{b}' = \mathbf{b} - \frac{\langle \mathbf{b}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}\mathbf{a} = \mathbf{b} - \frac{\mathbf{b}^T\mathbf{a}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{(0)(1) + (1)(2) + (1)(1)}{(1)^2 + (2)^2 + (1)^2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

We may scale the vector $\mathbf{b}'$ to get integer elements and use $\mathbf{b}'' = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$. Second, we remove the components of **a** and $\mathbf{b}''$ from $\mathbf{e}_1$. (Note that we could have also chosen the vector $\mathbf{e}_2$ or $\mathbf{e}_3$. Indeed, if we get the zero vector with our choice of $\mathbf{e}_1$, we will try another one.)

$$\mathbf{n} = \mathbf{e}_1 - \frac{\langle \mathbf{e}_1, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}\mathbf{a} - \frac{\langle \mathbf{e}_1, \mathbf{b}'' \rangle}{\langle \mathbf{b}'', \mathbf{b}'' \rangle}\mathbf{b}''$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{6}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}.$$

Similarly, we scale the vector **n** to get the orthogonal vector $\mathbf{n}' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$. So, the set of vector(s) that spans the subspace that is orthogonal to the subspace spanned by the vectors **a** and **b** is

$$\{ \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T \}.$$

2. A control system is described by

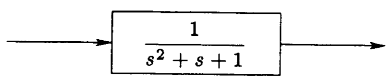
$$Y(s) = \begin{bmatrix} \frac{s+1}{s^2+s+1} & \frac{2s}{s^2+s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},$$

where $U = \mathcal{L}[u] = [U_1 \;\; U_2]^T$ and $Y = \mathcal{L}[y]$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than two state variables.

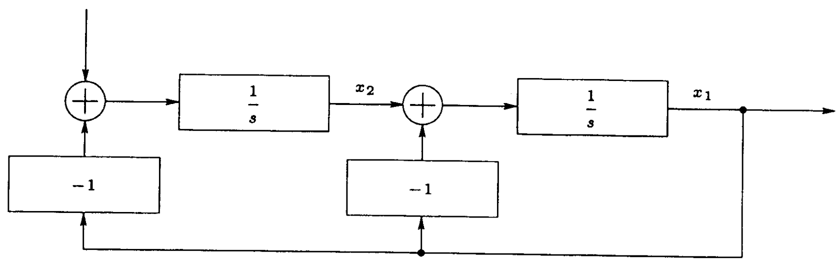
Solution: From the transfer matrix, we observe that there is only one output to the system. As a result, we may be able to factor out the common denominator and realize the system such that

$$Y(s) = [s^2+s+1]^{-1} \begin{bmatrix} s+1 & 2s \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

When the denominator polynomial is realized in the observer realization form, the two elements in the numerator matrix can be generated independently as the feedforward terms in the observer realization form.

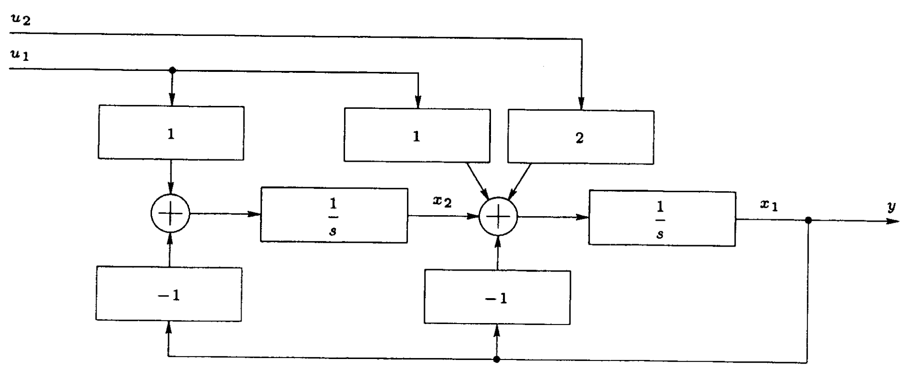


(a) The feedback portion.



(b) Observer realization form.

When the two elements in the numerator matrix are generated as the feedforward terms, we get



Next, we assign the state variables as shown in the figure and obtain

$$\dot{x}_1 = -x_1 + x_2 + u_1 + 2u_2,$$

$$\dot{x}_2 = -x_1 + u_1,$$

and

$$y = x_1.$$

After expressing the above equations in matrix form, we get the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

3. An autonomous control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 5 & -3 & -1 \\ 8 & -6 & 0 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}$ is the state variable.

(a) Determine the transformation that would put the system into a diagonal or Jordan form.

Solution: In order to find the desired transformation, we need to first determine the eigenvalues and the eigenvectors of the system. For the state matrix

$$A = \begin{bmatrix} 5 & -3 & -1 \\ 8 & -6 & 0 \\ 1 & -1 & -1 \end{bmatrix},$$

the eigenvalues are obtained from the solution to the characteristic equation $\det(sI - A) = 0$, where I is the appropriately dimensioned identity matrix. In our case,

$$\det(sI - A) = s^3 + 2s^2 - 4s - 8 = (s+2)^2(s-2).$$

So, the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -2$, and $\lambda_3 = 2$.

Next, we determine an eigenvector $\mathbf{v}_i$ associated with the eigenvalue λ_i from the equation $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ or $(A - \lambda_i I)\mathbf{v}_i = 0$ for $i = 1, 2, 3$.

