EE 431 | **Exam#2** 75 minutes | **Dec. 09, 1997**

1. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & -3 \\ 4 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t),$$

where $\mathbf{x}$ and $\mathbf{u}$ are the input and the state variables, respectively. Determine the control $\mathbf{u}$ that would transfer the states from $\mathbf{x}(0_-) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to $\mathbf{x}(0_+) = \begin{bmatrix} 4 & -1 \end{bmatrix}^T$. (20pts)

2. A control system is realized in a state-space form, such that

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t),$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{u}(t),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively. Determine whether or not this realization is minimal. (25pts)

3. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 2 \\ -8 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} (1-t) & t \end{bmatrix} \mathbf{x}(t).$$

(a) Determine whether or not $\mathbf{x}(0)$ can be obtained from $y(t)$ for $0 \le t \le 1$. (25pts)

(b) Determine whether or not $\mathbf{x}(1) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ can be reached from $\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. (05pts)

4. A linear control system is described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

where $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. The exponential function of At is given by

$$e^{At} = \begin{bmatrix} e^{-t} & 2te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ te^{-t} & (t^2-3t)e^{-t} & e^{-t} \end{bmatrix}.$$

Determine the Lyapunov function $L(\mathbf{x})$ for the autonomous system, such that

$$\frac{\mathrm{d}L(\mathbf{x})}{\mathrm{d}t} = \mathbf{x}^T \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

(25pts)

#1)

$$\dot{x} = \begin{bmatrix} -5 & -3 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} u$$

$$x(0_-) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x(0_+) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

We need impulse inputs to change the states instantaneously. Since the system is of order 2, $u(t) = u_0\delta(t) + u_1\delta'(t)$ should be sufficient if the system is controllable. After substituting $u(t)$ in the solution for $x(t)$, we get

$$x(t) = e^{At}\left\{ x(0_-) + \begin{bmatrix} B & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix} \right\}$$

In our case,

$$\begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = x(0_+) - x(0_-)$$

or $\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} B & AB \end{bmatrix}^{\dagger} \left(x(0_+) - x(0_-) \right)$

$$= \begin{bmatrix} 1 & 2 & -5 & -10 \\ 0 & 0 & 4 & 8 \end{bmatrix}^{\dagger} \left(\begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 2 & -5 & -10 \\ 0 & 0 & 4 & 8 \end{bmatrix}^{\#} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -5 & 4 \\ -10 & 8 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -5 & -10 \\ 0 & 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -5 & 4 \\ -10 & 8 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -5 & 4 \\ -10 & 8 \end{bmatrix} \begin{bmatrix} 130 & -100 \\ -100 & 80 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -5 & 4 \\ -10 & 8 \end{bmatrix} \frac{1}{400} \begin{bmatrix} 80 & 100 \\ 100 & 130 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 8 & 10 \\ 16 & 20 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 11 \\ 22 \\ -1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad u(t) = \begin{bmatrix} 11/20 \\ 11/10 \end{bmatrix} \delta(t) - \begin{bmatrix} 1/20 \\ 1/10 \end{bmatrix} \delta'(t)$$

#2

$$\dot{x} = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \end{bmatrix} u$$

Minimal realization when system is controllable and observable

OBSERVABILITY from $\mathcal{O}(C,A) = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}$

or $\mathcal{O}(C,A) = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 2 & 3 \\ 16 & 16 & 18 \end{bmatrix} \leftarrow$ full rank so observable

CONTROLLABILITY from $\mathcal{C}(A,B) = [B \; \cdots \; A^{n-1}B]$

or $\mathcal{C}(A,B) = \left[\begin{array}{cc|cc|cc} 0 & 1 & 2 & 3 & 6 & 19 \\ 1 & 0 & 0 & 5 & 10 & 15 \\ 0 & 1 & 2 & 3 & 6 & 19 \end{array}\right]$ same row

rank $\mathcal{C}(A,B) = 2$

i.e. not controllable

$\Rightarrow$ NOT MINIMAL

Indeed, in this case

$$(sI-A)^{-1} = \frac{1}{s^3+s^2-22s-40}\begin{bmatrix} s^2+2 & 2s+8 & 4s-2 \\ 6s-3 & s^2+s-12 & -s+23 \\ 3s+12 & 2s+8 & s^2+s-12 \end{bmatrix}$$

poles $= -4, -2, 5$

and

$$C(sI-A)^{-1}B+D = \frac{1}{s^2-3s-10}\begin{bmatrix} 2s^2-5s-21 \\ s^2-2s-5 \end{bmatrix}$$

pole at -4 gets canceled out by C & B

#3 $\dot{x} = \begin{bmatrix} -5 & 2 \\ -8 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = [\,(1-t) \quad t\,]\, x$

(a) $x(0)$ obtainability from $y(t)$ is the property of observability.

