MATH 3304 - FINAL EXAM SUMMER 2015

Name: ______________________________

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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (30 points) Find an explicit solution of the initial value problem $y' = \frac{x^2}{y}; y(0) = -1$.

Solution: We use separation of variables to write

$$\int y dy = \int x^2 dx,$$

so

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C.$$

Isolate y and take square roots to get

$$y = \pm\sqrt{\frac{2}{3}x^3 + c}.$$

Since our initial condition is $y(0) = -1$ we take the negative solution and apply the initial conditions to see

$$-1 = y(0) = -\sqrt{C}.$$

Hence $1 = C$ and we have derived the explicit solution

$$y(x) = -\sqrt{1 + \frac{2}{3}x^3}.$$

2. (30 points) Find the general solution of $y' = 4 - \frac{2y}{50+t}$.

Solution: First rewrite the equation in standard form:

$$y' + \frac{2}{50+t}y = 4,$$

so it is clear that the integrating factor is

$$\mu = \exp\left(\int \frac{2}{50+t}dt\right) = e^{2\log(50+t)} = (50+t)^2.$$

Multiply by the integrating factor to get

$$(50+t)^2y' + 2(50+t)y = 4(50+t)^2,$$

and the left hand side factors yielding

$$\frac{d}{dt}\left[(50+t)^2y\right] = 4(50+t)^2.$$

Integration with respect to t shows

$$(50+t)^2y = \int 4(50+t)^2dt = \frac{4}{3}(50+t)^3 + C.$$

Therefore the general solution is

$$y(t) = \frac{4}{3}(50+t) + \frac{C}{(50+t)^2}.$$

Note: Wolframalpha's solution expands $(50+t)^3$ *to* $t^3 + 150t^2 + 7500t + 125000$ *and does not cancel* $\frac{(50+t)^3}{(50+t)^2}$ *as we did.*

3. (30 points) Solve $4y'' + y = 2\sec\left(\frac{t}{2}\right)$ on the interval $-\pi < t < \pi$.

Solution: First solve the homogeneous equation $4y''+y = 0$ by observing the characteristic equation $4r^2 + 1 = 0$ and solving it to get $r = \pm\frac{i}{2}$. Therefore the homogeneous solution is

$$y_h(t) = c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right).$$

We will write $y_1(t) = \cos\left(\frac{t}{2}\right)$ and $y_2(t) = \sin\left(\frac{t}{2}\right)$. We will proceed using variation of parameters. First recall that variation of parameters is only defined for an equation in standard form (i.e. coefficient of y'' must be 1), so multiply our DE by $\frac{1}{4}$ to get

$$y'' + \frac{1}{4}y = \frac{1}{2}\sec\left(\frac{t}{2}\right).$$

First compute the Wronskian

$$W\{y_1, y_2\}(t) = \det\begin{bmatrix} \cos(\frac{t}{2}) & \sin(\frac{t}{2}) \\ -\frac{1}{2}\sin(\frac{t}{2}) & \frac{1}{2}\cos(\frac{t}{2}) \end{bmatrix} = \frac{1}{2}.$$

Variation of parameters tells us to assume the particular solution is $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ and so compute

$$u_1(t) = -\int \frac{\frac{1}{2}\sin(\frac{t}{2})\sec(\frac{t}{2})}{W\{y_1, y_2\}(t)}dt = -\int \tan\left(\frac{t}{2}\right) dt = 2\log\left(\cos\left(\frac{t}{2}\right)\right).$$

Now compute

$$u_2(t) = \int \frac{\frac{1}{2}\cos(\frac{t}{2})\sec(\frac{t}{2})}{W\{y_1, y_2\}(t)} = t.$$

Hence the particular solution of the differential equation is

$$y_p(t) = 2\log\left(\cos\left(\frac{t}{2}\right)\right)\cos\left(\frac{t}{2}\right) + t\sin\left(\frac{t}{2}\right).$$

Therefore the general solution of the differential equations is

$$y(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right) + 2\log\left(\cos\left(\frac{t}{2}\right)\right)\cos\left(\frac{t}{2}\right) + t\sin\left(\frac{t}{2}\right).$$

