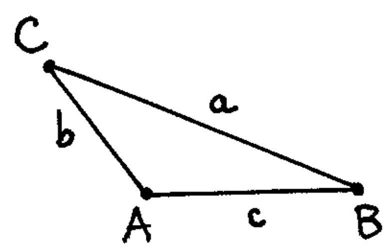
Sec. 6 : Oblique Triangles

Conventions: In an arbitrary triangle ABC, we will denote the side opposite vertex A by a, the side opposite vertex B by b, and the side opposite vertex C by c. Furthermore, we will use the same symbol to denote both a side and the length of that side, and the same symbol to denote a vertex and the measure of the interior angle at that vertex.



The Law of Sines: Let ABC be any triangle. Then

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

The Law of Cosines: Let ABC be any triangle. Then

$$a^2 + b^2 - 2ab\cos(C) = c^2,$$

$$b^2 + c^2 - 2bc\cos(A) = a^2,$$

$$c^2 + a^2 - 2ac\cos(B) = b^2.$$

Notes:

1. In a right triangle with hypotenuse c, the law of sines reduces to the right triangle definition of sine:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\underbrace{\sin(90^\circ)}_{1}} \iff \frac{a}{c} = \sin(A) \text{ and } \frac{b}{c} = \sin(B).$$

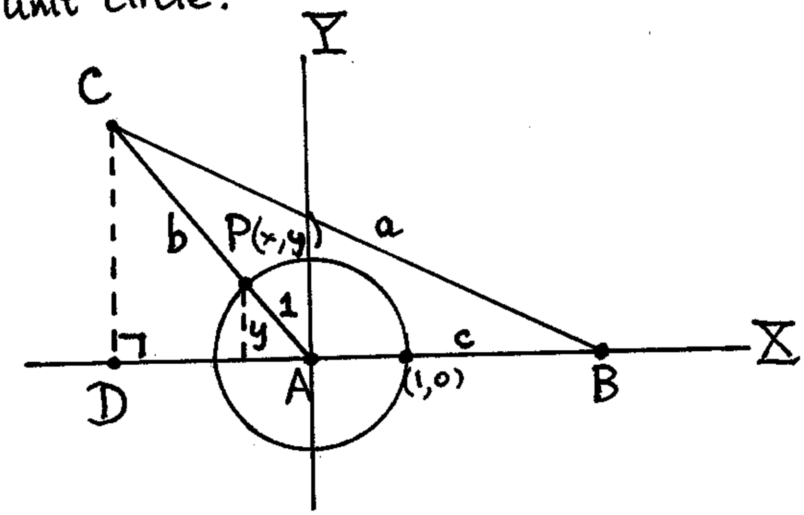
$$\frac{\text{opp}}{\text{hyp}}$$

2. The three formulas in the law of cosines can be expressed verbally as follows. "The square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides and the cosine of the angle opposite the first side."

3. In a right triangle with hypotenuse c, $\cos(C) = \cos(90^\circ) = 0$ so the law of cosines reduces to Pythagorus' theorem:

$$a^2 + b^2 - 2ab\cos(C) = c^2 \implies a^2 + b^2 = c^2.$$

<u>Proof of the law of sines:</u> Let A denote the largest angle in the triangle ABC and place A in standard position with side c along the positive X-axis. Drop a perpendicular from C to the X-axis and denote by D the foot of this perpendicular. Let P(x,y) be the point where the terminal side of A intersects the unit circle.



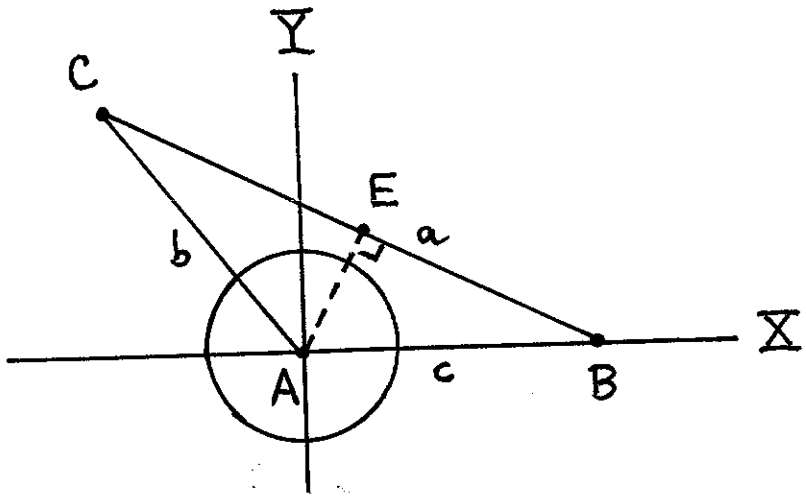
By similar triangles and the unit circle definition of sine,

$$\frac{\overline{CD}}{b} = \frac{y}{1} = \sin(A) \quad \text{so} \quad \overline{CD} = b\sin(A). \quad (*)$$

