EE 431 **Exam#1** **Oct. 26, 2004**

75 minutes

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \pi.$$

Determine $\sin(A)$ and $\cos(A)$. (20pts)

2. A control system is described by

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^2+s+1} \\ \dfrac{2s}{s^2+s+1} \end{bmatrix} U(s),$$

where $U = \mathcal{L}[u]$ and $Y = \mathcal{L}[y] = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}^T$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than two state variables. (25pts)

3. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine $\mathbf{x}(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, and $u(t)$ is the unit step function. (30pts)

4. The dynamical equations of a nonlinear control system are given by

$$\dot{x}_1(t) = -2x_1(t) + {x_2}^2(t),$$

$$\dot{x}_2(t) = {x_2}^3(t) - x_2(t),$$

where x_1 and x_2 are the state variables.

(a) Determine all the equilibrium states of the system. (05pts)

(b) Check the local stability of the system using Lyapunov's first method for all the equilibrium states. (20pts)

EE 431

Exam#1
Solutions

Oct. 26, 2004

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \pi.$$

Determine $\sin(A)$ and $\cos(A)$.

Solution: Any function, that has a non-trivial taylor series expansion, of an nth order matrix can be determined either by its taylor series expansion, where

$$f(A) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i} \right]_{x=0} A^i,$$

or by the use of the Cayley-Hamilton's theorem and application of the eigenvectors, where

$$f(A) = \sum_{i=0}^{n-1} \alpha_i A^i$$

for some scalars α_i, $i = 0, \ldots, n-1$. The scalars are determined by the application of the eigenvectors to the above equation that results in the set of equations

$$f(\lambda_1) = \alpha_0 + \alpha_1 \lambda_1 + \ldots + \alpha_{n-1} \lambda_1^{n-1}$$

$$\vdots$$

$$f(\lambda_n) = \alpha_0 + \alpha_1 \lambda_n + \ldots + \alpha_{n-1} \lambda_n^{n-1},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. In our case, $n = 2$; so

$$f(A) = \alpha_0 I + \alpha_1 A;$$

and since the matrix A is in upper-diagonal form, the eigenvalues are easily observed from the diagonal, such that $\lambda_1 = \lambda_2 = 0 = \lambda$. However, when an eigenvalue is repeated, we get the same equation more than once. In such a case, we use the derivatives of the equations for the repeated eigenvalue with respect to the eigenvalue, or

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda_i^k} \left(f(\lambda_i) = \alpha_0 + \alpha_1 \lambda_i + \ldots + \alpha_{n-1} \lambda_i^{n-1} \right)$$

for $k = 1, \ldots, r$, where r is the number of repetitions of the eigenvalue λ_i. In our case, the set of equations becomes

$$f(\lambda) = \alpha_0 + \alpha_1 \lambda,$$

and

$$\mathrm{d}f(\lambda)/\mathrm{d}\lambda = \alpha_1.$$

For $f = \sin$, and $\lambda = 0$; we get

$$\sin(0) = \alpha_0 + \alpha_1 (0),$$

and

$$\cos(0) = \alpha_1.$$

Solving the above set of equations simultaneously gives $\alpha_0 = 0$ and $\alpha_1 = 1$. As a result, we get

$$\sin(A) = A.$$

For $f = \cos$; we get

$$\cos(0) = \alpha_0 + \alpha_1(0),$$

and

$$-\sin(0) = \alpha_1.$$

Similarly, solving the above set of equations gives $\alpha_0 = 1$ and $\alpha_1 = 0$. So, we get

$$\cos(A) = I.$$

2. A control system is described by

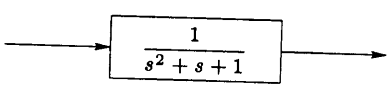
$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^2+s+1} \\ \dfrac{2s}{s^2+s+1} \end{bmatrix} U(s),$$

where $U = \mathcal{L}[u]$ and $Y = \mathcal{L}[y] = [\, Y_1 \;\; Y_2 \,]^T$ are the input and the output variables, respectively. Obtain a state-space representation of the system with no more than two state variables.

