Mathematics 5222 Midterm Exam Spring 2015 Name: Dr. Grow

This is a closed-book, closed-notes examination. You will have 75 minutes to complete your solutions to the problems on this exam.

1.(20 pts.) In a space V_2 of dimension two, write out completely the following expressions and perform any simplifications that can be made.

(a) $\delta_{ij}x^i x^j$ (b) δ_i^i (c) $g_{ij} = \dfrac{\partial y^k}{\partial x^i}\dfrac{\partial y^k}{\partial x^j}$

(The symbols δ_{ij} and δ_i^j denote Kronecker deltas.)

(d) If $a_j^i x^j = b^i$ is a system of n linear equations in the n unknowns x^i and $a = \left|a_j^i\right| \neq 0$, verify that $x^k = \dfrac{b^\alpha A_\alpha^k}{a}$ is the solution of the system. (Here A_i^j denotes the cofactor of a_i^j.)

2.(20 pts.) Write the transformation law for the components of the following under admissible coordinate transformations.

(a) a covariant vector; (b) a contravariant vector; (c) a mixed tensor of rank two.

(d) Write out explicitly the laws of transformation for the components of a contravariant vector in two-dimensional euclidean space when S is the transformation from polar coordinates x^1, x^2 to rectangular cartesian coordinates y^1, y^2 given by

$$S: \begin{cases} y^1 = x^1\cos\left(x^2\right) \\ y^2 = x^1\sin\left(x^2\right) \end{cases}$$

where $x^1 > 0$ and $0 \leq x^2 < 2\pi$.

3.(20 pts.) Let R_{ijkl} be the components of a covariant tensor of rank four in a two-dimensional Riemannian space V_2.

(a) How many components does this tensor have?

(b) If the tensor obeys the symmetry relations $R_{ijkl} = -R_{jikl}$ and $R_{ijkl} = -R_{ijlk}$, how many distinct, possibly nonvanishing components does the tensor possess?

(c) Show that if the tensor obeys the symmetry law $R_{ijkl} = R_{klij}$ in addition to the symmetry laws in part (b), then it must obey the symmetry law $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ as well.

(Please support your answers to parts (a), (b), and (c) with reasons.)

4.(20 pts.) Let y^1, y^2, y^3 denote rectangular cartesian coordinates in three-dimensional euclidean space $\mathbb{E}_3$. Consider the surface V_2 in $\mathbb{E}_3$ given by

$$y^1 = x^1\cos\left(x^2\right),\ y^2 = x^1\sin\left(x^2\right),\ y^3 = x^1$$

where $x^1 > 0$ and $0 \leq x^2 < 2\pi$. In the x^1, x^2 coordinate system, compute:

(a) the metric tensor for V_2;

(b) the conjugate metric tensor for V_2;

(c) the nonvanishing Christoffel symbols of the second kind for V_2.

5.(20 pts.) Write the definitions of the covariant derivative of the following in a Riemannian space with covariant metric tensor a_{ij}:

(a) a contravariant vector; (b) a covariant tensor of rank two.

(c) Show that the covariant derivatives of the metric tensor, the Kronecker delta, and the conjugate metric tensor vanish identically in the space.

Bonus (20 pts.): Let V_N be a Riemannian space of dimension N with covariant metric tensor a_{ij}. Define the divergence of a vector field on V_N with (continuously differentiable) contravariant components A^i to be the scalar invariant $div(A) = A^i{}_{,i}$. Define the gradient of a (continuously differentiable) scalar invariant u on V_N to be the covariant tensor $grad(u)_i = \frac{\partial u}{\partial x^i}$. Define the Laplacian of a (twice continuously differentiable) scalar invariant u on V_N to be the scalar invariant $\Delta(u) = div(grad(u))$.

(a) Write out explicit formulas for the actions of the gradient, divergence, and Laplacian operators in terms of the metric tensor, the Christoffel symbols, and ordinary partial derivatives with respect to the coordinates.

(b) Show that these formulas reduce to the usual definitions of the gradient, divergence, and Laplacian operators with respect to rectangular cartesian coordinates $y^1, \ldots, y^N$ in a euclidean space of dimension N.

