Mathematics 6215 Homework Set 3 Name: Dr. Grow

Solve the following problems from Royden and Fitzpatrick's *Real Analysis*.

p. 139: # 1, 3, 4 (second part only), 5
p. 143: # 7, 10, 12, 13, 14
p. 149: # 26, 32, 33, 34. Read # 35.

Also, solve each of these problems.

A. Let $p_0, p_1, \ldots, p_n$ be real numbers such that each $p_i > 1$, and

$$\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p_0}.$$

If, for each integer i between 1 and n, f_i belongs to $L^{p_i}(a,b)$, must it be the case that the product $f_1 \ldots f_n$ belongs to $L^{p_0}(a,b)$? Justify your answer.

B. (i) Let $p \in (0,\infty)$. Show that a measurable function f belongs to $L^p(0,1)$ if and only if

$$\sum_{n=1}^{\infty} m\left(\left\{x \in (0,1) : |f(x)|^p \geq n\right\}\right) < \infty.$$

(ii) Use part (i) to help determine the set of all $p \in (0,\infty)$ such that if f is any nonnegative, measurable function on $(0,1)$ satisfying

$$m\left(\left\{x \in (0,1) : f(x) \geq t\right\}\right) \leq \frac{1}{1+t^2}$$

for all $t > 0$, then $f \in L^p(0,1)$.

C. Let f and g belong to $L^2(-\infty,\infty)$. Define the convolution product $f * g$ by

$$(f * g)(x) = \int_{(-\infty,\infty)} f(y) g(x-y)\, dm(y)$$

for all real x, and define the translate of f by t according to

$$f_t(x) = f(x-t)$$

for all real x.

(1) Show that $\lim_{t \to 0} \|f_t - f\|_2 = 0$.

(2) Show that $f * g \in L^\infty(-\infty,\infty)$. (Do not neglect to check that $f * g$ is a measurable function!)

p. 150, #32. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence in $L^\infty(E)$ and $\sum_{k=1}^{\infty} a_k$ a convergent series of positive numbers such that $\|f_{k+1} - f_k\|_\infty \leq a_k$ for all k.

(a) Prove that there is a subset E_0 of E which has measure zero and

$$|f_{n+k}(x) - f_k(x)| \leq \|f_{n+k} - f_k\|_\infty \leq \sum_{j=k}^{\infty} a_j$$

for all k, n and all $x \in E \setminus E_0$.

(b) Conclude that there is a function $f \in L^\infty(E)$ such that $f_n \to f$ uniformly on $E \setminus E_0$.

(a) Let $E_k = \{x \in E : |f_{k+1}(x) - f_k(x)| > a_k\}$ for $k = 1, 2, 3, \ldots$ Since $\|f_{k+1} - f_k\|_\infty \leq a_k$, $m(E_k) = 0$ for all k and hence $E_0 = \bigcup_{k=1}^{\infty} E_k$ has measure $m(E_0) = 0$. If n and k are positive integers and $x \in E \setminus E_0$ then

$$\begin{aligned} |f_{n+k}(x) - f_k(x)| &\leq \sum_{j=k}^{n+k-1} |f_{j+1}(x) - f_j(x)| \\ &\leq \sum_{j=k}^{k+n-1} a_j \\ &< \sum_{j=k}^{\infty} a_j . \end{aligned}$$

[Note: We did not show $|f_{n+k}(x) - f_k(x)| \leq \|f_{n+k} - f_k\|_\infty$ for all k, n, and $x \in E \setminus E_0$.]

(b) Because $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive numbers, part (a) shows that $\langle f_k \rangle_{k=1}^{\infty}$ is a uniformly Cauchy sequence of functions on the set $E \setminus E_0$ and hence (by completeness of $\mathbb{R}$)

$$g(x) = \lim_{n\to\infty} f_n(x) = f_1(x) + \sum_{k=1}^{\infty}(f_{k+1}(x) - f_k(x)) \qquad (x \in E \setminus E_0)$$

exists and $f_n \to g$ uniformly on $E \setminus E_0$. Define a function f on E by

$$f(x) = \begin{cases} g(x) & \text{if} \quad x \in E \setminus E_0, \\ 0 & \text{if} \quad x \in E_0. \end{cases}$$

Clearly $f_n \to f$ uniformly on $E \setminus E_0$ and

$$f(x) = f_1(x) + \sum_{k=1}^{\infty}(f_{k+1}(x) - f_k(x)) \qquad (x \in E \setminus E_0)$$

so $|f(x)| \leq |f_1(x)| + \sum_{k=1}^{\infty} a_k \leq \|f_1\|_\infty + \sum_{k=1}^{\infty} a_k < \infty$ a.e.

on E. Hence $f \in L^\infty(E)$.

p. 150, #33. Use the preceding problem to show that $L^\infty(E)$ is a Banach space.

