Mathematics 322 Midterm Exam Name: Dr. Grow

Spring 2001

1.(20 pts.) Let $y^1, \ldots, y^N$ be a fixed set of coordinates in a space V_N, and form the set of N^2 quantities whose components in the $x^1, \ldots, x^N$ coordinate system are given by

$$T(i,j) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}.$$

Determine completely the tensor character (or lack thereof) of the set of quantities $T(i,j)$.

2.(20 pts.) Let R_{ijkl} be the components of a covariant tensor of rank four in a three-dimensional Riemannian space V_3.

(a) How many components does the tensor have?

(b) If the tensor obeys the symmetry relations

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij},$$

how many independent components does the tensor possess?

(c) In addition to the symmetry relations in (b), if the tensor obeys

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$

how many independent components does the tensor possess?

(It should be understood that you must support your answers to (b) and (c) with reasons.)

3.(20 pts.) In spherical coordinates r, θ, ϕ in three dimensional euclidean space, a vector field is such that the vector at each point points toward the origin and has magnitude equal to twice the distance of the point from the origin. Find the contravariant components and covariant components of this vector field in the spherical coordinate system and in the rectangular cartesian system.

Note: You may find useful the following identities relating spherical and rectangular cartesian coordinates:

$$y^1 = r\sin(\phi)\cos(\theta), \quad y^2 = r\sin(\phi)\sin(\theta), \quad y^3 = r\cos(\phi).$$

4.(20 pts.) Let V_2 be a Riemannian space with positive definite metric tensor a_{mn}.

(a) If θ denotes the angle between the x^1 and x^2 coordinate curves at a point P in V_2, show that

$$\cos(\theta) = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}.$$

(b) If the metric tensor in V_2 is given by

$$a_{11} = a_{22} = \frac{4(x^1)^2(x^2)^2}{\omega}, \quad a_{12} = a_{21} = \frac{2x^1x^2[4 - (x^1)^2 - (x^2)^2]}{\omega}$$

where $\omega = [(x^2 + 2)^2 - (x^1)^2][(x^1)^2 - (x^2 - 2)^2]$, find all points in V_2 where the coordinate curves are orthogonal. (Assume $x^1 > 0$ and $x^2 > 0$ at all points in V_2.)

5. (20 pts.) Let V_2 be the surface in three dimensional euclidean space given by

$$y^1 = x^1\cos(x^2), \quad y^2 = x^1\sin(x^2), \quad y^3 = (x^1)^2/2$$

where $0 \le x^2 < 2\pi$, $0 \le x^1 < \infty$, and equip V_2 with the metric tensor that it inherits as a surface in $\mathbb{E}_3$.

(a) Show that the metric tensor of V_2 is given by

$$a_{11} = 1 + (x^1)^2, \quad a_{22} = (x^1)^2, \quad \text{and} \quad a_{12} = a_{21} = 0.$$

(b) Compute the Christoffel symbols of the first kind in V_2.

(c) Compute the conjugate to the metric tensor in V_2.

(d) Compute the Christoffel symbols of the second kind in V_2.

(e) Write (but do not solve) the differential equations that define the geodesics in V_2.

Bonus (10 pts.): Compute the geodesics in V_2.