(c) Compute the actions of the gradient, divergence, and Laplacian operators in cylindrical coordinates in three-dimensional euclidean space. (Recall that the cartesian coordinates in $\mathbb{E}_3$ are related to cylindrical coordinates by the transformation formulas

$$y^1 = x^1 \cos(x^2),\ y^2 = x^1 \sin(x^2),\ y^3 = x^3.)$$

#1. (a) $\delta_{ij}x^ix^j = \sum_{i=1}^{2}\sum_{j=1}^{2}\delta_{ij}x^ix^j = \overset{1}{\cancel{\delta_{11}}}x^1x^1 + \overset{0}{\cancel{\delta_{12}}}x^1x^2 + \overset{0}{\cancel{\delta_{21}}}x^2x^1 + \overset{1}{\cancel{\delta_{22}}}x^2x^2$

$$= \boxed{(x^1)^2 + (x^2)^2}$$

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(b) $\delta^i_i = \sum_{i=1}^{2}\delta^i_i = \delta^1_1 + \delta^2_2 = 1+1 = \boxed{2}$

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(c) $g_{ij} = \dfrac{\partial y^k}{\partial x^i}\dfrac{\partial y^k}{\partial x^j} = \sum_{k=1}^{2}\dfrac{\partial y^k}{\partial x^i}\dfrac{\partial y^k}{\partial x^j} = \dfrac{\partial y^1}{\partial x^i}\dfrac{\partial y^1}{\partial x^j} + \dfrac{\partial y^2}{\partial x^i}\dfrac{\partial y^2}{\partial x^j}$

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$$\therefore \boxed{g_{11} = \left(\frac{\partial y^1}{\partial x^1}\right)^2 + \left(\frac{\partial y^2}{\partial x^1}\right)^2, \quad g_{12} = g_{21} = \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2} \quad g_{22} = \left(\frac{\partial y^1}{\partial x^2}\right)^2 + \left(\frac{\partial y^2}{\partial x^2}\right)^2.}$$

(d) $a^i_j x^j = a^i_j\left(\dfrac{b^\alpha A^j_\alpha}{a}\right) = \dfrac{b^\alpha}{a}\left(a^i_j A^j_\alpha\right) = \dfrac{b^\alpha}{a}\left(\delta^i_\alpha a\right) = \dfrac{b^i a}{a} \overset{\checkmark}{=} b^i$

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#2.

(a) The transformation law for the covariant components of a vector is

4 $$\bar{T}_i = T_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^i} \qquad \left(\text{or more explicitly } \bar{T}_i(\bar{x}) = T_\alpha(x)\frac{\partial x^\alpha}{\partial \bar{x}^i}\right).$$

(b) The transformation law for the contravariant components of a vector is

4 $$\bar{T}^i = T^\alpha \frac{\partial \bar{x}^i}{\partial x^\alpha} \qquad \left(\text{or more explicitly } \bar{T}^i(\bar{x}) = T^\alpha(x)\frac{\partial \bar{x}^i}{\partial x^\alpha}\right).$$

(c) The transformation law for the components of a mixed tensor of rank two is

4 $$\bar{T}_i^{\,j} = T_\alpha^{\,\beta} \frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial \bar{x}^j}{\partial x^\beta} \qquad \left(\text{or more explicitly } \bar{T}_i^{\,j}(\bar{x}) = T_\alpha^{\,\beta}(x)\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial \bar{x}^j}{\partial x^\beta}\right)$$

(d) In E_2 with polar coordinates x^1, x^2 and rectangular cartesian

8 coordinates y^1, y^2 related by the transformation

$$S: \begin{cases} y^1 = x^1\cos(x^2) & (x^1 > 0) \\ y^2 = x^1\sin(x^2) & (0 \le x^2 < 2\pi) \end{cases}$$

the contravariant components of a vector transform according to

$$T^i(y) = T^\alpha(x)\frac{\partial y^i}{\partial x^\alpha} = T^1(x)\frac{\partial y^i}{\partial x^1} + T^2(x)\frac{\partial y^i}{\partial x^2}.$$ Therefore

$$\boxed{T^1(y) =} T^1(x)\frac{\partial y^1}{\partial x^1} + T^2(x)\frac{\partial y^1}{\partial x^2} = \boxed{T^1(x)\cos(x^2) - T^2(x)x^1\sin(x^2)},$$

$$\boxed{T^2(y) =} T^1(x)\frac{\partial y^2}{\partial x^1} + T^2(x)\frac{\partial y^2}{\partial x^2} = \boxed{T^1(x)\sin(x^2) + T^2(x)x^1\cos(x^2)}.$$