In the right triangle CDB, $\sin(B) = \frac{\text{opp}}{\text{hyp}} = \frac{\overline{CD}}{a}$ so $\overline{CD} = a\sin(B)$ (**). Equating the two expressions for $\overline{CD}$ in (*) and (**) yields $a\sin(B) = b\sin(A)$, and rearranging gives

$$(***) \qquad \frac{a}{\sin(A)} = \frac{b}{\sin(B)}.$$

Next, choose a point E on side a such that AE is perpendicular to a. (This is possible since B and C are acute angles.)



Then by right triangle trigonometry,

$$\frac{\overline{AE}}{b} = \sin(C) \qquad \text{and} \qquad \frac{\overline{AE}}{c} = \sin(B)$$

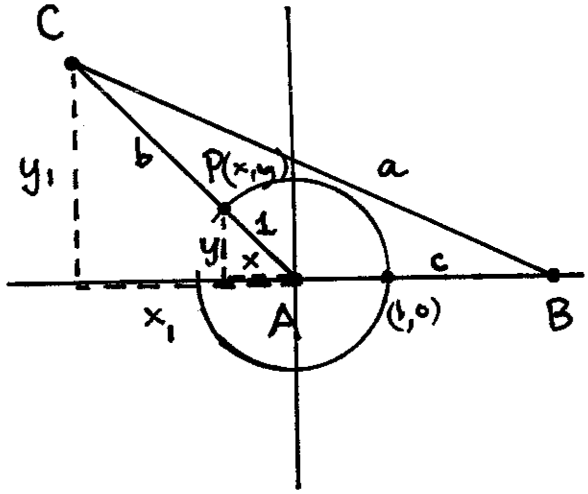
so $b\sin(C) = \overline{AE} = c\sin(B)$. Rearranging gives

$$(****) \qquad \frac{b}{\sin(B)} = \frac{c}{\sin(C)} .$$

Combining (***) and (****) leads to the desired conclusion:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} .$$

<u>Proof of the law of cosines</u>: Place the angle A of the triangle ABC in standard position so that side c lies along the positive X-axis. Let (x_1, y_1) denote the coordinates of C. Let $P(x,y)$ denote the



point of intersection of the terminal side of angle A with the unit circle. By similar triangles and the unit circle definitions of the trig functions

$$\frac{y_1}{b} = \frac{y}{1} = \sin(A)$$

and

$$\frac{x_1}{b} = \frac{x}{1} = \cos(A).$$

Therefore the coordinates of C are $(b\cos(A), b\sin(A))$. The coordinates of B are $(c, 0)$. By the formula for the distance between two points in the plane we have

$$a^2 = \overline{CB}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (c - b\cos(A))^2 + (0 - b\sin(A))^2.$$

Simplifying yields

$$a^2 = c^2 - 2bc\cos(A) + b^2\cos^2(A) + b^2\sin^2(A)$$

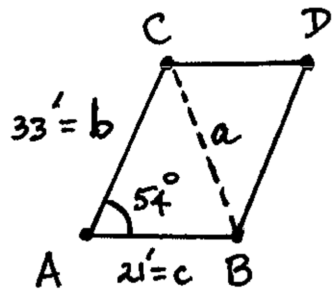
$$a^2 = c^2 - 2bc\cos(A) + b^2(\cos^2(A) + \sin^2(A))$$

$$a^2 = b^2 + c^2 - 2bc\cos(A).$$

The angle A and its opposite side a were not assumed to possess any singular properties. Therefore the same argument can be applied to derive the other two formulas in the law of cosines.

Example 1: Two sides of a parallelogram make an angle of 54°. The lengths of the two sides are 21 and 33 feet. Find the length of each diagonal.

