Solve the following problems from Royden and Fitzpatrick’s *Real Analysis*.

p. 139: # 1, 3, 4 (second part only), 5
p. 143: # 7, 10, 12, 13, 14
p. 149: # 26, 32, 33, 34. Read # 35.

Also, solve each of these problems.

A. Let \( p_0, p_1, \ldots, p_n \) be real numbers such that each \( p_i > 1 \), and

\[
\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p_0}.
\]

If, for each integer \( i \) between 1 and \( n \), \( f_i \) belongs to \( L^p(a,b) \), must it be the case that the product \( f_1 \ldots f_n \) belongs to \( L^p(a,b) \)? Justify your answer.

B. (i) Let \( p \in (0, \infty) \). Show that a measurable function \( f \) belongs to \( L^p(0,1) \) if and only if

\[
\sum_{n=1}^{\infty} m \left( \left\{ x \in (0,1) : |f(x)|^p \geq n \right\} \right) < \infty.
\]

(ii) Use part (i) to help determine the set of all \( p \in (0,\infty) \) such that if \( f \) is any nonnegative, measurable function on \((0,1)\) satisfying

\[
m \left( \left\{ x \in (0,1) : f(x) \geq t \right\} \right) \leq \frac{1}{1+t^2}
\]

for all \( t > 0 \), then \( f \in L^p(0,1) \).

C. Let \( f \) and \( g \) belong to \( L^2(-\infty, \infty) \). Define the convolution product \( f * g \) by

\[
(f * g)(x) = \int_{(-\infty,\infty)} f(y)g(x-y)dm(y)
\]

for all real \( x \), and define the translate of \( f \) by \( t \) according to

\[
f_t(x) = f(x-t)
\]

for all real \( x \).

1. Show that \( \lim_{t \to 0} \| f_t - f \|_2 = 0 \).

2. Show that \( f * g \in L^\infty(-\infty, \infty) \). (Do not neglect to check that \( f * g \) is a measurable function!)
0.150, #32. Let \( \langle f_n \rangle_{n=1}^{\infty} \) be a sequence in \( L^\infty(E) \) and \( \sum_{k=1}^{\infty} a_k \) a convergent series of positive numbers such that \( \| f_{n+k} - f_k \|_{\infty} \leq a_k \) for all \( k \).

(a) Prove that there is a subset \( E_0 \) of \( E \) which has measure zero and
\[
\| f_{n+k}(x) - f_k(x) \|_{\infty} \leq \sum_{j=k}^{\infty} a_j
\]
for all \( k, n \) and all \( x \in E \setminus E_0 \).

(b) Conclude that there is a function \( f \in L^\infty(E) \) such that \( f_n \to f \) uniformly on \( E \setminus E_0 \).

\[(a) \quad \text{Let } E_k = \{ x \in E : |f_{k+1}(x) - f_k(x)| > a_k \} \text{ for } k=1,2,3, \ldots \quad \text{Since}
\]
\[
\| f_{k+1} - f_k \|_{\infty} \leq a_k \quad m(E_k) = 0 \quad \text{for all } k \quad \text{and hence } E_0 = \bigcup_{k=1}^{\infty} E_k
\]
has measure \( m(E_0) = 0 \). If \( n \) and \( k \) are positive integers and \( x \in E \setminus E_0 \) then
\[
|f_{n+k}(x) - f_k(x)| \leq \sum_{j=k}^{n+k-1} |f_{j+1}(x) - f_j(x)|
\leq \sum_{j=k}^{\infty} a_j
\leq \sum_{j=k}^{\infty} a_j.
\]

[Note: He did not show \( |f_{n+k}(x) - f_k(x)| \leq \| f_{n+k} - f_k \|_{\infty} \) for all \( k, n \), and \( x \in E \setminus E_0 \).]

(b) Because \( \sum_{k=1}^{\infty} a_k \) is a convergent series of positive numbers, part
\[(a) \quad \text{shows that } \langle f_k \rangle_{k=1}^{\infty} \text{ is a uniformly Cauchy sequence of functions}
\]
on the set $E \setminus E_0$ and hence (by completeness of $\mathbb{R}$)

$$g(x) = \lim_{n \to \infty} f_n(x) = f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) \quad (x \in E \setminus E_0)$$

exists and $f_n \to g$ uniformly on $E \setminus E_0$. Define a function $f$ on $E$ by

$$f(x) = \begin{cases} 
  g(x) & \text{if} \quad x \in E \setminus E_0, \\
  0 & \text{if} \quad x \in E_0.
\end{cases}$$

Clearly $f_n \to f$ uniformly on $E \setminus E_0$ and

$$f(x) = f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) \quad (x \in E \setminus E_0)$$

so

$$|f(x)| \leq |f_1(x)| + \sum_{k=1}^{\infty} a_k \leq \|f_1\|_\infty + \sum_{k=1}^{\infty} a_k < \infty \quad \text{a.e.}$$

on $E$. Hence $f \in L^\infty(E)$. 
Use the preceding problem to show that $L^\infty(E)$ is a Banach space.

