Sec. 3: Trigonometric Identities (Theory)

A trigonometric identity is a relation between trig functions that holds at all values where each of the trig functions involved is defined.

The Fundamental Trig Identities

Reciprocal

$$\boxed{\sec(t) = \frac{1}{\cos(t)}}$$

$$\boxed{\csc(t) = \frac{1}{\sin(t)}}$$

$$\boxed{\cot(t) = \frac{1}{\tan(t)}}$$

Quotient

$$\boxed{\tan(t) = \frac{\sin(t)}{\cos(t)}}$$

$$\cot(t) = \frac{\cos(t)}{\sin(t)}$$

Even/Odd

$$\boxed{\cos(-t) = \cos(t)}$$

$$\boxed{\sin(-t) = -\sin(t)}$$

$$\tan(-t) = -\tan(t)$$

$$\cot(-t) = -\cot(t)$$

$$\sec(-t) = \sec(t)$$

$$\csc(-t) = -\csc(t)$$

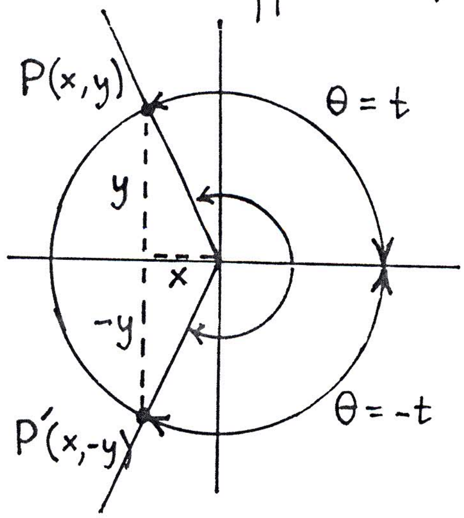
Pythagorean

$$\boxed{\cos^2(t) + \sin^2(t) = 1}$$

$$1 + \tan^2(t) = \sec^2(t)$$

$$1 + \cot^2(t) = \csc^2(t)$$

You should memorize the seven boxed fundamental trig identities. The first five boxed identities follow directly from the definitions. The two boxed even/odd identities are apparent from the diagram below.



$$\cos(-t) = x = \cos(t)$$

$$\sin(-t) = -y = -\sin(t)$$

The remaining seven unboxed fundamental trig identities can easily be derived from the boxed identities.

1. The second quotient identity follows from the third reciprocal identity and the first quotient identity:

$$\cot(t) = \frac{1}{\tan(t)} = \frac{1}{\frac{\sin(t)}{\cos(t)}} = \frac{\cos(t)}{\sin(t)}.$$

2. The second Pythagorean identity follows from the first Pythagorean identity by dividing through by $\cos^2(t)$:

$$\cos^2(t) + \sin^2(t) = 1 \quad \Rightarrow \quad \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

$$\Rightarrow \quad 1 + \left(\frac{\sin(t)}{\cos(t)}\right)^2 = \left(\frac{1}{\cos(t)}\right)^2 \quad \Rightarrow \quad 1 + \tan^2(t) = \sec^2(t).$$

3. The third even/odd identity follows from the first quotient identity and the first two even/odd identities:

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t).$$

The Sum and Difference Identities

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\boxed{\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\boxed{\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)}$$

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

You should memorize the two boxed sum identities. The remaining sum and difference identities can be derived from the two boxed sum identities and the boxed fundamental identities.

1. The cosine difference identity follows from the cosine sum identity and the even/odd identities:

$$\begin{aligned}\cos(A-B) &= \cos(A+(-B)) \\ &= \cos(A)\cos(-B) - \sin(A)\sin(-B) \\ &= \cos(A)\cos(B) + \sin(A)\sin(B)\end{aligned}$$

2. The tangent sum identity follows from the tangent quotient identity and the sine and cosine sum identities:

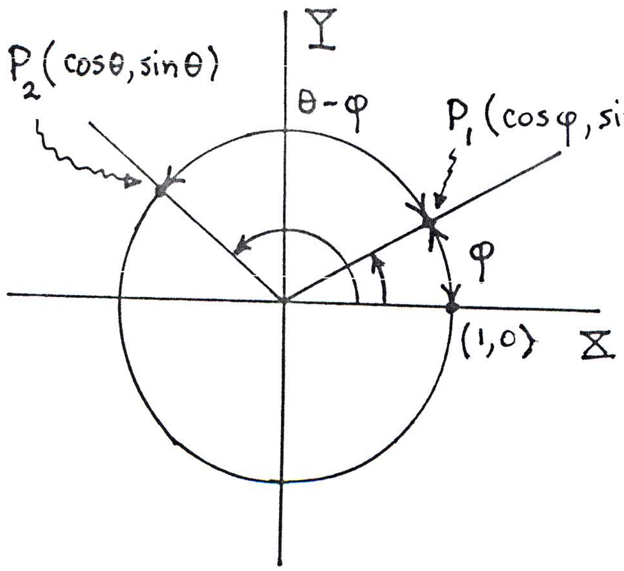
$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$= \frac{\dfrac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B)}}{\dfrac{\cos(A)\cos(B) - \sin(A)\sin(B)}{\cos(A)\cos(B)}}$$

$$= \frac{\dfrac{\sin(A)}{\cos(A)} + \dfrac{\sin(B)}{\cos(B)}}{1 - \dfrac{\sin(A)}{\cos(A)} \cdot \dfrac{\sin(B)}{\cos(B)}}$$

$$= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

Proof of the cosine difference identity:



