EE 432 | **Exam#1** | Mar. 02, 2006

Solutions

1. Find the extremal of the functional

$$J(x) = \int_0^1 \dot{x}^2 \,\mathrm{d}t$$

with $x(0) = 0$, $x(1) = 1/4$, and

$$K(x) = \int_0^1 x \,\mathrm{d}t = 1.$$

Solution: The extremal to the cost function

$$J(x, \dot{x}) = \int_{t_0}^{t_f} \Phi(t, x, \dot{x}) \,\mathrm{d}t = \int_0^1 \dot{x}^2 \,\mathrm{d}t$$

with the integral constraint

$$K(x, \dot{x}) = \int_{t_0}^{t_f} \Lambda(t, x, \dot{x}) \,\mathrm{d}t = \int_0^1 x \,\mathrm{d}t = 1$$

is the solution to the Euler-Lagrange's equation

$$\big(\Phi_x - \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\dot{x}}\big) + \lambda\big(\Lambda_x - \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_{\dot{x}}\big) = 0,$$

for a constant λ provided that the extremal exists and it is not the extremal of $K(x, \dot{x})$ as well. In our case,

$$\big(0 - \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{x})\big) + \lambda\big(1 - \frac{\mathrm{d}}{\mathrm{d}t}(0)\big) = 0,$$

or

$$-2\ddot{x} + \lambda = 0.$$

The solution to the above differential equation is

$$x(t) = (\lambda/4)t^2 + c_1 t + c_2,$$

for some constants c_1 and c_2. We need to determine the unknown constants from the boundary conditions. At $t = 0$, $x(0) = 0$, so

$$\big[(\lambda/4)t^2 + c_1 t + c_2\big]_{t=0} = 0,$$

or $c_2 = 0$. At $t = 1$, $x(1) = 1/4$, so

$$\big[(\lambda/4)t^2 + c_1 t + c_2\big]_{t=1} = 0,$$

or $c_1 = (1 - \lambda)/4$. In other words,

$$x(t) = (\lambda/4)t^2 + (1 - \lambda)/4t.$$

We need to determine the constant λ from the additional constraint, where

$$K(x) = \int_0^1 x\,\mathrm{d}t = \int_0^1 \left((\lambda/4)t^2 + (1-\lambda)/4t\right)\,\mathrm{d}t$$
$$= \left[(\lambda/12)t^3 + (1-\lambda)/8t^2\right]_{t=0}^{t=1} = \left[(\lambda/12) + (1-\lambda)/8\right] - \left[0\right] = 1,$$

or $\lambda = -21$. Therefore, the optimal solution is

$$x(t) = -(21/4)t^2 + (11/2)t \text{ for } 0 \le t \le 1.$$

2. Consider the cost function

$$J(\boldsymbol{x}, u) = \boldsymbol{x}^T(T)\begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1 \end{bmatrix}\boldsymbol{x}(T) + \int_0^T \left(\boldsymbol{x}^T\begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix}\boldsymbol{x} + (1/2)\,u^2\right)\,\mathrm{d}t.$$

and a continuous-time linear control-system described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t),$$

where u and $\boldsymbol{x}$ are the control and the state variables, respectively.

(a) Obtain the optimal feedback control that minimizes the cost function J for $T = 1$.

Solution: Since the finite-time cost function is quadratic in the state and the input variables, the optimal control can be expressed in state-feedback form, such that

$$u(t) = -R^{-1}B^TP(t)\boldsymbol{x}(t),$$

where R is from the cost function

$$J = \frac{1}{2}\boldsymbol{x}^T(t_f)S\boldsymbol{x}(t_f) + \int_0^{t_f}\frac{1}{2}\left(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right)\mathrm{d}t,$$

and P is the solution to the riccati equation

$$\dot{P}(t) = -A^TP(t) - P(t)A + P(t)BR^{-1}B^TP(t) - Q$$

with the end condition

$$P(t_f) = S$$

for the control system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t).$$

In our case, we have

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},\quad R = 1, \text{ and } S = \begin{bmatrix} 1 & 1/3 \\ 1/3 & 2 \end{bmatrix}.$$