By the example on pp. 137–138, we know that $(L^\infty(E), \|\cdot\|_\infty)$ is a normed linear space. It remains only to show that $(L^\infty(E), \|\cdot\|_\infty)$ is complete. To this end, let $\langle f_n \rangle_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(E)$. Choose a subsequence $\langle f_{n_k} \rangle_{k=1}^\infty$ such that $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq \frac{1}{2^k}$ for all $k \geq 1$. By the preceding problem, there is a function f in $L^\infty(E)$ and set $E_0 \subseteq E$ with measure $m(E_0) = 0$ such that $f_{n_k} \to f$ uniformly on $E \setminus E_0$. We claim that $f_n \to f$ in $L^\infty(E)$. To see this, let $\varepsilon > 0$ and choose an integer $N_0 \geq 1$ such that $\|f_m - f_n\|_\infty < \frac{\varepsilon}{2}$ for all $m, n \geq N_0$. Choose an integer $n_k \geq N_0$ such that

$$|f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2} \text{ for all } x \in E \setminus E_0.$$

Since $m(E_0) = 0$, it follows that $\|f - f_{n_k}\|_\infty < \frac{\varepsilon}{2}$. Consequently, for all $n \geq N_0$ we have

$$\|f - f_n\|_\infty \leq \|f - f_{n_k}\|_\infty + \|f_{n_k} - f_n\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

p. 150, #34: Prove that ℓ^p is a Banach space for $1 \le p < \infty$.

Let $\{\xi^{(j)}\}_{j=1}^{\infty} = \{ \{\xi_\nu^{(j)}\}_{\nu=1}^{\infty} \}_{j=1}^{\infty}$ be a Cauchy sequence in ℓ^p.

Choose a subsequence $\{\xi^{(j_k)}\}_{k=1}^{\infty}$ such that $\| \xi^{(j_{k+1})} - \xi^{(j_k)} \|_p < \frac{1}{2^k}$

for $k = 1, 2, \ldots$. Because

$$|\xi_{\nu_0}^{(j_{k+1})} - \xi_{\nu_0}^{(j_k)}| \le \left(\sum_{\nu=1}^{\infty} |\xi_\nu^{(j_{k+1})} - \xi_\nu^{(j_k)}|^p \right)^{1/p} < \frac{1}{2^k}$$

for all $\nu_0 = 1, 2, \ldots$ and all $k = 1, 2, \ldots$ we have $\sum_{k=1}^{\infty} |\xi_{\nu_0}^{(j_{k+1})} - \xi_{\nu_0}^{(j_k)}| < 1$.

Therefore, for each $\nu = 1, 2, \ldots$ we may define a real (or complex) number by

$$\xi_\nu = \xi_\nu^{(j_1)} + \sum_{k=1}^{\infty} \left(\xi_\nu^{(j_{k+1})} - \xi_\nu^{(j_k)} \right) \quad \left(= \lim_{k \to \infty} \xi_\nu^{(j_k)} \right)$$

and put $\xi = \{\xi_\nu\}_{\nu=1}^{\infty}$. Note that

Completeness of $\mathbb{R}$ (or $\mathbb{C}$) is being used here.

$\xi_\nu - \xi_\nu^{(j_k)} = \sum_{l=k}^{\infty} (\xi_\nu^{(j_{l+1})} - \xi_\nu^{(j_k)})$ for all $\nu = 1,2,\ldots$ and $k=1,2,\ldots$ Therefore, by Minkowski's inequality for l^p, $\xi = \{\xi_\nu\}_{\nu=1}^{\infty}$ satisfies

$$\|\xi - \xi^{(j_k)}\|_p \le \sum_{l=k}^{\infty} \|\xi^{(j_{l+1})} - \xi^{(j_l)}\|_p \le \sum_{l=k}^{\infty} \frac{1}{2^l} = \frac{1}{2^{k-1}}.$$

Note in particular that $\xi \in l^p$.

We claim that $\xi^{(j)} \to \xi$ in l^p. To see this, let $\varepsilon > 0$ and choose an integer $N_0 \ge 1$ such that $\|\xi^{(m)} - \xi^{(n)}\|_p < \frac{\varepsilon}{2}$ for all $m, n \ge N_0$. Let $n \ge N_0$ and choose an integer $j_k \ge N_0$ such that $\frac{1}{2^{k-1}} < \frac{\varepsilon}{2}$. Then

$$\|\xi - \xi^{(n)}\|_p \le \|\xi - \xi^{(j_k)}\|_p + \|\xi^{(j_k)} - \xi^{(n)}\| < \frac{1}{2^{k-1}} + \frac{\varepsilon}{2} < \varepsilon.$$

<u>Theorem 7, p.148</u>: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^p(0,1)$, $1 \le p < \infty$, which converges almost everywhere to a function f in $L^p(0,1)$. Show that $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^p(0,1)$ if and only $\|f_n\|_p \longrightarrow \|f\|_p$.

