Winter 1999

Mathematics 322 Midterm Exam Name:

This is an open-book, open-notes examination which you will have 75 minutes to complete.

1.(20 pts.) Let $\mathcal{M}$ be an M-dimensional manifold and, for r and s in $\{1, \ldots, M\}$, define the components of (the constant field) δ on $\mathcal{M}$ by $\delta(r,s) = 0$ if $r \neq s$ and $\delta(s,s) = 1$. Show that δ is a tensor on $\mathcal{M}$ and identify its rank and contravariant and covariant types.

2.(20 pts.) If $T(i,j)dx^i dx^j$ is an invariant on $\mathcal{M}$, what can we conclude about the set of quantities $T(i,j,)$? Why?

3.(20 pts.) Let T be a covariant tensor of rank 2 (i.e. a bilinear real-valued function on VxV) over a vector space V with bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$.

(a) If $f_j = a^i{}_j e_i$, derive the transformation law

$$T^f{}_{ij} = a^k{}_i a^l{}_j T^e{}_{kl}$$

for the components of T with respect to the bases f and e.

(b) If V is the tangent space at a point m of a manifold $\mathcal{M}$, and the bases of V are the coordinate vector fields with respect to two systems of coordinates x^i and x'^i at m:

$$e_i = \frac{\partial}{\partial x^i}(m) \quad \text{and} \quad f_j = \frac{\partial}{\partial x'^j}(m),$$

then express the transformation law for the components of T that was obtained in part (a) of this problem.

(c) Compare and contrast the equation from part (b) of this problem with equation 1.403 on page 13 of Synge and Schild.

4.(20 pts.) Let $\mathcal{M}$ be a three-dimensional manifold and let R be a rank four tensor on $\mathcal{M}$.

(a) How many components does R have?

(b) Suppose that R obeys the symmetry relations

$$R_{iklm} = -R_{ikml} = -R_{kilm}.$$

Show that R has 9 independent components.

(c) Show that the additional symmetry relation

$$R_{iklm} = R_{lmik}$$

further reduces the number of independent components to 6.

(d) Finally, if the components also satisfy the relation

$$R_{iklm} + R_{ilmk} + R_{imkl} = 0,$$

how many independent components does R have? Why?

5.(20 pts.) Let $\mathcal{M}$ be a two-dimensional manifold.

(a) If A^i and B^j are components of two contravariant vector fields A and B on $\mathcal{M}$, show that the four quantities

(*) $$A^iB^j$$

are the components of a contravariant tensor field of second rank on $\mathcal{M}$.

(b) Show that not every contravariant tensor field C of second rank on $\mathcal{M}$ can be formed in accordance with (*) from two contravariant vector fields A and B on $\mathcal{M}$.

(c) Nevertheless, show that the components of every second rank contravariant tensor field C on $\mathcal{M}$ can be expressed as

$$C^{ij} = A^iB^j + E^iD^j$$

for two appropriately selected pairs A,B and E,D of contravariant vector fields on $\mathcal{M}$.