Math 3304 Fall 2015 Final Exam

Your printed name: Dr. Grow

Your instructor's name: ______________________

Your section (or Class Meeting Days and Time): ______________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. The final exam consists of this cover page, 10 pages of problems containing 10 numbered problems, and 1 page of Laplace transforms.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. **Express all solutions in real-valued, simplified form.**
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

Problem	1	2	3	4	5	6	7	8	9	10	Sum
Points Earned											
Max. Points	20	20	20	20	20	20	20	20	20	20	200

1. [20] Find the explicit solution of the initial value problem

$$y' + 2ty = ty^2 \quad y(0) = 4.$$

First-order Nonlinear

$$\frac{dy}{dt} = ty^2 - 2ty = t(y^2 - 2y)$$ ← Variables separable.

PFD.

$$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}$$

$$1 = A(y-2) + By$$

$y = 0$: $1 = -2A$

$y = 2$: $1 = 2B$

$$\therefore \frac{1}{y(y-2)} = \frac{1/2}{y-2} - \frac{1/2}{y}$$

$$\int \frac{dy}{y(y-2)} = \int t\,dt$$

$$\int \left(\frac{\frac{1}{2}}{y-2} - \frac{\frac{1}{2}}{y} \right) dy = \frac{t^2}{2} + c_1$$

$$\frac{1}{2} \ln|y-2| - \frac{1}{2} \ln|y| = \frac{t^2}{2} + c_1$$

$$\ln \left| \frac{y-2}{y} \right| = t^2 + c$$

$$\frac{y-2}{y} = \pm e^{t^2+c} = Ke^{t^2} \quad (K = \pm e^c)$$

$$y - 2 = Kye^{t^2}$$

$$\left[y(1-Ke^{t^2}) = \right] \quad y - Kye^{t^2} = 2$$

$$y = \frac{2}{1-Ke^{t^2}}.$$

$$4 = y(0) = \frac{2}{1-K} \Rightarrow 1 - K = \frac{2}{4} \Rightarrow \frac{1}{2} = K$$

$$y = \frac{2}{1 - \frac{1}{2}e^{t^2}}$$

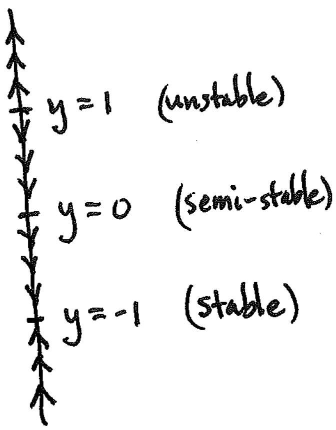
$$\boxed{y(t) = \frac{4}{2 - e^{t^2}}}$$

2. [20] For the differential equation $y' = y^2\,(y^2 - 1)$,

a) Determine the equilibrium solutions (critical points) of the differential equation.
b) Sketch the phase line (or phase portrait). Be sure to **show your work.**
c) Classify each equilibrium point as either asymptotically stable, unstable, or semi-stable.
d) If $y(t)$ denotes the solution of the differential equation satisfying the initial condition $y(0) = 0$, determine $\lim_{t\to\infty} y(t)$.

(a) $y = c = \text{constant} \Rightarrow y' = 0$. Therefore the equilibrium solutions satisfy $0 = y' = y^2(y^2-1) = y^2(y-1)(y+1)$ so $\boxed{y=0,\ y=1,\ \text{and}\ y=-1}$ are the equilibrium solutions.

(b) and (c)



Interval	Sign of $y' = y^2(y-1)(y+1)$
$-\infty < y < -1$	$(+)(-)(-) = +$
$-1 < y < 0$	$(+)(-)(+) = -$
$0 < y < 1$	$(+)(-)(+) = -$
$1 < y < \infty$	$(+)(+)(+) = +$

(d) Since the equilibrium solution $y=0$ of $y' = y^2(y^2-1)$ also satisfies the initial condition $y(0)=0$, we have $y(t)=0$ for all real t.

Therefore $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 0 = \boxed{0}$.