Since P is symmetric, let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

Substituting all these matrices into the riccati equation, we get

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_2 & \dot{p}_3 \end{bmatrix} = -\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$+\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_2 & \dot{p}_3 \end{bmatrix} = \begin{bmatrix} p_1 & -p_1+p_2 \\ p_2 & -p_2+p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ -p_1+p_2 & -p_2+p_3 \end{bmatrix}$$

$$+\begin{bmatrix} p_1 & 0 \\ p_2 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

We get

$$\dot{p}_1 = 2p_1 + {p_1}^2 - 3,$$
$$\dot{p}_2 = -p_1 + 2p_2 + p_1 p_2,$$

and

$$\dot{p}_3 = -2p_2 + 2p_3 + {p_2}^2 - 3$$

from the $(1,1)$, $(1,2)$ (or $(2,1)$), and $(2,2)$ terms of the matrix equation, respectively. From the equation in the $(1,1)$ term, we have

$$\dot{p}_1 = 2p_1 + {p_1}^2 - 3 = (p_1+3)(p_1-1)$$

with $p_1(1) = 1$. Solving the above differential equation, we get

$$\int \left(\frac{1}{(p_1+3)(p_1-1)}\right) \mathrm{d}p_1 = \int \mathrm{d}t,$$

$$\int \left(\frac{-1/4}{p_1+3} - \frac{1/4}{p_1-1}\right) \mathrm{d}p_1 = t + a,$$

$$\ln\left(\frac{p_1-1}{p_1+3}\right) = 4t + b,$$

or

$$\frac{p_1-1}{p_1+3} = c\, e^{4t}.$$

Substituting $p_1(1) = 1$, we get $c = 0$, or

$$\frac{p_1-1}{p_1+3} = 0.$$

So, $p_1(t) = 1$ for $0 \le t \le 1$.

Similarly, from the equation in the $(1,2)$ (or $(2,1)$) term, we have

$$\dot{p}_2 = -p_1 + 2p_2 + p_1 p_2 = 3p_2 - 1$$

with $p_2(1) = 1/3$. Solving the above differential equation, we get

$$p_2(t) = d\, e^{3t} + 1/3.$$

Substituting $p_2(1) = 1/3$, we get $d = 0$. So, $p_2(t) = 1/3$ for $0 \leq t \leq 1$.

Finally, from the equation in the $(2,2)$ term, we have

$$\dot{p}_3 = -2p_2 + 2p_3 + {p_2}^2 - 3 = 2p_3 - 32/9$$

with $p_3(1) = 2$. Solving the above differential equation, we get

$$p_3(t) = e\, e^{2t} + 16/9.$$

Substituting $p_3(1) = 2$, we get $e = (2/9)e^{-2}$. So,

$$p_3(t) = (2/9)\left(8 + e^{2(t-1)}\right)$$

for $0 \leq t \leq 1$. As a result,

$$P(t) = \begin{bmatrix} 1 & 1/3 \\ 1/3 & (2/9)\left(8 + e^{2(t-1)}\right) \end{bmatrix}$$

for $0 \leq t \leq 1$. Therefore, the optimal control is

$$u(t) = -R^{-1}B^T P \boldsymbol{x}(t) = -(1)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 1/3 & (2/9)\left(8 + e^{2(t-1)}\right) \end{bmatrix} \boldsymbol{x}(t),$$

or

$$u(t) = -\begin{bmatrix} 1 & 1/3 \end{bmatrix} \boldsymbol{x}(t) = -\begin{bmatrix} 1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ for } 0 \leq t \leq 1.$$

(b) Determine the optimal cost J^* for an arbitrary initial state, when $T = 1$.

