Mathematics 322 Final Exam Name: Dr. Grow

Spring 2001

This is an open book examination. Unless otherwise noted, y^i $(1 \le N)$ will denote rectangular cartesian coordinates in Euclidean N-space, $\mathbb{E}_N$.

1.(12 pts.) In $\mathbb{E}_2$ consider the contravariant vector at the point $(y^1, y^2) = (4,3)$ with components $T^1 = 2$ and $T^2 = 5$ in y coordinates.

(a) What are the covariant components T_1 and T_2 of the vector in y coordinates?

(b) Make the change of coordinates

$$x^1 = y^1 y^2$$
$$x^2 = (y^1)^2 + (y^2)^2.$$

What are the contravariant components of T in x coordinates?

2.(12 pts.) In V_4, the tensor A_{rst} is skew-symmetric in the last pair of indices and satisfies the relation

$$A_{rst} + A_{str} + A_{trs} = 0.$$

How many independent components does A have? Give reasons for your answer.

3.(12 pts.) In $\mathbb{E}_3$ recall that rectangular cartesian and spherical coordinates are related by

$$y^1 = r\sin(\phi)\cos(\theta), \quad y^2 = r\sin(\phi)\sin(\theta), \quad y^3 = r\cos(\phi).$$

In spherical coordinates in $\mathbb{E}_3$, a vector field is such that the vector at each point points along the parametric line of θ, in the sense of θ increasing, and its magnitude is $k\sin(\phi)$, where k is a constant. Find the contravariant and covariant components of this vector field.

4.(28 pts.) Consider the surface $\mathfrak{M}$ in $\mathbb{E}_3$ given parametrically by

$$y^1 = R\sin(x^1)\cos(x^2), \quad y^2 = R\sin(x^1)\sin(x^2), \quad y^3 = R\cos(x^1)$$

where $0 \le x^1 < \pi$, $0 \le x^2 < 2\pi$, and R is a positive constant.

(a) Show that the metric tensor of $\mathfrak{M}$ is

$$a_{11} = R^2, \quad a_{22} = R^2\sin^2(x^1), \quad a_{21} = a_{12} = 0.$$

(b) Compute the nonvanishing Christoffel symbols of the second kind for $\mathfrak{M}$.

(c) Write (BUT DO NOT SOLVE) the differential equations that define the geodesics in $\mathfrak{M}$.

(d) Compute the covariant curvature tensor for $\mathfrak{M}$.

5.(12 pts.) Let T_r and S^r be components of covariant and contravariant vector fields, respectively, in V_N. If $T_r S^r$ is constant on a curve C in V_N, and if T_r is propagated parallelly along C, what can you conclude about S^r on C?

6. (12 pts.) Are the relations

(i) $T_{|rs} = T_{|sr}$

(ii) $T_{r|sk} = T_{r|ks}$

true

(a) in curvilinear coordinates in a Euclidean space?

(b) in a general Riemannian space?

(Of course, reasons for your answers are required.)

7. (12 pts.) Explain in detail how you would determine if a surface in $\mathbb{E}_3$, given parametrically by

$$y^1 = f^1(u^1, u^2), \quad y^2 = f^2(u^1, u^2), \quad y^3 = f^3(u^1, u^2),$$

is flat.