3. [20] Find the general solution of the following differential equation

$$y' = \frac{y + e^t}{2}.$$

First-order, linear.

$$y' - \tfrac{1}{2}y = \tfrac{1}{2}e^t$$

Integrating factor: $\mu(t) = e^{\int p(t)\,dt} = e^{\int -\frac{1}{2}dt} = e^{-t/2}.$

$$e^{-t/2}\left(y' - \tfrac{1}{2}y\right) = e^{-t/2}\left(\tfrac{1}{2}e^t\right)$$

$$e^{-t/2}y' - \tfrac{1}{2}e^{-t/2}y = \tfrac{1}{2}e^{t/2}$$

$$\frac{d}{dt}\left(e^{-t/2}y\right) = \tfrac{1}{2}e^{t/2}$$

$$e^{-t/2}y = \int \tfrac{1}{2}e^{t/2}\,dt = e^{t/2} + c$$

$$y = \left(e^{t/2} + c\right)e^{t/2}$$

$$\boxed{y(t) = e^t + ce^{t/2}}$$

where c is an arbitrary constant.

$$y'' - \frac{1}{t}y' - \frac{3}{t^2}y = t \leftarrow g(t)$$

4. [20] Find the general solution of the following differential equation

$$t^2y'' - ty' - 3y = t^3, \quad t > 0$$

by using variation of parameters.

$y = t^m$ in $t^2y'' - ty' - 3y = 0$ leads to $t^2 m(m-1)t^{m-2} - tmt^{m-1} - 3t^m = 0$

$$t^m\left[m(m-1) - m - 3\right] = 0$$

Therefore $0 = m^2 - 2m - 3 = (m-3)(m+1)$ so $m = 3$ or $m = -1$. Consequently $y_h(t) = c_1 t^3 + c_2 t^{-1}$ where c_1, c_2 are arbitrary constants. A particular solution is $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{vmatrix}$

$= -t - 3t = -4t$,

$$u_1 = \int -\frac{y_2 g}{W}\,dt = \int -\frac{t^{-1} \cdot t}{-4t}\,dt = \frac{1}{4}\int \frac{1}{t}\,dt = \frac{1}{4}\ln(t) + \cancel{c}^{\,0},$$

$$u_2 = \int \frac{y_1 g}{W}\,dt = \int \frac{t^3 \cdot t}{-4t}\,dt = -\frac{1}{4}\int t^3\,dt = -\frac{1}{16}t^4 + \cancel{c}^{\,0},$$

Thus $y_p(t) = \frac{t^3}{4}\ln(t) - \frac{t^4}{16} \cdot t^{-1} = \frac{t^3}{4}\ln(t) - \frac{t^3}{16}$

The general solution is

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = c_1 t^3 + c_2 t^{-1} + \frac{1}{4}t^3\ln(t) - \frac{1}{16}t^3,$$

or $\boxed{y(t) = c_3 t^3 + c_2 t^{-1} + \frac{1}{4}t^3\ln(t),}$ $\quad \left(c_3 = c_1 - \frac{1}{16}\right)$

where c_3 and c_2 are arbitrary constants.

5. [20] Find a particular solution of the following differential equation

$$y''' - y'' = 2t.$$

(third-order, linear, nonhomogeneous)

$y = e^{rt}$ in $y''' - y'' = 0$ leads to $r^3 - r^2 = 0 \Rightarrow r^2(r-1) = 0$ so $r = 0$ (mult. 2) or $r = 1$. Therefore $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = e^t$ is a F.S.S. Note that $g(t) = 2t$ solves the homogeneous equation so we must modify the usual trial form for a particular solution and use $y_p(t) = t^2(At+B)$ or equivalently $y_p(t) = At^3 + Bt^2$. Then $y_p' = 3At^2 + 2Bt$, $y_p'' = 6At + 2B$, and $y_p''' = 6A$.

We want $y_p''' - y_p'' = 2t$ so $6A - (6At + 2B) = 2t$ or $-6At + (6A - 2B) = 2t + 0$.

Therefore $-6A = 2$ and $6A - 2B = 0$ so $A = -\frac{1}{3}$ and $B = -1$. A particular solution is $y_p(t) = t^2\left(-\frac{1}{3}t - 1\right) = \boxed{-\frac{t^3}{3} - t^2}$

6. [20] Solve the initial value problem

$$y'' + y = \delta(t-\pi) + u_\pi(t), \quad y(0) = 1, \quad y'(0) = 0$$

and calculate $y(2\pi)$.