Solution:



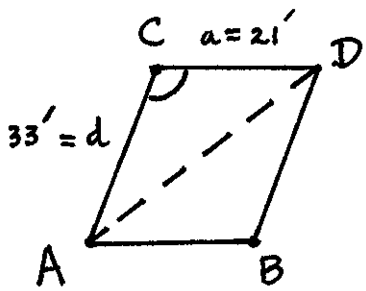
Since we know two sides and the included angle, we apply the law of cosines to the triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$= (33)^2 + (21)^2 - 2(33)(21)\cos(54^\circ)$$

$$\doteq 715.33$$

Therefore $\boxed{a \doteq 26.75 \text{ feet}}$.



To find the second diagonal using the law of cosines, we will need angle **C**:

$$C = 180^\circ - 54^\circ = 126^\circ.$$

Then applying the law of cosines gives

$$c^2 = a^2 + d^2 - 2ad\cos(C)$$

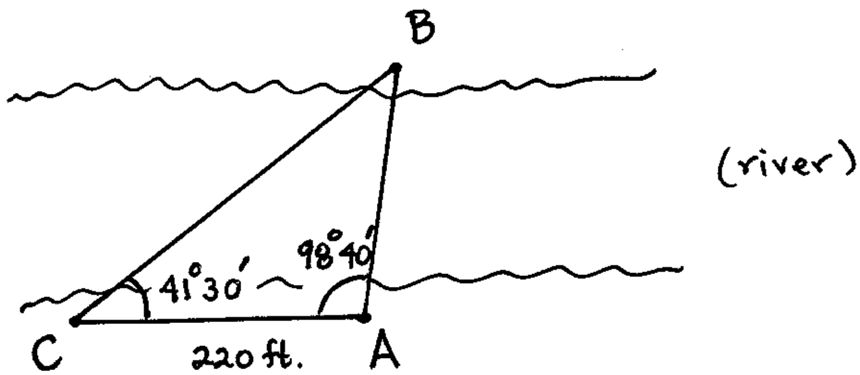
$$= (21)^2 + (33)^2 - 2(21)(33)\cos(126^\circ)$$

$$\doteq 2842.30$$

Therefore, the second diagonal is $\boxed{c \doteq 53.31 \text{ feet.}}$

Example 2: To find the distance between two points A and B on opposite sides of a river, we measure the distance from A to C to be 220 feet, the angle CAB to be $98°40'$, and the angle ACB to be $41°30'$. Compute the distance $\overline{AB}$.

Solution:



Since we know two angles and the included side, we apply the law of sines to find $\overline{AB} = c$:

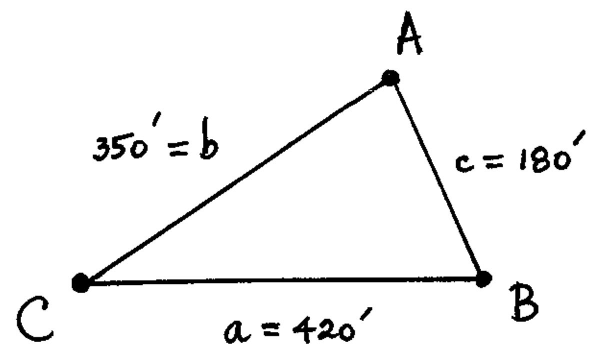
$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)} \Rightarrow c = \frac{b\sin(C)}{\sin(B)}$$

$$= \frac{(220)\sin(41°30')}{\sin(39°50')}$$

$$\doteq \boxed{227.6 \text{ feet}}$$

Example 3: A triangular plot of land has sides of lengths 420 feet, 350 feet, and 180 feet. Approximate the smallest angle between the sides.

Solution:



Since we are given the three sides of triangle ABC, we use the law of cosines to find the angle C opposite the shortest side c:

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Solving for cos (C) gives

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(420)^2 + (350)^2 - (180)^2}{2(420)(350)}$$

$$\doteq .9064626$$

Therefore $C = \arccos(.9064626) \doteq \boxed{24.98^\circ \doteq 24^\circ 59'.}$

Let ABC be any triangle. Then the area of ABC is:

(a) one-half the product of the lengths of any two sides and the sine of the angle between them;

(b) $\sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle: $s = \dfrac{a+b+c}{2}$.

<u>Example 4</u>: Find the areas of the triangles satisfying the given data.

(a) $a = 27$, $b = 36$, $c = 49$.

(b) $a = 87$, $c = 14$, $B = 74°$.