By the example on pp. 137-138, we know that $(L^\infty(E), \| \cdot \|_\infty)$ is a normed linear space. It remains only to show that $(L^\infty(E), \| \cdot \|_\infty)$ is complete. To this end, let $\langle f_n \rangle_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(E)$. Choose a subsequence $\langle f_{n_k} \rangle_{k=1}^\infty$ such that $\| f_{n_{k+1}} - f_{n_k} \|_\infty \leq \frac{1}{2^k}$ for all $k \geq 1$. By the preceding problem, there is a function $f$ in $L^\infty(E)$ and set $E_0 \subseteq E$ with measure $m(E_0) = 0$ such that $f_{n_k} \to f$ uniformly on $E \setminus E_0$. We claim that $f_n \to f$ in $L^\infty(E)$. To see this, let $\varepsilon > 0$ and choose an integer $N_0 \geq 1$ such that $\| f_m - f_n \|_\infty < \frac{\varepsilon}{2}$ for all $m, n \geq N_0$. Choose an integer $n_k \geq N_0$ such that

$$|f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2} \text{ for all } x \in E \setminus E_0.$$  

Since $m(E_0) = 0$, it follows that $\| f - f_{n_k} \|_\infty < \frac{\varepsilon}{2}$. Consequently, for all $n \geq N_0$ we have

$$\| f - f_n \|_\infty \leq \| f - f_{n_k} \|_\infty + \| f_{n_k} - f_n \|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Theorem 3.42: Prove that $L^p$ is a Banach space for $1 \leq p < \infty$.

Let $\{x_j^{(i)}\}_{j=1}^{\infty} = \{\{x_v^{(j)}\}_{v=1}^{\infty}\}_{j=1}^{\infty}$ be a Cauchy sequence in $L^p$.

Choose a subsequence $\{x_v^{(j_k)}\}_{k=1}^{\infty}$ such that $\|x_v^{(j_{k+1})} - x_v^{(j_k)}\|_p < \frac{1}{2^k}$ for $k = 1, 2, \ldots$. Because

$$|x_v^{(j_{k+1})} - x_v^{(j_k)}| \leq \left( \sum_{v=1}^{\infty} |x_v^{(j_{k+1})} - x_v^{(j_k)}|^p \right)^{1/p} < \frac{1}{2^k}$$

for all $v = 1, 2, \ldots$ and all $k = 1, 2, \ldots$ we have $\sum_{k=1}^{\infty} |x_v^{(j_{k+1})} - x_v^{(j_k)}| < 1$.

Therefore, for each $v = 1, 2, \ldots$ we may define a real (or complex) number

$$x_v = x_v^{(1)} + \sum_{k=1}^{\infty} (x_v^{(j_{k+1})} - x_v^{(j_k)}) \quad (= \lim_{k \to \infty} x_v^{(j_k)})$$

and put $x = \{x_v\}_{v=1}^{\infty}$. Note that completeness of $R$ (or $C$) is being used here.
\[ z_v - x_v = \sum_{k=1}^{\infty} (x_v^{(k+1)} - x_v^{(k)}) \] for all \( v = 1, 2, \ldots \) and \( k = 1, 2, \ldots \). Therefore, by Minkowski's inequality for \( L^p \), \( \xi = \{x_v\}_v \) satisfies
\[
\| \xi - \xi(0) \|_p \leq \sum_{k=1}^{\infty} \| \xi^{(k+1)} - \xi^{(k)} \|_p \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{k-1}}.
\]
Note in particular that \( \xi \in L^p \).

We claim that \( \xi^{(1)} \rightarrow \xi \) in \( L^p \). To see this, let \( \varepsilon > 0 \) and choose an integer \( N_0 \geq 1 \) such that
\[
\| \xi^{(m)} - \xi^{(n)} \|_p < \frac{\varepsilon}{2}
\]
for all \( m, n > N_0 \). Let \( n > N_0 \) and choose an integer \( j_k \geq N_0 \)
such that \( \frac{1}{2^{k-1}} < \frac{\varepsilon}{2} \). Then
\[
\| \xi - \xi^{(n)} \|_p \leq \| \xi - \xi^{(j_k)} \|_p + \| \xi^{(j_k)} - \xi^{(n)} \|_p < \frac{1}{2^{k-1}} + \frac{\varepsilon}{2} < \varepsilon.
\]

**Theorem 7.4.148**: Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of functions in \( L^p(0,1) \), \( 1 \leq p < \infty \), which converges almost everywhere to a function \( f \) in \( L^p(0,1) \). Show that \( \{f_n\}_{n=1}^{\infty} \) converges to \( f \) in \( L^p(0,1) \) if and only if \( \|f_n\|_p \rightarrow \|f\|_p \).