Proof: By Fatou's lemma, we have (*) $\|f\|_p \le \liminf_{n\to\infty} \|f_n\|_p$.
Suppose that $f_n \to f$ in $L^p(0,1)$. Then

$$(**) \quad \limsup_{n\to\infty} \|f_n\|_p \le \limsup_{n\to\infty}(\|f\|_p + \|f_n - f\|_p) = \|f\|_p.$$

It follows from (*) and (**) that $\|f\|_p = \lim_{n\to\infty} \|f_n\|_p$.

Conversely, suppose $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Consider the sequence $g_n(x) = 2^p(|f_n(x)|^p + |f(x)|^p) - |f_n(x) - f(x)|^p$. Then each g_n is measurable, $g_n \geq 0$, and $g_n(x) \to 2^{p+1}|f(x)|^p$ a.e. in $(0,1)$. By Fatou's lemma,

$$\int_0^1 2^{p+1}|f(x)|^p dx \leq \liminf_{n\to\infty}\left(\int_0^1 2^p|f_n(x)|^p dx + \int_0^1 2^p|f(x)|^p dx - \int_0^1 |f_n(x) - f(x)|^p dx\right)$$

$$= \int_0^1 2^{p+1}|f(x)|^p dx + \liminf_{n\to\infty}\left(-\int_0^1 |f_n(x) - f(x)|^p dx\right)$$

$$= \int_0^1 2^{p+1}|f(x)|^p dx - \limsup_{n\to\infty}\int_0^1 |f_n(x) - f(x)|^p dx.$$

Therefore $\limsup_{n\to\infty}\int_0^1 |f_n(x) - f(x)|^p dx \leq 0$, and hence $f_n \to f$ in $L^p(0,1)$.

(B) (i) Let f be a measurable function on $(0,1)$, $p \in (0,\infty)$, and $E_n = \{x \in (0,1) : |f(x)|^p \geq n\}$ $(n=1,2,\ldots)$. Then $f \in L^p(0,1)$ if and only if $\sum_{n=1}^{\infty} m(E_n) < \infty$.

Note that $n \cdot m(E_n \setminus E_{n+1}) \leq \int_{E_n \setminus E_{n+1}} |f(x)|^p dx \leq (n+1) \cdot m(E_n \setminus E_{n+1})$

for $n=1,2,\ldots$ so the Monotone Convergence Theorem implies

$$(*) \qquad \sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1}) \leq \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} |f(x)|^p dx = \int_{E_1} |f(x)|^p dx$$

$$\leq \sum_{n=1}^{\infty} (n+1) m(E_n \setminus E_{n+1}).$$

Suppose $\sum_{n=1}^{\infty} m(E_n) < \infty$. Then, since $\{m(E_n)\}_{n=1}^{\infty}$ is a decreasing sequence, $N m(E_N) \to 0$ as $N \to \infty$. Thus

$$\sum_{n=1}^{\infty} (n+1) m(E_n \setminus E_{n+1}) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} (n+1)[m(E_n) - m(E_{n+1})] \right)$$

$$= \lim_{N\to\infty} \left(m(E_1) - (N+1) m(E_{N+1}) + \sum_{n=1}^{N} m(E_n) \right)$$

$$= m(E_1) + \sum_{n=1}^{\infty} m(E_n) < \infty.$$

Therefore $(*)$ shows that $\int_0^1 |f(x)|^p dx = \int_{E_1} |f(x)|^p dx + \int_{(0,1)\setminus E} |f(x)|^p dx < \infty.$

Conversely, suppose $f \in L^p(0,1)$. Then $n \cdot m(E_n) \le \int_{E_n} |f(x)|^p dx$ for $n = 1, 2, \ldots$ and the dominated convergence theorem implies $\int_{E_n} |f(x)|^p dx \to 0$ as $n \to \infty$. Consequently $n \cdot m(E_n) \to 0$ as $n \to \infty$.