Solution: For the finite-time quadratic cost function with the state-feedback control, the optimal cost

$$J^* = \frac{1}{2}\boldsymbol{x}^T(0)P(0)\boldsymbol{x}(0),$$

where $\boldsymbol{x}$ is the state variable, and P is the solution to the riccati equation. In our case,

$$P(t) = \begin{bmatrix} 1 & 1/3 \\ 1/3 & (2/9)\left(8 + e^{2(t-1)}\right) \end{bmatrix}$$

for $0 \leq t \leq 1$. So,

$$P(0) = \begin{bmatrix} 1 & 1/3 \\ 1/3 & (2/9)\left(8 + e^{-2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0.3333 \\ 0.3333 & 1.8079 \end{bmatrix}$$

and for $\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^T$, the optimal cost

$$J^* = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix} \begin{bmatrix} 1 & 0.3333 \\ 0.3333 & 1.8079 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = {x_1}^2(0) + 0.6667x_1(0)x_2(0) + 1.8079{x_2}^2(0).$$

3. Find the optimal control that will minimize the cost function

$$J(x_1, x_2) = \int_0^T (1/2)\left({x_1}^2 + {x_2}^2\right)\,\mathrm{d}t,$$

and transfer the initial state $\boldsymbol{x}(0) = \begin{bmatrix} -4 & 0 \end{bmatrix}^T$ to the final state $\boldsymbol{x}(T) = \begin{bmatrix} 4 & 0 \end{bmatrix}^T$ for the control system described by

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t),$$

where u and $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ are the control and the state variables, respectively, provided that $|u(t)| \le 1$ for $t \ge 0$.

Solution: In this problem, the finite-time cost function is quadratic in the state variables, but the input variable is missing. As a result, we need to use the Pontryagin's optimality condition to determine the optimal control.

The Hamiltonian for a system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t).$$

with the cost function

$$J(\boldsymbol{x}, \boldsymbol{u}) = \int_0^T \phi(t, \boldsymbol{x}, \boldsymbol{u})\,\mathrm{d}t,$$

is given by

$$H\big(t, \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\lambda}\big) = \phi(t, \boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{\lambda}^T\big(A\boldsymbol{x} + B\boldsymbol{u}\big),$$

where $\boldsymbol{u}$ and $\boldsymbol{x}$ are the input and the state variables, respectively, and $\boldsymbol{\lambda}$ is the langrange multiplier. In our case,

$$H\big(t, \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{\lambda}\big) = (1/2){x_1}^2 + (1/2){x_2}^2 + \lambda_1 x_1 + \lambda_2 u,$$

where $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}^T$.

The optimality conditions in terms of the Hamiltonian are

$$\dot{\boldsymbol{x}} = H_{\boldsymbol{\lambda}}; \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = u;$$

$$\dot{\boldsymbol{\lambda}} = -H_{\boldsymbol{x}}; \quad \dot{\lambda}_1 = -x_1, \quad \dot{\lambda}_2 = -x_2 - \lambda_1;$$

and

$$\Big[H\Big]_{\substack{u=u^* \\ \boldsymbol{x}=\boldsymbol{x}^* \\ \boldsymbol{\lambda}=\boldsymbol{\lambda}^*}} \le \Big[H\Big]_{\substack{\boldsymbol{x}=\boldsymbol{x}^* \\ \boldsymbol{\lambda}=\boldsymbol{\lambda}^*}}; \quad (1/2){x_1^*}^2 + (1/2){x_2^*}^2 + \lambda_1^* x_2^* + \lambda_2^* u^* \le (1/2){x_1^*}^2 + (1/2){x_2^*}^2 + \lambda_1^* x_2^* + \lambda_2^* u.$$

$$\text{or } \lambda_2^* u^* \le \lambda_2^* u,$$

where $(\cdot)^*$ designates the optimal values. From the last optimality condition, we get