#3. 5 (a) The number of components of $R_{ijk\ell}$ in V_2 is $\boxed{2^4 = 16}$.

(b) Suppose the tensor obeys $R_{ijk\ell} = -R_{jik\ell}$ and $R_{ijk\ell} = -R_{ij\ell k}$.

10 For a fixed pair $k, \ell \in \{1,2\}$, there is one independent possibly nonvanishing component, $R_{12k\ell}$. To see this observe the following:

$R_{ijk\ell}$
(k, ℓ fixed)

$i \backslash j$	1	2
1	0	$R_{12k\ell}$
2	$-R_{12k\ell}$	0

$R_{iik\ell} \overset{(b)1}{=} -R_{iik\ell}$ implies $R_{iik\ell} = 0$.
(That is, $R_{11k\ell} = 0 = R_{22k\ell}$.)
$R_{ijk\ell} \overset{(b)1}{=} -R_{jik\ell}$ so $R_{21k\ell} = -R_{12k\ell}$

For a fixed pair $i, j \in \{1,2\}$, a similar argument using $R_{ijk\ell} = -R_{ij\ell k}$ shows that there is one independent possibly nonvanishing component, R_{ij12}, and $R_{ij21} = -R_{ij12}$, $R_{ij11} = 0 = R_{ij22}$. Combining these two cases we see that there are only four possibly nonzero components:

$$R_{1212},\ R_{1221},\ R_{2112},\ R_{2121}.$$

Among these we have $R_{1221} = -R_{1212}$ and $R_{2121} = -R_{2112}$ so only $\boxed{\text{two}}$ components are independent, say R_{1212} and R_{2112}.

(c) 5 Suppose the tensor obeys $R_{ijk\ell} = R_{k\ell ij}$ in addition to the symmetry laws in part (b). Then $R_{2112} = R_{1221} = -R_{1212}$ so the tensor has only $\boxed{\text{one}}$ independent possibly nonzero component, say R_{1212}.

Consider the desired symmetry law,

$$(*) \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad .$$

If all four indices $i, j, k,$ and l are identical then $(*)$ clearly holds because $R_{\alpha\alpha\alpha\alpha} = 0$ so each term in $(*)$ is zero. Similarly, if three indices among $i, j, k,$ and l are identical then

$$0 = R_{\beta\alpha\alpha\alpha} = R_{\alpha\beta\alpha\alpha} = R_{\alpha\alpha\beta\alpha} = R_{\alpha\alpha\alpha\beta}$$

so again $(*)$ holds because each term in $(*)$ is zero. Since the indices $i, j, k,$ and l must belong to the two-element set $\{1, 2\}$, it is impossible to have three or four distinct indices among $i, j, k,$ and l. Therefore, it only remains to consider the case when there are exactly two distinct indices among $i, j, k,$ and l. We enumerate the possibilities below:

$$R_{ijji} + R_{ijij} + \cancelto{0}{R_{iijj}} \overset{(b)\,2}{=} R_{ijji} - R_{ijji} \overset{\checkmark}{=} 0$$

$$R_{ijij} + \cancelto{0}{R_{iijj}} + R_{ijji} \overset{(b)\,2}{=} R_{ijij} - R_{ijij} \overset{\checkmark}{=} 0$$

$$\cancelto{0}{R_{iijj}} + R_{ijji} + R_{ijij} \overset{(b)\,2}{=} -R_{ijij} + R_{ijij} \overset{\checkmark}{=} 0.$$

Therefore $(*)$ holds for all $i, j, k,$ and l in $\{1, 2\}$.

