1. (16 pts.) If \( \mathbf{v} = (v_x, v_y, v_z) \) belongs to \( E^3 \) define \( \mathbf{Tv} \approx (-2v_x + 3v_z, -v_x, v_x + 2v_y) \).

(a) Why is \( \mathbf{T} \) a tensor?
(b) Determine the matrix of Cartesian components of the tensor \( \mathbf{T} \).
(c) Complete the blanks.

\[
\mathbf{Sv} \approx \frac{1}{2}(\mathbf{T} + \mathbf{T}^T)\mathbf{v} \\
\mathbf{Av} \approx \frac{1}{2}(\mathbf{T} - \mathbf{T}^T)\mathbf{v} \\
\]

(d) Determine the matrices of the Cartesian components of \( \mathbf{S} \) and \( \mathbf{A} \).
(e) Find the Cartesian components of the vector \( \mathbf{w} \) such that \( \mathbf{Av} = \mathbf{w} \times \mathbf{v} \) for all \( \mathbf{v} \) in \( E^3 \).

2. (16 pts.) Consider the transformation of coordinates

\[
x = u - v^2 \\
y = u + v
\]

in \( E^2 \) where \( -\infty < u < \infty \) and \(-\frac{1}{2} < v \). Compute:

(a) the base vectors \( \mathbf{g}_1 = \mathbf{g}_u \) and \( \mathbf{g}_2 = \mathbf{g}_v \);
(b) the reciprocal base vectors;
(c) the six Christoffel symbols \( \Gamma^i_{11} = \Gamma^u_{uu}, \Gamma^i_{12} = \Gamma^u_{uv}, \) etc.;
(d) the roof (contravariant) components \( a^u = a^1 \) and \( a^v = a^2 \) of the acceleration vector;
(e) the physical components \( a^{(u)} \) and \( a^{(v)} \) of the acceleration vector.

In addition, if \( \mathbf{f} = u\nu v\mathbf{g}_\nu \), write down the **component form** of Newton's Second Law in the \( uv \) coordinate system.

3. (16 pts.) (a) Write the definitions in general coordinates of \( E^3 \) for \( \text{div} \mathbf{A} \) and \( \nabla^2 I \).

(b) Express \( \text{div} \mathbf{A} \) and \( \nabla^2 I \) in the cylindrical coordinate system \( x^1 = r, x^2 = \theta, x^3 = z \). Recall that the Cartesian coordinates \( y^1, y^2, y^3 \) in \( E^3 \) are related to cylindrical coordinates by \( y^1 = x^1 \cos(x^2), y^2 = x^1 \sin(x^2), y^3 = x^3 \).

4. (16 pts.) (a) Show that the covariant derivatives of the metric and conjugate metric tensors are identically zero in any Riemannian space.

(b) Let \( V_H \) denote an arbitrary Riemannian space and \( g_{\bar{\mu}} \) its metric tensor. If \( B_\ell \) is a covariant vector in \( V_H \), use part (a) and the definition of the covariant derivative to show that \( (g^{\bar{\mu}}B_\ell)_m = g^{\bar{\mu}}B_{m,\ell} \).

5. (16 pts.) Let \( y^1, y^2, y^3 \) denote Cartesian coordinates in \( E^3 \). Consider the surface \( \mathcal{M} \) in \( E^3 \) given by

\[
y^1 = x^1 \cos(x^2), \quad y^2 = x^1 \sin(x^2), \quad y^3 = x^1
\]

where \( 0 < x^1 < \infty \) and \( 0 \leq x^2 < 2\pi \).

(a) Show that the metric which \( \mathcal{M} \) inherits from \( E^3 \) is \( g_{11} = 2, g_{22} = (x^1)^2, g_{12} = g_{21} = 0 \).

(b) Compute the nonvanishing Christoffel symbols of the second kind for \( \mathcal{M} \).

(c) Write, BUT DO NOT SOLVE, the differential equations that define the geodesics in \( \mathcal{M} \).
(d) Is the coordinate curve: \( x^1 = s, \ x^2 = \text{constant} \), a geodesic in \( \mathcal{M} \)? Why or why not?

6.(16 pts.) (a) Explain in detail how you would determine if a surface in \( E^3 \), given parametrically by

\[
y^1 = f^1(x^1, x^2), \ y^2 = f^2(x^1, x^2), \ y^3 = f^3(x^1, x^2)
\]

is flat.
(b) Determine whether or not the surface \( \mathcal{M} \) in problem 5 is flat.