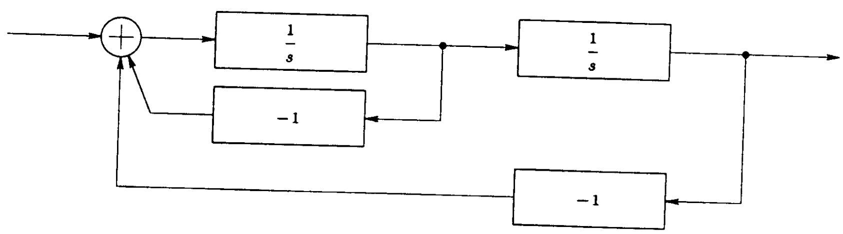
Solution: From the transfer matrix, we observe that there is only one input to the system. As a result, we may be able to factor out the common denominator and realize the system such that

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s+1 \\ 2s \end{bmatrix} \left[s^2+s+1\right]^{-1} U(s).$$

When the denominator polynomial is realized in the controller realization form, the two elements in the numerator matrix can be generated independently as the feedforward terms in the controller realization form.

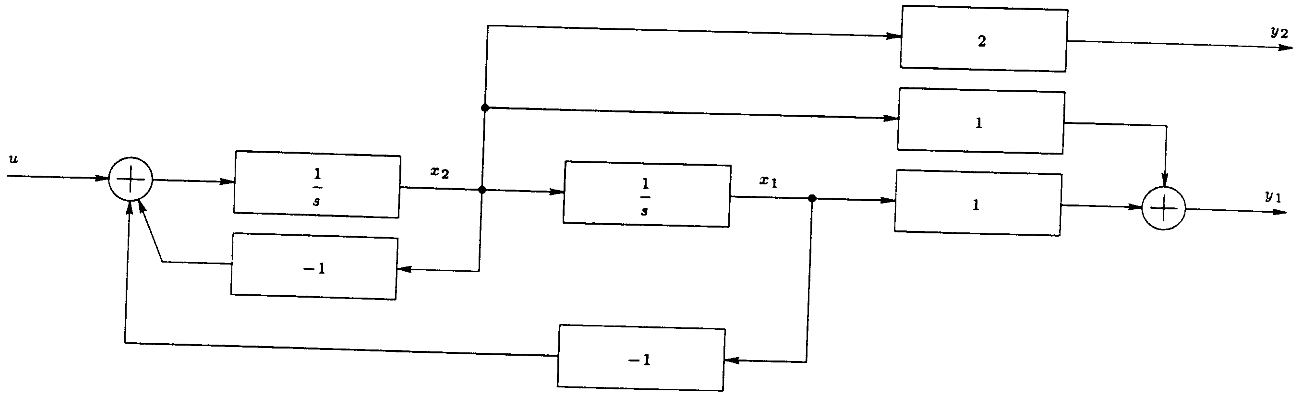


(a) The feedback portion.



(b) Controller realization form.

When the two elements in the numerator matrix are generated as the feedforward terms, we get



Next, we assign the state variables as shown in the figure and obtain

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - x_2 + u,$$

and

$$y_1 = x_1 + x_2,$$
$$y_2 = 2x_2.$$

After expressing the above equations in matrix form, we get the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t).$$

3. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine $\mathbf{x}(t)$ for $t \geq 0$; when $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, and $u(t)$ is the unit step function.

Solution: The solution to a control system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\, d\tau$$

for $t \geq 0$. To determine e^{At}, we may use a few different methods. However, in our case the matrix A is in jordan form, and e^{At} may be written directly, where

$$e^{At} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}.$$

Therefore, the state-variable is given by

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & (t-\tau)e^{-(t-\tau)} & 0 \\ 0 & e^{-(t-\tau)} & 0 \\ 0 & 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (1)\, d\tau \\
&= \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} \\ 0 \\ e^{-2(t-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-(t-\tau)} \\ 0 \\ (1/2)e^{-2(t-\tau)} \end{bmatrix}_{\tau=0}^{\tau=t} \\
&= \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} 1-e^{-t} \\ 0 \\ (1/2)(1-e^{-2t}) \end{bmatrix},
\end{aligned}$$

or

$$\mathbf{x}(t) = \begin{bmatrix} 1-e^{-t} \\ 0 \\ (1+e^{-2t})/2 \end{bmatrix} \quad \text{for } t \geq 0.$$

4. The dynamical equations of a nonlinear control system are given by

$$\dot{x}_1(t) = -2x_1(t) + {x_2}^2(t),$$
$$\dot{x}_2(t) = {x_2}^3(t) - x_2(t),$$

where x_1 and x_2 are the state variables.