#4. (a) The metric tensor for V_2 that it inherits by virtue of being a
7 surface in E_3 is given by $a_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}$. We compute as follows:

$$a_{11} = \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} + \frac{\partial y^3}{\partial x^1}\frac{\partial y^3}{\partial x^1} = (\cos x^2)^2 + (\sin x^2)^2 + 1 = 2.$$

$$a_{22} = \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} + \frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2} + \frac{\partial y^3}{\partial x^2}\frac{\partial y^3}{\partial x^2} = (-x^1\sin x^2)^2 + (x^1\cos x^2)^2 = (x^1)^2$$

$$a_{21} = a_{12} = \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2} + \frac{\partial y^3}{\partial x^1}\frac{\partial y^3}{\partial x^2} = (\cos x^2)(-x^1\sin x^2) + (\sin x^2)(x^1\cos x^2) = 0.$$

Therefore the metric tensor for V_2 is $[a_{ij}] = \begin{bmatrix} 2 & 0 \\ 0 & (x^1)^2 \end{bmatrix}$.

(b) Expressed as matrices, the conjugate metric tensor is the inverse matrix
6 of the metric tensor. Therefore, the conjugate metric tensor a^{ij} is:

$$[a^{ij}] = [a_{ij}]^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/(x^1)^2 \end{bmatrix}.$$

(c) (Method I) Using the definition $[rs,t] = \frac{1}{2}\left(\frac{\partial a_{rt}}{\partial x^s} + \frac{\partial a_{st}}{\partial x^r} - \frac{\partial a_{rs}}{\partial x^t}\right)$,
7 suspending summation conventions, and noting $a_{ij} = 0$ if $i \neq j$, we find:

$$[rr,r] = \frac{1}{2}\left(\frac{\partial a_{rr}}{\partial x^r} + \frac{\partial a_{rr}}{\partial x^r} - \frac{\partial a_{rr}}{\partial x^r}\right) = \frac{1}{2}\frac{\partial a_{rr}}{\partial x^r} = 0.$$

$$(r \neq t) \quad [rr,t] = \frac{1}{2}\left(\cancelto{0}{\frac{\partial a_{rt}}{\partial x^r}} + \cancelto{0}{\frac{\partial a_{rt}}{\partial x^r}} - \frac{\partial a_{rr}}{\partial x^t}\right) = -\frac{1}{2}\frac{\partial a_{rr}}{\partial x^t} = \begin{cases} 0 & \text{if } r=1 \text{ and } t=2, \\ -x^1 & \text{if } r=2 \text{ and } t=1. \end{cases}$$

$$(r \neq t) \quad [rt,r] = [tr,r] = \frac{1}{2}\left(\frac{\partial a_{rr}}{\partial x^t} + \cancelto{0}{\frac{\partial a_{tr}}{\partial x^r}} - \cancelto{0}{\frac{\partial a_{rt}}{\partial x^r}}\right) = \frac{1}{2}\frac{\partial a_{rr}}{\partial x^t} = \begin{cases} x^1 & \text{if } r=2 \text{ and } t=1, \\ 0 & \text{if } r=1 \text{ and } t=2. \end{cases}$$

Therefore the only nonvanishing Christoffel symbols of the first kind in V_2 are

$$[21,2] = [12,2] = x^1 \quad \text{and} \quad [22,1] = -x^1.$$

The definition $\left\{ {t \atop rs} \right\} = a^{t\alpha}[rs,\alpha]$ implies the only nonvanishing

Christoffel symbols of the second kind are

$$\left\{ {2 \atop 21} \right\} = \left\{ {2 \atop 12} \right\} = a^{2\alpha}[12,\alpha] = \frac{1}{(x^1)^2}(x^1) = \boxed{\frac{1}{x^1}}$$

and

$$\left\{ {1 \atop 22} \right\} = a^{1\alpha}[22,\alpha] = \frac{1}{2}(-x^1) = \boxed{-\frac{x^1}{2}}.$$