4. (30 points) Find the general solution of $y^{(4)} - y''' + y'' = 0$.
Solution: The characteristic equation associated with this differential equation is

$$r^4 - r^3 + r^2 = r^2(r^2 - r + 1) = 0.$$

We see that $r = 0$ is a double root of the equation and the quadratic formula shows that there are complex roots $r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence the general solution is

$$y(t) = c_1 + tc_2 + c_3 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_4 e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

5. (30 points) Find the solution of the initial value problem

$$y'' + y = \delta(t-17); y(0) = 1, y'(0) = 0.$$

Solution: Take the Laplace transform of the equation to get

$$(s^2\mathscr{L}\{y\} - sy(0) - y'(0)) + \mathscr{L}\{y\} = e^{-17s},$$

and apply the initial conditions and simplify yielding

$$\mathscr{L}\{y\} = \frac{e^{-17s}}{s^2+1} + \frac{s}{s^2+1}.$$

Taking inverse Laplace transforms yields

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{e^{-17s}}{s^2+1}\right\}(t) + \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) \\ &= u_{17}(t)\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-17) + \cos(t) \\ &= u_{17}(t)\sin(t-17) + \cos(t), \end{aligned}$$

which is the solution of the DE (note: Wolfram alpha uses $\theta(t-c)$ to denote our function $u_c(t)$ and it expresses the soltuion using $\sin(17-t)$ instead of $\sin(t-17)$).

6. (30 points) Use the Laplace transform to solve the integral equation

$$y(t) + 5\int_0^t e^{-4(t-\tau)} y(\tau) d\tau = 1.$$

Solution: Let $f(t) = e^{-4t}$ and hence $\mathscr{L}\{f\} = \dfrac{1}{s+4}$. We may express the integral equation as

$$y(t) + 5(f * y)(t) = 1.$$

Taking the Laplace transform and using the convolution theorem on the second term yields

$$\mathscr{L}\{y\} + 5\frac{\mathscr{L}\{y\}}{s+4} = \frac{1}{s}.$$

Factor $\mathscr{L}\{y\}$ on the left hand side to get

$$\left(1 + \frac{5}{s+4}\right)\mathscr{L}\{y\} = \frac{1}{s}.$$

Get a common denominator on the left and simplify to get

$$\mathscr{L}\{y\} = \frac{s+4}{s(s+9)}.$$

Apply partial fractions to the right hand side to get

$$\frac{s+4}{s(s+9)} = \frac{5}{9}\frac{1}{s+9} + \frac{4}{9}\frac{1}{s}.$$

Thus taking the inverse Laplace transform of $\mathscr{L}\{y\}$ yields

$$y = \frac{5}{9}\mathscr{L}^{-1}\left\{\frac{1}{s+9}\right\} + \frac{4}{9}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{5}{9}e^{-9t} + \frac{4}{9}.$$

7. (30 points) Find the solution of the initial value problem

$$y''(t) + 4y(t) = \begin{cases} 0 & ; t < \frac{\pi}{2} \\ -8 & ; t \geq \frac{\pi}{2}, \end{cases}$$

with initial conditions $y(0) = 2, y'(0) = 0$.
Solution: We may express the right hand side using the step function $-8u_{\frac{\pi}{2}}(t)$. This yields the DE

$$y''(t) + 4y(t) = -8u_{\frac{\pi}{2}}(t).$$

Take the Laplace transform of this DE to get

$$(s^2\mathscr{L}\{y\} - sy(0) - y'(0)) + 4\mathscr{L}\{y\} = \frac{-8e^{-\frac{\pi}{2}s}}{s}.$$

Apply initial conditions and simplify to get

$$\mathscr{L}\{y\} = -\frac{8e^{-\frac{\pi}{2}s}}{s(s^2+4)} + \frac{2s}{s^2+4}.$$