(a) Determine all the equilibrium states of the system.

Solution: Equilibrium states of the system $\mathbf{x}_e = \begin{bmatrix} x_{1e} & x_{2e} \end{bmatrix}^T$ are the states that would make $\dot{\mathbf{x}}_e = 0$. Setting $\dot{x}_1 = \dot{x}_2 = 0$ in the system equations, we get

$$-2x_{1e} + {x_{2e}}^2 = 0,$$
$${x_{2e}}^3 - x_{2e} = 0.$$

From the second equation, we get

$${x_{2e}}^3 - x_{2e} = x_{2e}({x_{2e}}^2 - 1) = 0,$$

or $x_{2e} = 0$ or $x_{2e} \pm 1$. For $x_{2e} = 0$, we get $x_{1e} = 0$ from the first equation; and for $x_{2e} = \pm 1$, we get $x_{1e} = 1/2$. Therefore, the equilibrium states are $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\begin{bmatrix} (1/2) & 1 \end{bmatrix}^T$, and $\begin{bmatrix} (1/2) & -1 \end{bmatrix}^T$.

(b) Check the local stability of the system using Lyapunov's first method for all the equilibrium states.

Solution: Lyapunov's first stability theorem checks the stability of the linearized system about the equilibrium states to decide on the local stability. For a nonlinear system described by

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix},$$

the linearized system is given by

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \vdots \\ \dot{\tilde{x}}_n \end{bmatrix} = \begin{bmatrix} \partial f_1(x_1, \dots, x_n)/\partial x_1 & \cdots & \partial f_1(x_1, \dots, x_n)/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n(x_1, \dots, x_n)/\partial x_1 & \cdots & \partial f_n(x_1, \dots, x_n)/\partial x_n \end{bmatrix}_* \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix}.$$

In our case, $f_1(x_1, x_2) = -2x_1 + x_2{}^2$, and $f_2(x_1, x_2) = x_2{}^3 - x_2$. The linearized system is given by

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \partial f_1(x_1, x_2)/\partial x_1 & \partial f_1(x_1, x_2)/\partial x_2 \\ \partial f_2(x_1, x_2)/\partial x_1 & \partial f_2(x_1, x_2)/\partial x_2 \end{bmatrix}_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2x_2 \\ 0 & 3x_2{}^2 - 1 \end{bmatrix}_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

The $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ equilibrium state case: In this case, the linearized system becomes

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Since the state-transition matrix is diagonal, we observe that the eigenvalues are -1 and -2. From Lyapunov's first stability theorem, if all the eigenvalues of the linearized system have negative real parts, then the equilibrium state is locally, asymptotically stable. Therefore, the equilibrium state $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is locally, asymptotically stable.

The $\begin{bmatrix} (1/2) & 1 \end{bmatrix}^T$ equilibrium state case: In this case, the linearized system becomes

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Since the state-transition matrix is upper-diagonal, we observe that the eigenvalues are -1 and 2. From Lyapunov's first stability theorem, if at least one of the eigenvalues of the linearized system has a positive real part, then the equilibrium state is locally unstable. Therefore, the equilibrium state $\begin{bmatrix} (1/2) & 1 \end{bmatrix}^T$ is locally unstable.

The $\begin{bmatrix} (1/2) & -1 \end{bmatrix}^T$ equilibrium state case: In this case, the linearized system becomes

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Since the state-transition matrix is also upper-diagonal, we observe that the eigenvalues are again -1 and 2. Therefore, the equilibrium state $\begin{bmatrix} (1/2) & -1 \end{bmatrix}^T$ is locally unstable.