Since the forcing term $g(t) = \delta(t-\pi) + u_\pi(t)$ contains a Dirac delta, we use the Laplace transform method.

$$\mathcal{L}\{y'' + y\}(s) = \mathcal{L}\{\delta(\cdot - \pi) + u_\pi(\cdot)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + \mathcal{L}\{y\}(s) = e^{-\pi s} + \frac{e^{-\pi s}}{s}$$

by entries 10, 15, and 11 in the Laplace transform table. Then

$$(s^2+1)\mathcal{L}\{y\}(s) = s + e^{-\pi s} + \frac{e^{-\pi s}}{s}$$

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)}$$

P.F.D.

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$s = 0$: $1 = A$

$s = i$: $1 = (Bi + C)i$

$1 + 0i = -B + Ci$

$\therefore$ $B = -1$ and $C = 0$,

so $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\}$$

$$= \cos(t) + u_\pi(t)f(t-\pi) + u_\pi(t)g(t-\pi)$$

by entries 5 and 12 in the Laplace transform table.

Here $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$ and $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos(t)$

by entries 4, 1, and 5 in the Laplace transform table. Thus,

$$\boxed{y(t) = \cos(t) + u_\pi(t)\sin(t-\pi) + u_\pi(t)[1 - \cos(t-\pi)]}$$

solves the IVP.

$$y(2\pi) = \overset{1}{\cos(2\pi)} + \overset{1}{\cancel{u_\pi(2\pi)}}\overset{0}{\sin(2\pi-\pi)} + \overset{1}{\cancel{u_\pi(2\pi)}}[1 - \overset{-1}{\cos(2\pi-\pi)}] = \boxed{3}.$$

7. [20] Solve the integral equation

$$y + \int_0^t \tau\, y(t-\tau)\, d\tau = (t-1)u_1(t).$$

We write the integral term on the LHS as $(f*y)(t)$ where $f(t) = t$.
Taking the Laplace transform of both sides yields

$$\mathcal{L}\{ y + f*y \}(s) = \mathcal{L}\{ f(t-1)u_1(t) \}(s)$$

Using entries 14, 12, and 3 in the Laplace transform table gives

$$\mathcal{L}\{y\}(s) + \mathcal{L}\{f\}(s)\mathcal{L}\{y\}(s) = e^{-s}\mathcal{L}\{f\}(s)$$

and

$$\mathcal{L}\{y\}(s) + \frac{1}{s^2}\mathcal{L}\{y\}(s) = e^{-s} \cdot \frac{1}{s^2} .$$

Then

$$s^2\mathcal{L}\{y\}(s) + \mathcal{L}\{y\}(s) = e^{-s}$$

and

$$\mathcal{L}\{y\}(s) = \frac{e^{-s}}{s^2+1}$$

so

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s^2+1} \right\} = \boxed{u_1(t)\sin(t-1)}$$

by entries 12 and 4 in the Laplace transform table.

8. [20] In the absence of other factors, the population of mosquitoes in a certain area increases at a rate proportional to the current population, and the population doubles each week. There are 2,000,000 mosquitoes in the area initially, and predators eat 20,000 mosquitoes/day. If $P(t)$ denotes the population of mosquitoes after t days, then SET UP, BUT DO NOT SOLVE, an initial value problem that models the population of mosquitoes.

In the absence of other factors, $\frac{dP}{dt} = kP$ and $P(7) = 2P(0)$.

Then $\frac{dP}{P} = k\,dt \Rightarrow \ln(P(t)) = \int \frac{dP}{P} = \int k\,dt = kt + c$. It follows that $P(t) = e^{kt+c} = Ae^{kt}$ where $A = e^c$ is an arbitrary constant. Clearly $P(0) = A$ so $P(t) = P(0)e^{kt}$. Then $2P(0) = P(7) = P(0)e^{7k}$ implies $2 = e^{7k}$ so $\ln(2) = 7k$ and $k = \frac{\ln(2)}{7}$ (= rate constant for unrestrained population growth).