(Method II) Since $a_{ij} = 0$ if $i \neq j$, we may use the results of exercise 3 in Section 31:

$$\left\{ {i \atop ii} \right\} = \frac{1}{2}\frac{\partial}{\partial x^i}(\log a_{ii}), \quad \left\{ {i \atop ij} \right\} = \frac{1}{2}\frac{\partial}{\partial x^j}(\log a_{ii}), \quad \left\{ {i \atop jj} \right\} = -\frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i},$$

where the summation convention is suspended and $i \neq j$. Therefore:

$$\left\{ {1 \atop 11} \right\} = \frac{1}{2}\frac{\partial}{\partial x^1}(\log a_{11}) = \frac{1}{2}\frac{\partial}{\partial x^1}(\log 2) = 0$$

$$\left\{ {2 \atop 22} \right\} = \frac{1}{2}\frac{\partial}{\partial x^2}(\log a_{22}) = \frac{1}{2}\frac{\partial}{\partial x^2}(\log(x^1)^2) = 0$$

$$\left\{ {1 \atop 12} \right\} = \left\{ {1 \atop 21} \right\} = \frac{1}{2}\frac{\partial}{\partial x^2}(\log a_{11}) = \frac{1}{2}\frac{\partial}{\partial x^2}(\log 2) = 0.$$

$$\boxed{\left\{ {2 \atop 12} \right\} = \left\{ {2 \atop 21} \right\} =} \frac{1}{2}\frac{\partial}{\partial x^1}(\log a_{22}) = \frac{1}{2}\frac{\partial}{\partial x^1}(\log(x^1)^2) = \frac{\partial}{\partial x^1}(\log x^1) = \boxed{\frac{1}{x^1}}.$$

$$\boxed{\left\{ {1 \atop 22} \right\} =} -\frac{1}{2a_{11}} \cdot \frac{\partial a_{22}}{\partial x^1} = -\frac{1}{4}\frac{\partial (x^1)^2}{\partial x^1} = \boxed{-\frac{x^1}{2}}.$$

$$\left\{ {2 \atop 11} \right\} = -\frac{1}{2a_{22}} \cdot \frac{\partial g_{11}}{\partial x^2} = -\frac{1}{2(x^1)^2}\frac{\partial (2)}{\partial x^2} = 0.$$

#5. In a Riemannian space with metric tensor a_{ij}:

(a) the covariant derivative of a vector with contravariant components T^i

5 is

$$\boxed{T^i_{\ ,j} = \frac{\partial T^i}{\partial x^j} + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} T^\alpha \ ;}$$

(b) the covariant derivative of a covariant tensor T_{ij} of rank two is

5

$$\boxed{T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} T_{\alpha j} - \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} T_{i\alpha} \, .}$$

(c) The covariant derivative of the metric tensor is

10

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} a_{\alpha j} - \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} a_{i\alpha}$$

$$= \frac{\partial a_{ij}}{\partial x^k} - [ik,j] - [jk,i]$$

$$= \cancel{\frac{\partial a_{ij}}{\partial x^k}} - \frac{1}{2}\left(\cancel{\frac{\partial a_{ij}}{\partial x^k}} + \cancel{\frac{\partial a_{kj}}{\partial x^i}} - \cancel{\frac{\partial a_{ik}}{\partial x^j}} \right) - \frac{1}{2}\left(\cancel{\frac{\partial a_{ji}}{\partial x^k}} + \cancel{\frac{\partial a_{ki}}{\partial x^j}} - \cancel{\frac{\partial a_{jk}}{\partial x^i}} \right)$$

$$= 0 \, .$$

The covariant derivative of the Kronecker δ^j_i is

$$\delta^j_{i,k} = \cancel{\frac{\partial (\delta^j_i)}{\partial x^k}}^{\,0} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \delta^j_\alpha + \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} \delta^\alpha_i = - \begin{Bmatrix} j \\ ik \end{Bmatrix} + \begin{Bmatrix} j \\ ik \end{Bmatrix} = 0 \, .$$