$$u^* = -\operatorname{sgn}(\lambda_2).$$

In order to determine the optimal trajectory, we need to analyze the response when $u = \pm 1$. For $u = \pm 1$, we get

$$\begin{aligned} \dot{x}_1(t) &= x_2 \\ \dot{x}_2(t) &= \pm 1, \end{aligned}$$

or

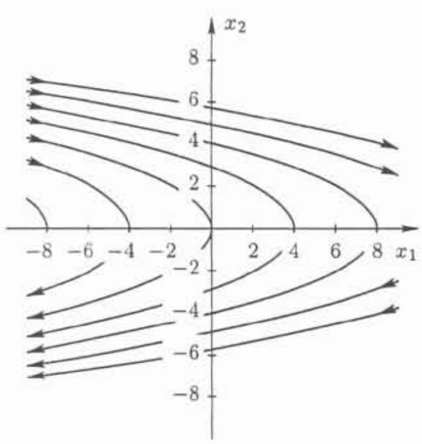
$$\begin{aligned} x_1(t) &= \pm t^2/2 + c_1 t + c_2 \\ x_2(t) &= \pm t + c_1, \end{aligned}$$

for $t \geq 0$ and for some constants c_1 and c_2. To get a state trajectory, we eliminate the time variable by solving for t. From the second equation, we have $t = \pm(x_2 - c_1)$, and

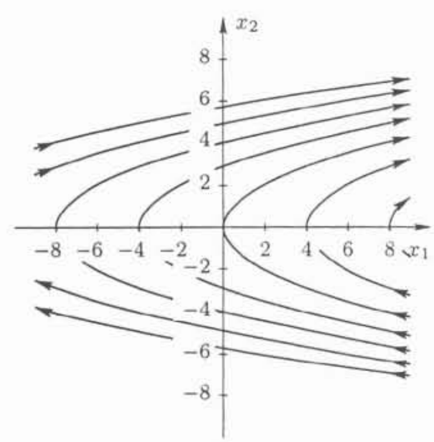
$$\begin{aligned} x_1 &= \pm(x_2 - c_1)^2/2 \pm c_1(x_2 - c_1) + c_2 \\ &= \pm(1/2)\big((x_2 - c_1)^2 + 2c_1(x_2 - c_1) + {c_1}^2\big) + c_3 \\ &= \pm(1/2)\big((x_2 - c_1) + c_1\big)^2 + c_3 \\ &= \pm(1/2){x_2}^2 + c_3 \end{aligned}$$

Therefore, the state trajectories are parabolas with vertices at $(c_3, 0)$. To see the direction of motion on the parabolas, we may check the extreme values of t as shown in the following table.

t	$(x_1, x_2)_{u=-1}$	$(x_1, x_2)_{u=+1}$
$-\infty$	$(-\infty, +\infty)$	$(+\infty, -\infty)$
$+\infty$	$(-\infty, -\infty)$	$(+\infty, +\infty)$



$u = -1$ case.



$u = 1$ case.

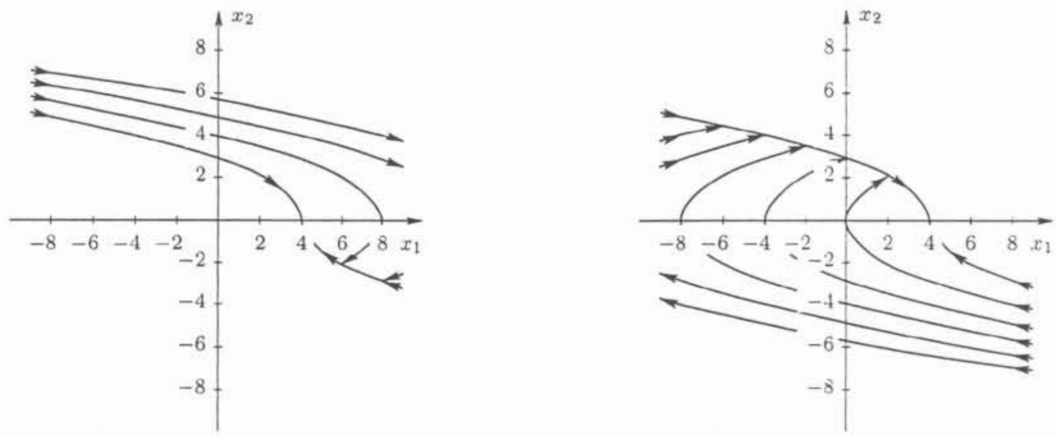
Since our destination is $\boldsymbol{x}(T) = \begin{bmatrix} 4 & 0 \end{bmatrix}^T$, the last switch is to be to the curves that go through $(4, 0)$, specifically