Using (*) we thus have

$$\sum_{n=1}^{\infty} m(E_n) = \lim_{N \to \infty} \left(-N m(E_{N+1}) + \sum_{n=1}^{N} m(E_n) \right)$$

$$= \lim_{N \to \infty} \left(\sum_{n=1}^{N} n \left(m(E_n) - m(E_{n+1}) \right) \right)$$

$$= \sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1})$$

$$\le \int_{E_1} |f(x)|^p dx \quad < \quad \infty .$$

(ii) Suppose f is a nonnegative measurable function on $(0,1)$ with the property that $m(\{x \in (0,1) : f(x) \ge t\}) < \frac{1}{1+t^2}$ for all $t > 0$. The claim that $f \in L^p(0,1)$ for all $p \in [1,2)$. To see this let $p \in [1,2)$. Then

$$\sum_{n=1}^{\infty} m(\{x \in (0,1) : (f(x))^p \ge n\}) = \sum_{n=1}^{\infty} m(\{x \in (0,1) : f(x) \ge n^{1/p}\})$$

$$< \sum_{n=1}^{\infty} \frac{1}{1 + n^{\frac{2}{p}}} < \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{p}}} < \infty .$$

Part (i) then implies that $f \in L^p(0,1)$.

We claim that the minimum value of p for which f fails to be in $L^p(0,1)$ is $p=2$. For let

$$f(x) = \frac{1}{2}\sqrt{\frac{1}{x}-1} \quad \text{for } x \in (0,1).$$

Then
$$\begin{aligned}
m(\{x\in(0,1): f(x) \geq t\}) &= m\left(\left\{x\in(0,1): \tfrac{1}{2}\sqrt{\tfrac{1}{x}-1} \geq t\right\}\right) \\
&= m\left(\left\{x\in(0,1): \tfrac{1}{x} \geq 4t^2+1\right\}\right) \\
&= m\left(\left\{x\in(0,1): \tfrac{1}{1+4t^2} \geq x\right\}\right) \\
&= \frac{1}{1+4t^2} \\
&< \frac{1}{1+t^2} \quad \text{for all } t>0.
\end{aligned}$$

However $f \notin L^2(0,1)$ because $\int_0^1 f^2(x)\,dx = \frac{1}{4}\int_0^1 \left(\frac{1}{x}-1\right)dx = +\infty$.

(c) (1) Let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. There exists a continuous function g, vanishing outside an interval I of finite length, such that $\|f-g\|_2 < \frac{\varepsilon}{3}$ (cf. #15, p. 93). Because g is uniformly continuous on $\mathbb{R}$ there exists $\delta \in (0,1)$ such that

$$|g(x)-g(y)| < \frac{\varepsilon}{3\sqrt{m(I)+1}} \quad \text{for all } x,y \in \mathbb{R} \text{ satisfying } |x-y| < \delta.$$

Let $|t| < \delta$. Then

$$\|f_t - f\|_2 \leq \|f_t - g_t\|_2 + \|g_t - g\|_2 + \|g - f\|_2$$

$$< \frac{\varepsilon}{3} + \left(\int_{-\infty}^{\infty} |g(x-t) - g(x)|^2 dx \right)^{1/2} + \frac{\varepsilon}{3}$$

$$\leq \frac{2\varepsilon}{3} + \left(\int_{I \cup (I+t)} \frac{\varepsilon^2}{3^2 (m(I)+1)} dx \right)^{1/2}$$

$$\leq \varepsilon .$$

That is, $\lim_{t \to 0} \|f_t - f\|_2 = 0$.

(2) Let $f, g \in L^2(\mathbb{R})$ and $x, y \in \mathbb{R}$. Then

$$|(f*g)(x) - (f*g)(y)| = \left| \int_{-\infty}^{\infty} f(t) [g(x-t) - g(y-t)] dt \right|$$

$$\leq \int_{-\infty}^{\infty} |f(t)| |g(x-t) - g(y-t)| dt$$

$$\leq \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |g(x-t) - g(y-t)|^2 dt \right)^{1/2}$$

$$= \|f\|_2 \cdot \left(\int_{-\infty}^{\infty} |g(\tau) - g(\tau + y - x)|^2 d\tau \right)^{1/2}$$

$$= \|f\|_2 \cdot \|g - g_{x-y}\|_2$$

From part (1) of this problem and the previous estimate, we see that $f*g$ is a (uniformly) continuous function on $\mathbb{R}$, and hence $f*g$ is a measurable function on $\mathbb{R}$. Also note that, for any $x \in \mathbb{R}$,

$$|(f*g)(x)| \leq \int_{-\infty}^{\infty} |f(y)||g(x-y)|\, dy$$

$$\leq \left(\int_{-\infty}^{\infty} |f(y)|^2 dy\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |g(x-y)|^2 dy\right)^{1/2}$$

$$= \|f\|_2 \cdot \|g\|_2 < \infty.$$

Consequently, $f*g$ is bounded on $\mathbb{R}$. It follows that $f*g \in L^{\infty}(\mathbb{R})$.