<u>Solution</u>: (a) Since all three sides are known, we can use Heron's formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where

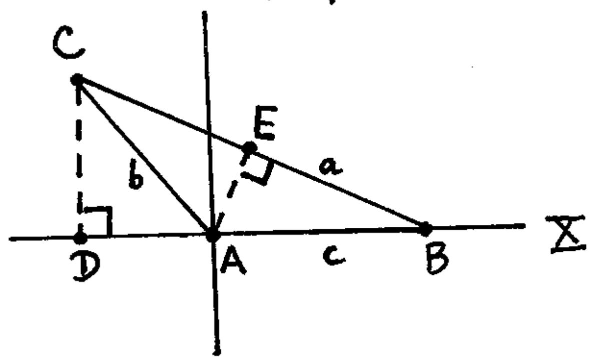
$$s = \frac{a+b+c}{2} = \frac{27+36+49}{2} = 56. \text{ Then}$$

$$\text{Area} = \sqrt{\underset{\substack{\uparrow\\ 8\cdot 7}}{56}(29)(\underset{\substack{\uparrow\\ 4\cdot 5}}{20})(7)} = 4\cdot 7\sqrt{2\cdot 29\cdot 5} = \boxed{28\sqrt{290} \doteq 476.8}$$

(b) We use the principle (a) in the theorem at the top of this page:

$$\text{Area} = \frac{1}{2}ac\sin(B) = \frac{1}{2}(87)(14)\sin(74°) = \boxed{609\sin(74°) \doteq 585.4}$$

Proof of (a): Let A denote the largest angle in the triangle ABC. Place A in standard position with side c along the positive X-axis. Drop a perpendicular from C to the X-axis and denote by D the foot of this perpendicular. Choose a point E on the side a such that AE is perpendicular to side a. Then unit circle and right triangle trigonometry give the following relations:



$$\frac{\overline{CD}}{b} = \sin(A) \quad \text{so} \quad \overline{CD} = b\sin(A), \qquad (*)$$

$$\frac{\overline{AE}}{c} = \sin(B) \quad \text{so} \quad \overline{AE} = c\sin(B), \qquad (**)$$

$$\frac{\overline{AE}}{b} = \sin(C) \quad \text{so} \quad \overline{AE} = b\sin(C). \qquad (***)$$

Using the basic formula Area = $\frac{1}{2}$(base)(height) for a triangle together with (*), (**), and (***) gives

$$\text{Area} = \tfrac{1}{2}(\overline{CD})(\overline{AB}) = \tfrac{1}{2}(b\sin(A))c = \tfrac{1}{2}bc\sin(A),$$

$$\text{Area} = \tfrac{1}{2}(\overline{AE})(\overline{BC}) = \tfrac{1}{2}(c\sin(B))a = \tfrac{1}{2}ac\sin(B),$$

$$\text{Area} = \tfrac{1}{2}(\overline{AE})(\overline{BC}) = \tfrac{1}{2}(b\sin(C))a = \tfrac{1}{2}ab\sin(C).$$

Proof of Heron's formula (b): From part (a), the area of triangle ABC is given by

$$(*) \qquad \text{Area} = \tfrac{1}{2} bc\sin(A).$$

But the law of cosines implies

$$(\dagger) \qquad \cos(A) = \frac{b^2+c^2-a^2}{2bc}.$$

Applying the Pythagorean identity and $(\dagger)$ gives

$$(**) \qquad \sin(A) = \sqrt{1-\cos^2(A)} = \sqrt{1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2} = \frac{\sqrt{(2bc)^2-(b^2+c^2-a^2)^2}}{2bc}.$$

Substituting from $(**)$ into $(*)$ and simplifying using the identities $x^2-y^2=(x-y)(x+y)$ and $x^2 \pm 2xy + y^2 = (x \pm y)^2$ leads to

$$\begin{aligned}
\text{Area} &= \frac{\sqrt{(2bc)^2-(b^2+c^2-a^2)^2}}{4} \\
&= \frac{1}{4}\sqrt{[2bc-(b^2+c^2-a^2)][2bc+b^2+c^2-a^2]} \\
&= \frac{1}{4}\sqrt{[a^2-(b-c)^2][(b+c)^2-a^2]} \\
&= \sqrt{\frac{[a-(b-c)][a+b-c][b+c-a][b+c+a]}{16}} \\
&= \sqrt{\left(\frac{a+b+c-2b}{2}\right)\left(\frac{a+b+c-2c}{2}\right)\left(\frac{a+b+c-2a}{2}\right)\left(\frac{a+b+c}{2}\right)} \\
&= \sqrt{(s-b)(s-c)(s-a)s}.
\end{aligned}$$