# Math 5222 Lecture 14 Problems

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**Problems**

1. Show that

$$R_{ijkl} = \frac{\partial}{\partial x^k}[jl, i] - \frac{\partial}{\partial x^l}[jk, i] + \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} [il, \alpha] - \begin{Bmatrix} \alpha \\ jl \end{Bmatrix} [ik, \alpha].$$

2. Show that

$$R_{ijkl} = \frac{1}{2}\left( \frac{\partial^2 g_{il}}{\partial x^j\, \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i\, \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j\, \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i\, \partial x^k} \right) + g^{\alpha\beta}([jk, \beta][il, \alpha] - [jl, \beta][ik, \alpha]).$$

3. Using the formula of Problem 2 show that

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

and

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

4. If $\phi$ is a scalar, then $g^{ij}\phi_{,ij}$ is a scalar and is equal to

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left( \sqrt{g}\, g^{ij} \frac{\partial \phi}{\partial x^j} \right).$$

5. Referring to Problem 4, show that $g^{ij}\phi_{,ij} = 0$ reduces to $\partial^2\phi/\partial x^i\, \partial x^i = 0$ when the $g_{ij}$ are the metric coefficients of $E_3$ referred to a cartesian frame. This implies that Laplace's equation in general curvilinear coordinates has the form $g^{ij}\phi_{,ij} = 0$, since this is a tensor equation.

6. Referring to Problem 5, show that Laplace's equation in polar coordinates has the form

$$\frac{\partial^2\phi}{(\partial y^1)^2} + \frac{1}{(y^1)^2}\frac{\partial^2\phi}{(\partial y^2)^2} + \frac{1}{(y^1 \sin y^2)^2}\frac{\partial^2\phi}{(\partial y^3)^2} + \frac{2}{y^1}\frac{\partial\phi}{\partial y^1} + \frac{1}{(y^1)^2}\cot y^2 \frac{\partial\phi}{\partial y^2} = 0.$$