Differentiate covariantly both sides of the identity $a_{i\alpha} a^{\alpha j} = \delta^j_i$

to obtain $(a_{i\alpha} a^{\alpha j})_{,k} = \delta^{j}_{i,k} = 0$. Applying the product rule and the first portion of part (c) of this problem gives

$$\cancelto{0}{g_{i\alpha,k}}\, a^{\alpha j} + a_{i\alpha} a^{\alpha j}{}_{,k} = 0$$

$$a_{i\alpha} a^{\alpha j}{}_{,k} = 0 .$$

Raising the index i gives

$$a^{i\ell} a_{i\alpha} a^{\alpha j}{}_{,k} = a^{i\ell} 0 = 0 .$$

But $a^{i\ell} a_{i\alpha} = \delta_{\alpha}^{\ell}$ so $a^{\ell j}{}_{,k} = \delta_{\alpha}^{\ell} a^{\alpha j}{}_{,k} = a^{i\ell} a_{i\alpha} a^{\alpha j}{}_{,k} = 0$.

Bonus: Let V_N be a Riemannian space of dimension N with metric tensor a_{ij}.

(a) The gradient of a C^1 scalar invariant u on V_N is the vector with
6 covariant components

$$\boxed{\operatorname{grad}(u)_i = \frac{\partial u}{\partial x^i}}.$$

The divergence of a C^1 vector field with contravariant components A^i is the scalar invariant

$$\boxed{\operatorname{div}(A) =} A^i_{,i} = \boxed{\frac{\partial A^i}{\partial x^i} + \left\{ {i \atop \beta\, i} \right\} A^\beta}.$$

The Laplacian of a C^2 scalar invariant u on V_N is the scalar invariant

$$\boxed{\Delta(u)} = \operatorname{div}(\operatorname{grad} u) = \operatorname{div}\left(a^{i\alpha} \frac{\partial u}{\partial x^\alpha}\right)$$

$$= \left(a^{i\alpha}\frac{\partial u}{\partial x^\alpha}\right)_{,i}$$

$$= a^{i\alpha}\left(\frac{\partial u}{\partial x^\alpha}\right)_{,i}$$

$$= \boxed{a^{i\alpha}\left(\frac{\partial^2 u}{\partial x^i \partial x^\alpha} - \left\{ {\beta \atop \alpha\, i} \right\} \frac{\partial u}{\partial x^\beta}\right)}$$

(b) Let $V_N = \mathbb{E}_N$ and $y^1, \ldots, y^N$ be rectangular cartesian coordinates in
7 $\mathbb{E}_N$. Then $a_{ij} = \delta_{ij}$ is the metric tensor for $\mathbb{E}_N$ in rectangular cartesian coordinates so $[ij,k] = \frac{1}{2}\left(\frac{\partial a_{ik}}{\partial y^j} + \frac{\partial a_{jk}}{\partial y^i} - \frac{\partial a_{ij}}{\partial y^k}\right) = 0$ for all i, j, and k in $\{1, \ldots, N\}$. Hence all the Christoffel symbols of

the second kind vanish so

$$\operatorname{div}(A) = A^i_{;i} = \frac{\partial A^i}{\partial y^i} + \left\{ {i \atop \beta\, i} \right\} A^\beta = \boxed{\frac{\partial A^1}{\partial y^1} + \frac{\partial A^2}{\partial y^2} + \cdots + \frac{\partial A^N}{\partial y^N}},$$

which is the usual definition of the divergence of a vector A with contravariant components $A^1, \ldots, A^N$. Likewise the Laplacian of a C^2 scalar invariant u is

$$\Delta u = a^{i\alpha}\left(\frac{\partial^2 u}{\partial y^i \partial y^\alpha} - \left\{ {\beta \atop \alpha\, i} \right\} \frac{\partial u}{\partial y^\beta} \right)$$

$$= \delta^{i\alpha} \frac{\partial^2 u}{\partial y^i \partial y^\alpha}$$

$$= \frac{\partial^2 u}{\partial y^i \partial y^i}$$

$$= \boxed{\frac{\partial^2 u}{(\partial y^1)^2} + \frac{\partial^2 u}{(\partial y^2)^2} + \cdots + \frac{\partial^2 u}{(\partial y^N)^2}}$$

which is the usual definition of the Laplacian of u. Clearly

$$\operatorname{grad}(u)_i = \frac{\partial u}{\partial y^i}$$

is the usual definition of the gradient of a scalar invariant.

