Mathematics 6215 | Midterm Exam Fall 2015 | Name:____________________

Directions: Complete any 5 of the following 6 problems. Each problem has the same value: 30 points. **CIRCLE** the numbers of the 5 problems you want me to grade on your exam.

1. (a) Define the inner and outer measure of a set $E \subseteq \mathbb{R}$.
   (b) Define the phrase "$E \subseteq \mathbb{R}$ is (Lebesgue) measurable".
   (c) Define the (Lebesgue) measure of a measurable set $E \subseteq \mathbb{R}$.
   (d) Define the phrase "$f : E \to [-\infty, \infty]$ is a measurable function".
   (e) Define the (Lebesgue) integral of a simple, measurable function φ on $E \subseteq \mathbb{R}$".
   (f) Define the (Lebesgue) integral of a measurable function $f : E \to [0, \infty]$.
   (g) Define the (Lebesgue) integral of a measurable function $f : E \to [-\infty, \infty]$.

2. (a) State Lebesgue's Monotone Convergence Theorem.
   (b) State Fatou's Lemma.
   (c) State Lebesgue's Dominated Convergence Theorem.
   (d) Give an example of a sequence for which Fatou's Lemma holds but the inequality is strict.
   (e) Give an example of a nonnegative, decreasing, measurable sequence of functions for which the conclusion of Lebesgue's Monotone Convergence Theorem does not hold.

3. Let $E \subseteq \mathbb{R}$ be measurable.
   (a) If $0 < p < \infty$, define the space $L^p(E)$ and its "norm".
   (b) Define the space $L^0(E)$ and its metric (provided $m(E) < \infty$).
   (c) Define the space $L^\infty(E)$ and its norm.
   (d) State Hölder's inequality and the conditions under which it holds.
   (e) State Minkowski's inequality and the conditions under which it holds.
   (f) State the inclusions, if any, among the spaces $L^p[0,1]$ where $0 \le p \le \infty$.
   (g) State the inclusions, if any, among the spaces $L^p(\mathbb{R})$ where $0 \le p \le \infty$.

4. (a) Define the Fourier transform of $f \in L^1(\mathbb{R})$.
   (b) State the Riemann-Lebesgue Lemma for functions in $L^1(\mathbb{R})$.
   (c) Define the Fourier transform of $f \in L^1[0,1]$.
   (d) Define the N^{th} Fourier partial sum $S_N f$ of $f \in L^1[0,1]$.
   (e) For which $L^p[0,1]$-spaces does $\|f - S_N f\|_p \to 0$ as $N \to \infty$?
   (f) State Parseval's identity for $f \in L^2[0,1]$.

5. (a) Give an example, if possible, of a normed linear space which is not an inner product space.
   (b) Give an example, if possible, of an inner product space which is not a normed linear space.
   (c) Define the phrase "$\langle x_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in the normed linear space $(X, \|\cdot\|)$".

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(d) Define the phrase "$\langle x_n \rangle_{n=1}^{\infty}$ is a convergent sequence in the normed linear space $(X, \| \cdot \|)$".

(e) Define the phrase "the normed linear space $(X, \| \cdot \|)$ is a Banach (or complete) space".

(f) Define the phrase "$(X, \langle \cdot , \cdot \rangle)$ is a Hilbert space".

(g) Give an example of a normed linear space which is not a Banach space.

(h) State the Riesz-Fischer Theorem.

6. (a) Define the phrase "Λ is a bounded linear functional on the normed linear space $(X, \| \cdot \|)$".

(b) Give an example of a bounded linear functional on $\mathbb{R}^n$ equipped with the Euclidean norm.

(c) Give an example of a bounded linear functional on $C[0,1]$ equipped with the uniform norm.

(d) Give an example of a bounded linear functional on $(L^\infty[0,1], \| \cdot \|_\infty)$.

(e) State the Riesz Representation Theorem for the L^p – spaces.