**Proof**: By Fatou's lemma, we have (4) \( \|f\|_p \leq \liminf_{n \to \infty} \|f_n\|_p \).

Suppose that \( f_n \rightarrow f \) in \( L^p(0,1) \). Then
\[
\limsup_{n \to \infty} \|f_n\|_p \leq \limsup_{n \to \infty} (\|f\|_p + \|f_n - f\|_p) = \|f\|_p.
\]
It follows from (1) and (2) that \( \|f\|_p = \lim_{n \to \infty} \|f_n\|_p \).

Conversely, suppose \( \|f_n\|_p \to \|f\|_p \) as \( n \to \infty \). Consider the sequence \( g_n(x) = 2^p \left( |f_n(x)|^p + |f(x)|^p - |f_n(x) - f(x)|^p \right) \). Then each \( g_n \) is measurable, \( g_n \geq 0 \), and \( g_n(x) \to 2^{p+1}|f(x)|^p \) a.e. in \((0,1)\).

By Fatou's lemma,

\[
\int_0^1 2^{p+1} |f(x)|^p \, dx \leq \liminf_{n \to \infty} \left( \int_0^1 2^p |f_n(x)|^p \, dx + \int_0^1 2^p |f(x)|^p \, dx - \int_0^1 |f_n(x) - f(x)|^p \, dx \right)
\]

\[
= \int_0^1 2^p |f(x)|^p \, dx + \liminf_{n \to \infty} \left( - \int_0^1 |f_n(x) - f(x)|^p \, dx \right)
\]

\[
= \int_0^1 2^{p+1} |f(x)|^p \, dx - \limsup_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^p \, dx.
\]

Therefore \( \limsup_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^p \, dx \leq 0 \), and hence \( f_n \to f \) in \( L^p(0,1) \).
(b) Let $f$ be a measurable function on $(0,1)$, $p \in (0,\infty)$, and $E_n = \{ x \in (0,1) : |f(x)|^p \geq n \}$ \((n=1,2,\ldots)\). Then $f \in L^p(0,1)$ if and only if \(\sum_{n=1}^{\infty} m(E_n) < \infty\).

Note that \(n \cdot m(E_n \setminus E_{n+1}) \leq \int_{E_n \setminus E_{n+1}} |f(x)|^p \, dx \leq (n+1) \cdot m(E_n \setminus E_{n+1})\) for $n=1,2,\ldots$ so the Monotone Convergence Theorem implies

\[
\sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1}) \leq \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} |f(x)|^p \, dx = \int_{E_1} |f(x)|^p \, dx \leq \sum_{n=1}^{\infty} (n+1)m(E_n \setminus E_{n+1}).
\]

Suppose \(\sum_{n=1}^{\infty} m(E_n) < \infty\). Then, since \(\{m(E_n)\}_{n=1}^{\infty}\) is a decreasing sequence, \(N m(E_N) \to 0\) as \(N \to \infty\). Thus

\[
\sum_{n=1}^{\infty} (n+1)m(E_n \setminus E_{n+1}) = \lim_{N \to \infty} \left( \sum_{n=1}^{N} (n+1)[m(E_n) - m(E_{n+1})]\right)
\]

\[
= \lim_{N \to \infty} \left( m(E_1) - (N+1)m(E_{N+1}) + \sum_{n=1}^{N} m(E_n) \right)
\]

\[
= m(E_1) + \sum_{n=1}^{\infty} m(E_n) < \infty.
\]

Therefore (b) shows that \(\int_{0}^{E_1} |f(x)|^p \, dx = \int_{E_1} |f(x)|^p \, dx + \int_{(0,1) \setminus E_1} |f(x)|^p \, dx < \infty\).
Conversely, suppose \( f \in L^p(0,1) \). Then \( n \cdot m(E_n) \leq \int_{E_n} |f(x)|^p \, dx \) for \( n=1, 2, \ldots \) and the dominated convergence theorem implies \( \int_{E_n} |f(x)|^p \, dx \to 0 \) as \( n \to \infty \). Consequently \( n \cdot m(E_n) \to 0 \) as \( n \to \infty \).