(c) If u is a C^1 scalar invariant on $\mathbb{E}_3$ then in cylindrical coordinates x^1, x^2, x^3, the gradient of u is the vector with covariant components

$$\boxed{\operatorname{grad}(u)_i = \frac{\partial u}{\partial x^i}}.$$

Using the formula $a_{ij}(x) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j}$ for the metric tensor, we easily find

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $a_{ij} = 0$ if $i \neq j$, exercises 2 and 3 in Section 31 imply:

$$\left\{ {i \atop jk} \right\} = 0 \quad \text{if } i, j, \text{ and } k \text{ are distinct};$$

$$\left\{ {i \atop ii} \right\} = \frac{1}{2}\frac{\partial}{\partial x^i}\left(\log a_{ii}\right) = 0;$$

$$\left\{ {i \atop ij} \right\} = \frac{1}{2}\frac{\partial}{\partial x^j}\left(\log a_{ii}\right) = \begin{cases} 1 & \text{if } i = 2 \text{ and } j = 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$\left\{ {i \atop jj} \right\} = -\frac{1}{2a_{ii}} \cdot \frac{\partial a_{jj}}{\partial x^i} = \begin{cases} -x^1 & \text{if } i = 1 \text{ and } j = 2, \\ 0 & \text{otherwise}; \end{cases}$$

where we suppose $i \neq j$. Consequently

$$\left\{ {2 \atop 21} \right\} = \left\{ {2 \atop 12} \right\} = 1, \quad \left\{ {1 \atop 22} \right\} = -x^1,$$

and all other Christoffel symbols of the second kind are zero.

Therefore the divergence of a contravariant vector in cylindrical coordinates is

$$\operatorname{div}(A) = A^i{}_{,i} = \frac{\partial A^i}{\partial x^i} + \overbrace{\begin{Bmatrix} i \\ \beta\, i \end{Bmatrix}}^{\substack{\text{only nonzero contribution} \\ \text{occurs when } i=2,\ \beta=1}} A^\beta$$

$$= \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} + \begin{Bmatrix} 2 \\ 1\,2 \end{Bmatrix} A^1$$

$$\boxed{\operatorname{div}(A) = \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} + A^1}\ .$$

The Laplacian of a C^2 scalar invariant u on $\mathbb{E}_3$ is

$$\Delta u = a^{i\alpha}\left(\frac{\partial^2 u}{\partial x^i \partial x^\alpha} - \begin{Bmatrix} \beta \\ \alpha\, i \end{Bmatrix} \frac{\partial u}{\partial x^\beta} \right)$$

$$= \cancelto{1}{a^{11}}\left(\frac{\partial^2 u}{(\partial x^1)^2} - \cancelto{0}{\begin{Bmatrix} \beta \\ 1\,1 \end{Bmatrix} \frac{\partial u}{\partial x^\beta}} \right) + \cancelto{\frac{1}{(x^1)^2}}{a^{22}}\left(\frac{\partial^2 u}{(\partial x^2)^2} - \overbrace{\begin{Bmatrix} \beta \\ 2\,2 \end{Bmatrix}}^{\text{only nonzero contribution occurs when } \beta=1} \frac{\partial u}{\partial x^\beta} \right) + \cancelto{1}{a^{33}}\left(\frac{\partial^2 u}{(\partial x^3)^2} - \cancelto{0}{\begin{Bmatrix} \beta \\ 3\,3 \end{Bmatrix}} \frac{\partial u}{\partial x^\beta} \right)$$

$$= \frac{\partial^2 u}{(\partial x^1)^2} + \frac{1}{(x^1)^2}\left(\frac{\partial^2 u}{(\partial x^2)^2} + x^1 \frac{\partial u}{\partial x^1} \right) + \frac{\partial^2 u}{(\partial x^3)^2}$$

ck here.

$$\boxed{\Delta u = \frac{1}{x^1} \frac{\partial}{\partial x^1}\left(x^1 \frac{\partial u}{\partial x^1} \right) + \frac{1}{(x^1)^2} \frac{\partial^2 u}{(\partial x^2)^2} + \frac{\partial^2 u}{(\partial x^3)^2}}$$

Another way to express the answer.