Observability of time varying system from

$$\theta(t_0,t_f) = \int_{t_0}^{t_f} \Phi(t_f,t_0)^T C^T(t)\, C(t)\, \Phi(t_f,t_0)\, dt$$

$\Phi(t,t_0) = e^{A(t-t_0)}$ in our case

The eigenvalues of A from $\det(\lambda I - A) = 0$

or $\det \begin{bmatrix} \lambda+5 & -2 \\ 8 & \lambda-3 \end{bmatrix} = (\lambda+5)(\lambda-3) - (-2)(8)$

$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

$= 0$ or $\lambda = -1, -1$

e^{At} from $e^{At} = aI + bA$ (since $n=2$)

and $e^{\lambda t} = a + b\lambda$

$t e^{\lambda t} = b$ (since λ is repeated)

or $e^{-t} = a - b$

$t e^{-t} = b$ $\Longrightarrow$ $a = (t+1)e^{-t}$

$$e^{At} = (t+1)e^{-t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} -5 & 2 \\ -8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-4t+1)e^{-t} & 2te^{-t} \\ -8te^{-t} & (4t+1)e^{-t} \end{bmatrix}$$

$$C(t)e^{At} = \begin{bmatrix} (1-t) & t \end{bmatrix}\begin{bmatrix} -4t+1 & 2t \\ -8t & 4t+1 \end{bmatrix}e^{-t}$$

$$= \begin{bmatrix} ((1-t)(-4t+1)-8t^2) & (1-t)2t+t(4t+1) \end{bmatrix}e^{-t}$$

$$= \begin{bmatrix} (-4t^2-5t+1) & (2t+3)t \end{bmatrix}e^{-t}$$

$$\vartheta(0,1) = \begin{bmatrix} \int_0^1 (-4t^2-5t+1)^2 e^{-2t}\,dt & \int_0^1 t(2t+3)(-4t^2-5t+1)e^{-2t}\,dt \\ \int_0^1 t(2t+3)(-4t^2-5t+1)e^{-2t}\,dt & \int_0^1 t^2(2t+3)^2 e^{-2t}\,dt \end{bmatrix}$$

If rank $\vartheta(0,1) = 2$, then the system is observable

(b) $x(1)$ being reached from $x(0)$ is determined by the controllability property

CONTROLLABILITY from $\mathcal{C}(A,B) = [B \; AB \cdots A^{n-1}B]$

In our case $\mathcal{C}(A,B) = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \leftarrow$ rank $=2$

so controllable

#4 For a linear system, Lyapunov function is

$L(x) = x^T P x$ where $A^T P + PA = -Q$

where Q is a positive definite matrix

when A is asympt. stable

Moreover, $P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$

since $Q = I$ and $e^{At} = \begin{bmatrix} e^{-t} & 2te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ te^{-t} & (t^2-3t)e^{-t} & e^{-t} \end{bmatrix}$

$$P = \int_0^\infty \begin{bmatrix} 1 & 0 & t \\ 2t & 1 & t^2-3t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2t & 0 \\ 0 & 1 & 0 \\ t & t^2-3t & 1 \end{bmatrix} e^{-2t}\, dt$$

$$= \int_0^\infty \begin{bmatrix} t^2+1 & t^3-3t^2+2t & t \\ t^3-3t^2+2t & 4t^2+1+(t^2-3t)^2 & t^2-3t \\ t & t^2-3t & 1 \end{bmatrix} e^{-2t}\, dt$$

$$= \begin{bmatrix} \int_0^\infty (t^2+1)e^{-2t}dt & \int_0^\infty (t^3-3t^2+2t)e^{-2t}dt & \int_0^\infty te^{-2t}dt \\ \int_0^\infty (t^3-3t^2+2t)e^{-2t}dt & \int_0^\infty (t^4-6t^3+13t^2+1)e^{-2t}dt & \int_0^\infty (t^2-3t)e^{-2t}dt \\ \int_0^\infty te^{-2t}dt & \int_0^\infty (t^2-3t)e^{-2t}dt & \int_0^\infty e^{-2t}dt \end{bmatrix}$$