Partial fractions yields

$$\frac{-8}{s(s^2+4)} = \frac{2s}{s^2+4} - \frac{2}{s},$$

and now take the inverse Laplace transform of $\mathscr{L}\{y\}$ to get

$$\begin{aligned} y(t) &= 2\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi}{2}s}}{s^2+4}\right\}(t) - 2\mathscr{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s}\right\}(t) + 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &= 2u_{\frac{\pi}{2}}(t)\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\left(t - \frac{\pi}{2}\right) - 2u_{\frac{\pi}{2}}(t)\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\left(t - \frac{\pi}{2}\right) + 2\cos(2t) \\ &= 2u_{\frac{\pi}{2}}(t)\cos\left(2\left(t - \frac{\pi}{2}\right)\right) - 2u_{\frac{\pi}{2}}(t) + 2\cos(2t) \end{aligned}$$

Since $\cos(2(t - \frac{\pi}{2})) = \cos(2t - \pi) = -\cos(2t)$, we may express the solution as

$$y(t) = -2u_{\frac{\pi}{2}}(t)\cos(2t) - 2u_{\frac{\pi}{2}}(t) + 2\cos(2t).$$

Written as a piecewise function, this is

$$y(t) = \begin{cases} 2\cos(2t) & ; t < \frac{\pi}{2} \\ -2 & ; t \geq \frac{\pi}{2}, \end{cases}$$

which is the solution of the initial value problem.

8. (35 points) Find real-valued solutions of the initial value problem

$$\vec{y}' = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix} \vec{y}; \vec{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution: Assume that $\vec{y} = \vec{k}e^{\lambda t}$. Plugging this into the DE yields the eigenvalue problem

$$\begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix} \vec{k} = \lambda \vec{k}.$$

The eigenvalues of this matrix are $\lambda_1 = i$ and $\lambda_2 = -i$. We will take the associated eigenvectors to be $\vec{k}^{(1)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$ and $\vec{k}^{(2)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$. Hence

$$\begin{aligned} \vec{y}^{(1)}(t) &= \vec{k}^{(1)} e^{\lambda_1 t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right) (\cos(t) + i \sin(t)) \\ &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(t) \right) + i \left(\begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t) \right). \end{aligned}$$

The real-valued solutions are

$$\tilde{\vec{y}}^{(1)} = \operatorname{Re}\left(\vec{y}^{(1)}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(t),$$

and

$$\tilde{\vec{y}}^{(2)} = \operatorname{Im}\left(\vec{y}^{(1)}\right) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t).$$

Hence the general solution is

$$\vec{y}(t) = c_1 \tilde{\vec{y}}^{(1)} + c_2 \tilde{\vec{y}}^{(2)} = c_1 \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ -2\cos(t) \end{bmatrix}.$$

Apply the initial conditions to see

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{y}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

From this equation we see $c_1 = 1$ and $c_2 = -1$. Therefore the solution of the initial value problem is

$$\vec{y}(t) = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix} - \begin{bmatrix} \sin(t) \\ -2\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) - \sin(t) \\ 2\sin(t) + 2\cos(t) \end{bmatrix}$$

9. (35 points) Find the general solution of the equation

$$\vec{y}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{y} + \begin{bmatrix} te^{2t} \\ te^{2t} \end{bmatrix},$$

given that $\psi(t) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix}$ is a fundamental matrix for the system.

Solution: Let $\vec{g}(s) = \begin{bmatrix} se^{2s} \\ se^{2s} \end{bmatrix}$. First compute

$$\det \psi(t) = e^{-t} + 4e^{-t} = 5e^{-t}.$$

Now compute

$$\psi^{-1}(s) = \frac{1}{5e^{-t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix}.$$

Hence

$$\psi^{-1}(s)\vec{g}(s) = \frac{1}{5} \begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} se^{2s} \\ se^{2s} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 \\ 5s \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix}.$$

Now we use variation of parameters to compute

$$\begin{aligned} \vec{y}_p(t) &= \psi(t) \int^t \begin{bmatrix} 0 \\ s \end{bmatrix} ds \\ &= \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{t^2}{2} \end{bmatrix} \\ &= \frac{t^2}{2} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore the general solution is

$$\vec{y}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{t^2}{2} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Table of Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$u_c(t)f(t)$	$e^{-cs}\mathscr{L}\{f(t+c)\}(s)$
$e^{ct}f(t)$	$F(s-c)$
$(f*g)(t)$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$