$$x_1 = -(1/2){x_2}^2 + 4, \text{ when } u = -1;$$

and

$$x_1 = (1/2){x_2}^2 + 4, \text{ when } u = 1.$$

To determine the control signal for each region, we choose the trajectories that intersect the above curves with different values of u as shown in the following figures. The first figure shows the region in the state trajectory, where the optimal control starts with $u = -1$; and when $x_1 = (1/2){x_2}^2 + 4$, that is shown by the thicker line in the first figure, the control is switched to $u = 1$. The second figure shows the region, where the optimal control starts with $u = 1$; and when $x_1 = -(1/2){x_2}^2 + 4$, that is shown by the thicker line in the second figure, the control is switched to $u = -1$.



(a) $u = -1$ to $u = 1$ case. (b) $u = 1$ to $u = -1$ case.

There is also the consideration of singularity intervals. We observe that the singularity intervals occur when $\lambda_2 = 0$ for a time period. In that time period, $\dot{\lambda}_2 = 0$, and as a result $\lambda_1 = -x_2$. In addition, we have

$$H^*|_{\substack{\lambda_1^*=-x_2^* \\ \lambda_2^*=0}} = \left[(1/2){x_1^*}^2 + (1/2){x_2^*}^2 + \lambda_1^* x_2^* + \lambda_2^* u^*\right]_{\substack{\lambda_1^*=-x_2^* \\ \lambda_2^*=0}} = (1/2)\left(x_1^* - x_2^*\right)\left(x_1^* + x_2^*\right) = 0,$$

since the final time is free. In either case, since $|u| \leq 1$, we get $|x_1| \leq 0$ and $|x_2| \leq 0$. Since those regions of state variables are not encountered to go from $\boldsymbol{x}(0) = \left[\,-4 \;\; 0\,\right]^T$ to $\boldsymbol{x}(T) = \left[\,4 \;\; 0\,\right]^T$: in our region of operation, we don't have a singular interval during our trajectory.

Starting at $(-4, 0)$, we first get on the trajectory $x_1 = (1/2){x_2}^2 + c_3$ with control $u = 1$; then switch to the trajectory $x_1 = -(1/2){x_2}^2 + 4$ with control $u = -1$ to reach the final destination $(4, 0)$. Solving for the constant c_3, such that the trajectory $x_1 = (1/2){x_2}^2 + c_3$ goes through the point $(-4, 0)$; we get $x_1 = (1/2){x_2}^2 - 4$. The intersection points of the trajectories $x_1 = (1/2){x_2}^2 - 4$ and $x_1 = -(1/2){x_2}^2 + 4$ are $(0, \pm 2\sqrt{2})$. In other words, the optimal switching solution is

$$\left\{ \begin{array}{l} x_1 = -4 \\ x_2 = 0 \end{array} \right\} \xrightarrow{u=+1} \left\{ \begin{array}{l} x_1 = 0 \\ x_2 = 2\sqrt{2} \end{array} \right\} \xrightarrow{u=-1} \left\{ \begin{array}{l} x_1 = 4 \\ x_2 = 0 \end{array} \right\}.$$