Using (i), we thus have

\[
\sum_{n=1}^{\infty} m(E_n) = \lim_{N \to \infty} \left( -N m(E_{N+1}) + \sum_{n=1}^{N} m(E_n) \right)
\]

\[
= \lim_{N \to \infty} \left( \sum_{n=1}^{N} n \left( m(E_n) - m(E_{N+1}) \right) \right)
\]

\[
= \sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1})
\]

\[
\leq \int_{E_1} |f(x)|^p \, dx < \infty.
\]

(ii) Suppose \( f \) is a nonnegative measurable function on \((0,1)\) with the property that \( m(\{ x \in (0,1) : f(x) \geq t \}) < \frac{1}{1+t^2} \) for all \( t > 0 \). The claim that \( f \in L^p(0,1) \) for all \( p \in [1,2) \). To see this let \( p \in [1,2) \). Then

\[
\sum_{n=1}^{\infty} m(\{ x \in (0,1) : (f(x))^p \geq n \}) = \sum_{n=1}^{\infty} m(\{ x \in (0,1) : f(x) \geq n^{\frac{1}{p}} \})
\]

\[
< \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{2}{p}}} < \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2-p}{p}}} < \infty.
\]
Part (i) then implies that $f \in L^p(0,1)$.

We claim that the minimum value of $p$ for which $f$ fails to be in $L^p(0,1)$ is $p = 2$. For let

$$f(x) = \frac{1}{2} \sqrt{\frac{1}{x} - 1} \quad \text{for } x \in (0,1).$$

Then

$$m\left(\{x \in (0,1) : f(x) \geq t\}\right) = m\left(\{x \in (0,1) : \frac{1}{2} \sqrt{\frac{1}{x} - 1} \geq t\}\right)$$

$$= m\left(\{x \in (0,1) : \frac{1}{x} \geq 4t^2 + 1\}\right)$$

$$= m\left(\{x \in (0,1) : \frac{1}{1+4t^2} \geq x\}\right)$$

$$= \frac{1}{1+4t^2}$$

$$< \frac{1}{1+t^2} \quad \text{for all } t > 0.$$

However, $f \notin L^2(0,1)$ because

$$\int_0^1 f^2(x) \, dx = \frac{1}{4} \int_0^1 \left(\frac{1}{x} - 1\right) \, dx = +\infty.$$

(2) (1) Let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. There exists a continuous function $g$, vanishing outside an interval $I$ of finite length, such that $\|f-g\|_2 < \frac{\varepsilon}{3}$ (cf. #15, p.93). Because $g$ is uniformly continuous on $\mathbb{R}$, there exists $\delta \in (0,1)$ such that

$$|g(x) - g(y)| < \frac{\varepsilon}{3\sqrt{m(I)} + 1} \quad \text{for all } x, y \in \mathbb{R} \text{ satisfying } |x - y| < \delta.$$
Let $|t| < \delta$. Then
\[
\| f_t - f \|_2 \leq \| f_t - g_t \|_2 + \| g_t - g \|_2 + \| g - f \|_2
\]
\[
= \frac{\varepsilon}{3} + \left( \int_{-\infty}^{\infty} |g(x-t) - g(x)|^2 \, dx \right)^{1/2} + \frac{\varepsilon}{3}
\]
\[
\leq \frac{2\varepsilon}{3} + \left( \int_{\mathbb{R}(I+t)} \frac{\varepsilon^2}{3^2(m(I)+1)} \, dx \right)^{1/2}
\]
\[
\leq \varepsilon.
\]

That is, \( \lim_{t \to 0} \| f_t - f \|_2 = 0 \).

(2) Let \( f, g \in L^2(\mathbb{R}) \) and \( x, y \in \mathbb{R} \). Then
\[
| (f * g)(x) - (f * g)(y) | = \left| \int_{-\infty}^{\infty} f(t) [ g(x-t) - g(y-t) ] \, dt \right|
\]
\[
\leq \int_{-\infty}^{\infty} |f(t)||g(x-t) - g(y-t)| \, dt
\]
\[
\leq \left( \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right)^{1/2} \cdot \left( \int_{-\infty}^{\infty} |g(x-t) - g(y-t)|^2 \, dt \right)^{1/2}
\]
\[
= \| f \|_2 \cdot \left( \int_{-\infty}^{\infty} |g(t) - g(t + y - x)|^2 \, dt \right)^{1/2}
\]
\[
= \| f \|_2 \cdot \| g - g_{x,y} \|_2
\]
From part (1) of this problem and the previous estimate, we see that $f \ast g$ is a (uniformly continuous) function on $\mathbb{R}$, and hence $f \ast g$ is a measurable function on $\mathbb{R}$. Also note that, for any $x \in \mathbb{R}$,

$$
| (f \ast g)(x) | \leq \int_{-\infty}^{\infty} |f(y)||g(x-y)| \, dy \\
\leq \left( \int_{-\infty}^{\infty} |f(y)|^2 \, dy \right)^{1/2} \cdot \left( \int_{-\infty}^{\infty} |g(x-y)|^2 \, dy \right)^{1/2} \\
= \| f \|_2 \cdot \| g \|_2 < \infty.
$$

Consequently, $f \ast g$ is bounded on $\mathbb{R}$. It follows that $f \ast g \in L^\infty(\mathbb{R})$. 