Problems

1. Show that

\[ R_{ijkl} = \frac{\partial}{\partial x^l} [jl, i] - \frac{\partial}{\partial x^l} [jk, il] + \left\{ \begin{array}{c} a \\ jk \\ il, il \\ [il, il] - [jl, il] \\ [jk, il] \end{array} \right\}. \]

2. Show that

\[ R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} \right) \]

\[ + g^{ap} ([jk, \beta][il, \alpha] - [jl, \beta][ik, \alpha]). \]

3. Using the formula of Problem 2 show that

\[ R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klji} \]

and

\[ R_{ijkl} + R_{iklj} + R_{jikl} = 0. \]

4. If \( \phi \) is a scalar, then \( g^{ij} \phi \) is a scalar and is equal to

\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial \phi}{\partial x^j} \right). \]

5. Referring to Problem 4, show that \( g^{ij} \phi \) is a scalar when the \( g_{ij} \) are the metric coefficients of \( E_3 \) referred to a cartesian frame. This implies that Laplace’s equation in general curvilinear coordinates has the form

\[ g^{ij} \phi_{,ij} = 0, \]

since this is a tensor equation.

6. Referring to Problem 5, show that Laplace’s equation in polar coordinates has the form

\[ \frac{\partial^2 \phi}{(\partial y^2)^2} + \frac{1}{(y^2)^2} \frac{\partial^2 \phi}{(\partial y^2)^2} + \frac{1}{(y^2 \sin \theta)^2} \frac{\partial^2 \phi}{(\partial y^2)^2} + \frac{2}{y^2} \frac{\partial \phi}{\partial y} + \frac{1}{(y^1)^2} \cot y^2 \frac{\partial \phi}{\partial y} = 0. \]