1. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(10pts)

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

(15pts)

2. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} \mathbf{u}(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Obtain its Kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system. (25pts)

3. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{2s+1}{s(s+1)^2} & \dfrac{2s+1}{(s+1)^2} \\[2ex] \dfrac{1}{s(s+1)^2} & \dfrac{1}{(s+1)^2} \end{bmatrix}.$$

Obtain its left coprime factorization, such that $H = D^{-1}N$ for some matrices D and N, and the matrix N is in Hermite form. (25pts)

4. A continuous-time linear control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Design an output feedback controller for the system, such that the 2% settling time is less than 2 seconds, and the output response is over-damped. (25pts)

1. A control system is described in state-space representation, such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For the following A, B, C, and D matrices, determine whether the system is asymptotically stable, marginally stable, or unstable; and whether it is bounded-input-bounded-output stable or not. Justify your answer.

(a)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since, the state matrix A is diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is diagonal, the two zero-valued eigenvalues don't effect each other, or they are not cascaded. Therefore, each of the eigenvalues λ_1 and λ_2 would generate a constant response resulting in a marginally stable response. Since there are no more eigenvalues, we conclude that the system is marginally stable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\mathbf{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\mathbf{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. In our case,

$$C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1/(s+1) \end{bmatrix} = \frac{1}{s+1}.$$

We realize that only one pole of the system is visible in the transfer function; and it has a negative real part. Therefore, the system is marginally stable, and it is bounded-input-bounded-output stable.

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } D = 0.$$

Solution: Since the state matrix A is upper diagonal, we observe the eigenvalues directly from the diagonal elements as $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$. The eigenvalue λ_3 has a negative real part, and it would generate an asymptotically stable response. The eigenvalues λ_1 and λ_2 are both zero, and each would generate a constant response individually. However, if they are cascaded, a constant response generated by the first one would result in a ramp response by the second one. In this case, since the state matrix A is in Jordan form, the two zero-valued eigenvalues are cascaded, and they effect each other. Therefore, the state corresponding to λ_2 would generate a constant response, and the state corresponding to λ_1 would then generate a ramp response. As a result, we conclude that the system is unstable.

In order to determine the bounded-input-bounded-output stability of the system, we may determine the transfer function and observe the poles of the system after all the reductions. The transfer function of the system is given by

$$\mathcal{L}\left[\mathbf{y}\right](s) = \left(C(sI - A)^{-1}B + D\right)\mathcal{L}\left[\mathbf{u}\right](s)$$

where $\mathcal{L}\left[(\cdot)\right](s)$ is the Laplace transform, and I is the appropriately dimensioned identity matrix. One method to determine the inverse of $(sI - A)$ is to use row operations on the augmented matrix $\begin{bmatrix} (sI-A) & I \end{bmatrix}$ to generate $\begin{bmatrix} I & (sI-A)^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 0 & 0 & s+1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 1/s & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/s & 1/s^2 & 0 \\ 0 & 1 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/(s+1) \end{array}\right].$$

Therefore,

$$C(sI-A)^{-1}B+D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 1/s^2 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1/(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s^2 \\ 1/s \\ 0 \end{bmatrix} = \frac{1}{s^2} + \frac{1}{s} + 0 = \frac{s+1}{s^2}.$$

Because of the repeated pole at zero, the impulse response would contain a ramp function; and as a result the system is not bounded-input-bounded-output stable.

In summary, the system is unstable, and it is not bounded-input-bounded-output stable.

2. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} \mathbf{u}(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Obtain its Kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system.

Solution: In order to determine the Kalman decomposition of a system, we first need to obtain the controllability and the observability matrices. In an nth order system that is described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively; the observability matrix for $n = 3$ is given by

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ \hline CA \\ \hline CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ \hline 1 & 0 & -1 \\ \hline 0 & 1 & -1 \end{bmatrix}.$$

The rank of the observability matrix is 3. As a result, there will be no unobservable part in the Kalman decomposition, and there is no need for a contribution to the transformation from the observability matrix.

The controllability matrix for $n = 3$ is given by

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} B \mid AB \mid A^2B \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & 2 & 0 & 2 & 0 & 4 \\ -1 & 0 & 0 & 2 & 0 & 4 \\ -1 & 0 & 0 & 2 & 0 & 4 \end{array}\right].$$

The rank of the controllability matrix is 2, since the second and the third rows are identical. Therefore, there are only 2 linearly independent columns. One of them is $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ from the second column, and the other one is $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ from the fourth or the sixth column.

These two vectors can easily be orthonormalized either by the Gram-Schmidt orthogonalization procedure or in this case by inspection to $\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\bar{\mathbf{v}}_2 = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$.

The controllable subspace of the transformation, that would separate the controllable and uncontrollable portions, is spanned by the orthonormal vectors $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$, and the uncontrollable subspace of the transformation is orthogonal to the controllable subspace. In our case, the dimension of the transformation is 3, and the dimension of the controllable subspace is 2. As a result, the dimension of the uncontrollable subspace is 1. In other words, we only need to determine only one normal vector that is linearly independent or orthogonal to $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$.