Accounting for predators as well, our model for the mosquito population, based on

$$\text{Net rate of change of mosquitoes} = \text{Rate of inflow of mosquitoes} - \text{Rate of outflow of mosquitoes},$$

is the IVP: $\boxed{\frac{dP}{dt} = \frac{\ln(2)}{7}P - 20{,}000 \quad \text{subject to } P(0) = 2{,}000{,}000.}$

(Here t is measured in days.)

Another correct solution is

$$\frac{dP}{d\tau} = \ln(2)P - 140{,}000, \quad P(0) = 2{,}000{,}000,$$

where τ is measured in weeks.

9. [20] Transform the differential equation

$$(*) \qquad y^{(4)} + y' - y = \sin t$$

into a system of first order equations and then a vector-matrix equation. "<u>Do not solve the system</u>".

Let $x_1(t) = y(t)$, $x_2(t) = y'(t)\ (= x_1'(t))$, $x_3(t) = y''(t)\ (= x_2'(t))$, and $x_4(t) = y'''(t)\ (= x_3'(t))$. From the DE $(*)$ we see that $x_4'(t) = y^{(4)}(t) = -y'(t) + y(t) + \sin(t) = -x_2(t) + x_1(t) + \sin(t)$. Therefore $(*)$ is equivalent to the system of first order differential equations:

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ x_3'(t) = x_4(t) \\ x_4'(t) = x_1(t) - x_2(t) + \sin(t) \, . \end{cases}$$

This system can be expressed in vector-matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(t) \end{bmatrix} .$$

10. [20] Given that $x^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ is a solution of the homogeneous system

$$x' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x,$$

find the general solution of the nonhomogeneous differential equation system

$$x' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x + \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Note that the eigenvalues λ of the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2,$$

so $\lambda = 2$ is an eigenvalue of multiplicity two. From the given information we observe that $\vec{k} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 2$; i.e. $A\vec{k} = 2\vec{k}$. A second solution of $\vec{x}' = A\vec{x}$ has the form $\vec{x}^{(2)}(t) = t\vec{k}e^{2t} + \vec{l}e^{2t}$ where $(A - 2I)\vec{l} = \vec{k}$. That is, $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which is equivalent to

$$\begin{cases} \cancel{-l_1 - l_2 = 1} & \text{Redundant} \\ l_1 + l_2 = -1 & \Rightarrow l_2 = -1 - l_1. \end{cases}$$

Thus, $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ -1-l_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} l_1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Without loss of generality, we may set $l_1 = 0$ so $\vec{l} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Therefore

$\vec{x}^{(2)}(t) = t\vec{k}e^{2t} + \vec{l}e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ and the general solution of the homogeneous system $\vec{x}' = A\vec{x}$ is $\vec{x}_h(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$ where c_1 and c_2 are arbitrary constants.

We use the method of undetermined coefficients to find a particular solution of the nonhomogeneous system. Since $\vec{g}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is not a solution of the homogeneous system, a trial particular solution is $\vec{x}_p(t) = \text{constant} = \begin{bmatrix} a \\ b \end{bmatrix}$ where a and b are constants to be determined such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{x}_p' = A\vec{x}_p + \vec{g}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

(OVER)

To solve $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ we use Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \Leftrightarrow \begin{cases} a - b = -1 \\ \quad 4b = 4 \end{cases}$$

so $b=1$ and $a=0$. I.e. $\vec{x}_p(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Putting together the pieces of information above, the general solution of the nonhomogeneous system is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + \vec{x}_p(t),$$

or

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{2t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{2t}\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}},$$

where c_1 and c_2 are arbitrary constants.

Short Table of Laplace Transforms

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n, \ n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
5.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
6.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
8.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
10.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
11.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
12.	$u_c(t)\,f(t-c)$	$e^{-cs}F(s)$
13.	$e^{ct}f(t)$	$F(s-c)$
14.	$(f * g)(t)$	$F(s)G(s)$
15.	$\delta(t-c)$	e^{-cs}

Math 3304

Final Exam

Fall 2015

number of exams = 28

mean = 129.8

median = 133.5

standard deviation = 45.9

Distribution of Scores

Range	Grade	Frequency
180 - 200	A	5
160 - 179	B	3
140 - 159	C	5
120 - 139	D	3
100 - 119	F	5
80 - 99	F	2
0 - 79	F	5