For $\lambda_{1,2} = -2$, we have

$$(A + 2I)\mathbf{v}_{1,2} = \begin{bmatrix} 7 & -3 & -1 \\ 8 & -4 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} {v_{1,2}}^1 \\ {v_{1,2}}^2 \\ {v_{1,2}}^3 \end{bmatrix} = 0,$$

or

$$\begin{aligned} 7{v_{1,2}}^1 - 3{v_{1,2}}^2 - {v_{1,2}}^3 &= 0, \\ 8{v_{1,2}}^1 - 4{v_{1,2}}^2 \phantom{- {v_{1,2}}^3} &= 0, \\ {v_{1,2}}^1 - {v_{1,2}}^2 + {v_{1,2}}^3 &= 0. \end{aligned}$$

From the second equation, we get ${v_{1,2}}^2 = 2{v_{1,2}}^1$; and from the first or the third equation, we get ${v_{1,2}}^3 = {v_{1,2}}^1$. Since, we have only one degree of freedom, we only obtain one eigenvector for the repeated eigenvalue -2, and the transformed system will be in the Jordan form. The eigenvector is in the form

$$\mathbf{v}_1 = \begin{bmatrix} \alpha & 2\alpha & \alpha \end{bmatrix}^T \text{ for some non-zero scalar } \alpha.$$

We obtain another linearly independent vector for the transformation from the equation $(A - \lambda_{1,2}I)\mathbf{v}_2 =$

$$(A+2I)\mathbf{v}_2 = \begin{bmatrix} 7 & -3 & -1 \\ 8 & -4 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} {v_2}^1 \\ {v_2}^2 \\ {v_2}^3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \end{bmatrix},$$

or

$$\begin{aligned} 7{v_2}^1 - 3{v_2}^2 - {v_2}^3 &= \alpha, \\ 8{v_2}^1 - 4{v_2}^2 \qquad &= 2\alpha, \\ {v_2}^1 - {v_2}^2 + {v_2}^3 &= \alpha. \end{aligned}$$

From the second equation, we get ${v_2}^2 = 2{v_2}^1 - \alpha/2$; and from the first or the third equation, we get ${v_2}^3 = {v_2}^1 + \alpha/2$. In this case, the eigenvector is in the form

$$\mathbf{v}_2 = \begin{bmatrix} \beta & (2\beta - \alpha/2) & (\beta + \alpha/2) \end{bmatrix}^T \text{ for some scalar } \beta.$$

For $\lambda_3 = 2$, we have

$$(A-2I)\mathbf{v}_3 = \begin{bmatrix} 3 & -3 & -1 \\ 8 & -8 & 0 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} {v_3}^1 \\ {v_3}^2 \\ {v_3}^3 \end{bmatrix} = 0,$$

or

$$\begin{aligned} 3{v_3}^1 - 3{v_3}^2 - {v_3}^3 &= 0, \\ 8{v_3}^1 - 8{v_3}^2 \qquad &= 0, \\ {v_3}^1 - {v_3}^2 - 3{v_3}^3 &= 0. \end{aligned}$$

From the second equation, we get ${v_3}^1 = {v_3}^2$; and from the first or the third equation, we get ${v_3}^3 = 0$. The eigenvector is in the form

$$\mathbf{v}_3 = \begin{bmatrix} \gamma & \gamma & 0 \end{bmatrix}^T \text{ for some non-zero scalar } \gamma.$$

The transformation that would put the system into a Jordan form is in general given by

$$T = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma \\ 2\alpha & 2\beta - \alpha/2 & \gamma \\ \alpha & \beta + \alpha/2 & 0 \end{bmatrix}.$$

One particular case is when $\alpha = 2$, $\beta = 0$, and $\gamma = 1$, so that the corresponding transformation is

$$T = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

(b) Determine $\mathbf{x}(t)$ for $t \geq 0$, when $\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$.

Solution: We may determine the state variable $\mathbf{x}(t)$ using various methods including Laplace transform, Cayley-Hamilton theorem, or block-diagonalizing transformation. For a given state matrix A and a transformation matrix T,

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = Te^{T^{-1}ATt}T^{-1}\mathbf{x}(0).$$

For a block-diagonalizing transformation T, the determination of $e^{T^{-1}ATt}$ is trivial. In our case, for

$$T = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

we get

$$T^{-1}AT = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and

$$e^{T^{-1}ATt} = \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

As a result, once we determine T^{-1}, we can obtain $\mathbf{x}(t)$ from the above formulation. One method to determine the inverse of T is to use row operations on the augmented matrix $\begin{bmatrix} T & I \end{bmatrix}$ to generate $\begin{bmatrix} I & T^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 4 & -1 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/2 & -1/2 & -1/2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & -1/2 & -1/2 \end{array}\right].$$

Then the state variable

$$\mathbf{x}(t) = Te^{T^{-1}ATt}T^{-1}\mathbf{x}(0)$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1/4 & 1/4 & 1/4 \\ 1/2 & -1/2 & 1/2 \\ 3/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} te^{-2t} \\ e^{-2t} \\ e^{2t} \end{bmatrix}.$$

Therefore,

$$\mathbf{x}(t) = \begin{bmatrix} 2te^{-2t} + e^{2t} \\ 4te^{-2t} - e^{-2t} + e^{2t} \\ 2te^{-2t} + e^{-2t} \end{bmatrix} \quad \text{for } t \geq 0.$$

4. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine the transfer function or the transfer matrix of the system.

Solution: The transfer matrix of a control system described in the state-state representation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

is

$$F(s) = C(sI - A)^{-1}B + D,$$

where

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad D = 0,$$

and I is the appropriately dimensioned identity matrix. So,

$$F(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ s+2 \end{bmatrix}$$

$$= \frac{1}{(s+2)^2} (s+2).$$

In other words, the transfer function is $F(s) = 1/(s+2)$.