To find this vector $\bar{\mathbf{v}}_3$, we may start with the usual basis vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

and remove the components of $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ from each of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, until we get a non-zero vector. Once we get one non-zero vector, we don't need to proceed any further.

Here, since $\bar{\mathbf{v}}_1 = \mathbf{e}_1$, there is no need to even try $\mathbf{e}_1$. So, we remove the components of $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ from $\mathbf{e}_3$. (Note that we could have also chosen the vector $\mathbf{e}_2$. Indeed, if we get the zero vector with our choice of $\mathbf{e}_3$, we will try $\mathbf{e}_2$.)

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{e}_3 - \frac{\langle \mathbf{e}_3 , \bar{\mathbf{v}}_1 \rangle}{\langle \bar{\mathbf{v}}_1 , \bar{\mathbf{v}}_1 \rangle}\bar{\mathbf{v}}_1 - \frac{\langle \mathbf{e}_3 , \bar{\mathbf{v}}_2 \rangle}{\langle \bar{\mathbf{v}}_2 , \bar{\mathbf{v}}_2 \rangle}\bar{\mathbf{v}}_2 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - (0) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (\sqrt{2}/2) \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{v}_3$ is non-zero, we have the desired vector. After normalizing the vector $\mathbf{v}_3$, we obtain $\bar{\mathbf{v}}_3 = \begin{bmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$. The transformation that will separate the controllable and the uncontrollable portions is $\mathbf{x} = T\bar{\mathbf{x}}$, where

$$T = \begin{bmatrix} \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \bar{\mathbf{v}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Since the columns of T are orthonormal to each other, $T^{-1} = T^T$; we have

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= T^T A T \bar{\mathbf{x}}(t) + T^T B \mathbf{u}(t), \\ \mathbf{y}(t) &= C T \bar{\mathbf{x}}(t) + D\mathbf{u}(t). \end{aligned}$$

Note here that if we had chosen any linearly independent vector instead of $\bar{\mathbf{v}}_3$, the transpose of the transformation matrix would not have been its inverse. For our system,

$$\dot{\bar{\mathbf{x}}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \bar{\mathbf{x}}(t)$$
$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \bar{\mathbf{x}}(t),$$

or

$$\dot{\bar{\mathbf{x}}}(t) = \begin{bmatrix} 1 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \bar{\mathbf{x}}(t) + \begin{bmatrix} 1 & 2 \\ -\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & -\sqrt{2} \end{bmatrix} \bar{\mathbf{x}}(t).$$

Since the first two dimensions contain the controllable subspace the Kalman decomposition of the system will be

$$\dot{\bar{\mathbf{x}}}(t) = \left[\begin{array}{c} \bar{A}_{\substack{\text{controllable} \\ \text{observable}}} \\ \hline \bar{A}_{\substack{\text{uncontrollable} \\ \text{observable}}} \end{array} \right] \bar{\mathbf{x}}(t) + \left[\begin{array}{c} \bar{B}_{\substack{\text{controllable} \\ \text{observable}}} \\ \hline 0 \end{array} \right] \mathbf{u}(t)$$
$$= \left[\begin{array}{ccc} 1 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 1 & 1 \\ \hline 0 & 0 & 2 \end{array} \right] \bar{\mathbf{x}}(t) + \left[\begin{array}{cc} 1 & 2 \\ -\sqrt{2} & 0 \\ \hline 0 & 0 \end{array} \right] \mathbf{u}(t),$$

$$\mathbf{y}(t) = \left[\bar{C}_{\substack{\text{controllable} \\ \text{observable}}} \right] \bar{\mathbf{x}}(t)$$
$$= \begin{bmatrix} 1 & 0 & -\sqrt{2} \end{bmatrix} \bar{\mathbf{x}}(t),$$

where $\bar{\mathbf{x}}$ is the transformed state variable.

3. The transfer matrix of a control system is given by

$$H(s) = \begin{bmatrix} \dfrac{2s+1}{s(s+1)^2} & \dfrac{2s+1}{(s+1)^2} \\ \dfrac{1}{s(s+1)^2} & \dfrac{1}{(s+1)^2} \end{bmatrix}.$$

Obtain its left coprime factorization, such that $H = D^{-1}N$ for some matrices D and N, and the matrix N is in Hermite form.

Solution: We first obtain an initial left factorization of $H = D_0^{-1}N_0$, such that

$$H(s) = \begin{bmatrix} \dfrac{2s+1}{s(s+1)^2} & \dfrac{2s+1}{(s+1)^2} \\ \dfrac{1}{s(s+1)^2} & \dfrac{1}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s(s+1)^2} & 0 \\ 0 & \dfrac{1}{s(s+1)^2} \end{bmatrix} \begin{bmatrix} 2s+1 & s(2s+1) \\ 1 & s \end{bmatrix}$$

$$= D_0^{-1}(s)N_0(s).$$

From the above factorization, we get

$$D_0 = \begin{bmatrix} s(s+1)^2 & 0 \\ 0 & s(s+1)^2 \end{bmatrix}.$$

Next, we form an augmented matrix from N_0 and D_0, and perform row operations until we obtain the Hermite form.

$$\left[\, N_0(s) \mid D_0(s) \,\right] = \left[\begin{array}{cc|cc} 2s+1 & s(2s+1) & s(s+1)^2 & 0 \\ 1 & s & 0 & s(s+1)^2 \end{array}\right].$$

Multiplying the second row by $-(2s+1)$ and adding it to the first row, we get

$$\left[\, N_1(s) \mid D_1(s) \,\right] = \left[\begin{array}{cc|cc} 0 & 0 & s(s+1)^2 & -(2s+1)s(s+1)^2 \\ 1 & s & 0 & s(s+1)^2 \end{array}\right].$$

Dividing the first row by $s(s+1)^2$, we get

$$\left[\, N_2(s) \mid D_2(s) \,\right] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -(2s+1) \\ 1 & s & 0 & s(s+1)^2 \end{array}\right].$$

The last row operation resulted in a $N(s)$ that is in Hermite form, so $N(s) = N_2(s)$, and $D(s) = D_2(s)$. Since

$$D^{-1}(s) = \begin{bmatrix} 1 & -(2s+1) \\ 0 & s(s+1)^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \dfrac{2s+1}{s(s+1)^2} \\ 0 & \dfrac{1}{s(s+1)^2} \end{bmatrix},$$

we get

$$H(s) = D^{-1}(s)N(s) = \begin{bmatrix} 1 & \dfrac{2s+1}{s(s+1)^2} \\ 0 & \dfrac{1}{s(s+1)^2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & s \end{bmatrix}.$$

4. A continuous-time linear control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Design an output feedback controller for the system, such that the 2% settling time is less than 2 seconds, and the output response is over-damped.

Solution: We determine the desired system closed-loop poles from the system requirements.

Given Requirements	General System Restrictions	Specific System Restrictions
2% settling-time for a unit-step input	$t_{2\%s} \leq 2\,\mathrm{s}$, or $\frac{4}{\sigma_o} \leq 2.$	$\sigma_o \geq 2$, since $t_{2\%s} = 4/\sigma_o$.
The output response is over-damped.	The closed-loop poles are distinct and real.	$\text{pole}_1 = p_{d_1} = -\sigma_1$, $\text{pole}_2 = p_{d_2} = -\sigma_2$; where $\sigma_1 \neq \sigma_2$.

From the given requirements, we choose $\sigma_1 = 2$, and $\sigma_2 = 4 > 2 = \sigma_1$. The desired characteristic polynomial p_{c_d} can be obtained from the desired-pole locations, where

$$p_{c_d}(s) = (s - (-2))(s - (-4)) = s^2 + 6s + 8.$$

The characteristic polynomial p_c under state-feedback gain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$, such that the input $u = -K\mathbf{x}$, can be determined from

$$\begin{aligned} p_c(s) &= \det\big(sI - (A - BK)\big) \\ &= \det\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) \\ &= s^2 + (k_2 + 2)z + (k_1 - 1). \end{aligned}$$

Setting $p_c(s) = p_{c_d}(s)$, we get

$$k_1 - 1 = 8,$$

or $k_1 = 9$; and

$$k_2 + 2 = 6,$$

or $k_2 = 4$. Therefore,

$$K = \begin{bmatrix} 9 & 4 \end{bmatrix}.$$

However, since only the output, not the state variable, is available, we need to design an observer and use the observer state variable $\hat{\mathbf{x}}$ instead of the state variable x.

The desired observer-characteristic polynomial p_{o_d} can be obtained from the desired observer-pole locations. Since there is no explicit specifications, we may choose the two desired observer-pole locations ourselves. Choosing both poles at -10 that is faster than the system poles, we get the desired observer characteristic polynomial

$$p_{o_d}(s) = (s+10)(s+10) = s^2 + 20s + 100.$$

The observer-characteristic polynomial p_o under the observer-error gain $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$ can be determined from

$$\begin{aligned} p_o(s) &= \det\big(sI - (A - LC)\big) \\ &= \det\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\right)\right) \\ &= s^2 + (l_1 + l_2 + 2)s + (3l_1 + l_2 - 1). \end{aligned}$$

Setting $p_o(s) = p_{o_d}(s)$, we get

$$3l_1 + l_2 - 1 = 100,$$

and

$$l_1 + l_2 + 2 = 20.$$

Solving the two equations for l_1 and l_2, we get

$$L = \begin{bmatrix} 41.5 \\ -23.5 \end{bmatrix}.$$

Therefore,

$$u(t) = \begin{bmatrix} -9 & -4 \end{bmatrix} \hat{\mathbf{x}}(t) \text{ for } t \geq 0,$$

where

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + Bu(t) - \begin{bmatrix} 41.5 \\ -23.5 \end{bmatrix}\big(y(t) - C\hat{\mathbf{x}}(t)\big),$$

and A, B, and C are the state, the input, and the output matrices of the system, respectively.