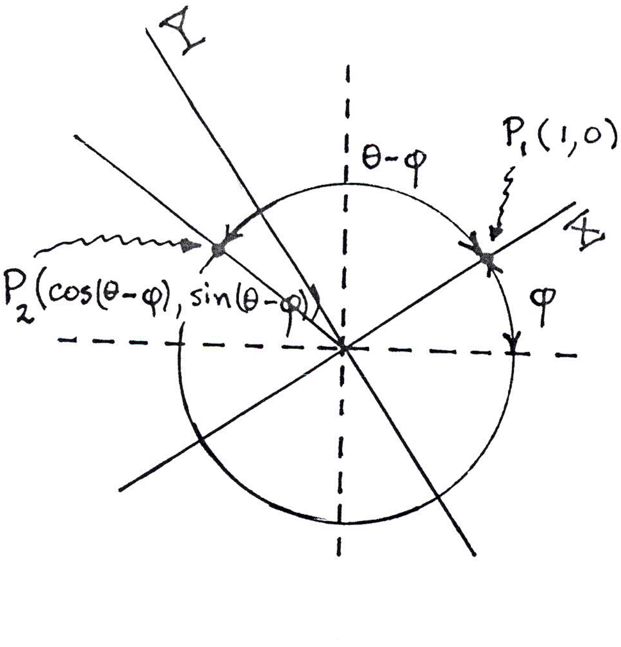
Construct angles in standard position with radian measures φ and θ, respectively. Relabeling if necessary, we may assume $\theta \geq \varphi$. Using the formula for the distance between two points in the plane we have

$$\overline{P_1P_2}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 = (\cos\theta - \cos\varphi)^2 + (\sin\theta - \sin\varphi)^2$$

$$= \cos^2(\theta) - 2\cos(\theta)\cos(\varphi) + \cos^2(\varphi) + \sin^2(\theta) - 2\sin(\theta)\sin(\varphi) + \sin^2(\varphi)$$

$$= \cos^2(\theta) + \sin^2(\theta) + \cos^2(\varphi) + \sin^2(\varphi) - 2\left(\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)\right)$$

$$= 2 - 2\left(\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)\right).$$



Now rotate the the X-Y coordinate axes by an angle with radian measure φ. In the new coordinates $P_1 = P_1(1,0)$ and $P_2 = P_2(\cos(\theta-\varphi), \sin(\theta-\varphi))$. As before, using the distance formula we have

$$\overline{P_1P_2}^2 = (\cos(\theta-\varphi) - 1)^2 + (\sin(\theta-\varphi) - 0)^2$$

$$= \cos^2(\theta-\varphi) - 2\cos(\theta-\varphi) + 1 + \sin^2(\theta-\varphi) = 2 - 2\cos(\theta-\varphi).$$

Equating the two expressions for $\overline{P_1P_2}^2$ yields

$$2-2\cos(\theta-\varphi) = 2-2\left(\cos(\theta)\cos(\varphi)+\sin(\theta)\sin(\varphi)\right)$$

and simplifying gives

$$\boxed{\cos(\theta-\varphi) = \cos(\theta)\cos(\varphi)+\sin(\theta)\sin(\varphi).}$$

<u>Proof of the cosine sum identity</u>:

$$\begin{aligned}
\cos(\theta+\varphi) &= \cos(\theta-(-\varphi)) \\
&= \cos(\theta)\cos(-\varphi)+\sin(\theta)\sin(-\varphi) && \text{(by cosine difference)} \\
&= \cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi) && \text{(by even/odd)}
\end{aligned}$$

<u>Proof of the sine sum identity</u>:

Set $\theta = \pi/2$ and $\varphi = t$ in the cosine difference identity to obtain

$$\begin{aligned}
(*) \quad \cos\left(\tfrac{\pi}{2}-t\right) &= \cos\left(\tfrac{\pi}{2}\right)\cos(t)+\sin\left(\tfrac{\pi}{2}\right)\sin(t) \\
&= 0\cdot\cos(t) + 1\cdot\sin(t) \\
&= \sin(t).
\end{aligned}$$

Thus

$$\begin{aligned}
\sin\left(\tfrac{\pi}{2}-\theta\right) &= \cos\left(\tfrac{\pi}{2}-\left(\tfrac{\pi}{2}-\theta\right)\right) && \left(\text{take } t=\tfrac{\pi}{2}-\theta \text{ in } (*)\right) \\
&= \cos(\theta). \quad (**)
\end{aligned}$$

Therefore

$$\begin{aligned}
\sin(\theta+\varphi) &= \cos\left(\tfrac{\pi}{2}-\theta-\varphi\right) && (\text{take } t=\theta+\varphi \text{ in } (*)) \\
&= \cos\left(\tfrac{\pi}{2}-\theta\right)\cos(\varphi)+\sin\left(\tfrac{\pi}{2}-\theta\right)\sin(\varphi) && \text{(by cosine difference)} \\
&= \sin(\theta)\cos(\varphi)+\cos(\theta)\sin(\varphi) && (\text{by } (*) \text{ and } (**)).
\end{aligned}$$

Double - Angle Identities

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 2\cos^2(A) - 1\\ &= 1 - 2\sin^2(A)\end{aligned}$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

These identities are special cases of the sum identities and need not be memorized. For example:

$$\sin(2A) = \sin(A+A) = \sin(A)\cos(A) + \cos(A)\sin(A) = 2\sin(A)\cos(A).$$

$$\begin{aligned}\cos(2A) &= \cos(A+A) = \cos(A)\cos(A) - \sin(A)\sin(A) = \cos^2(A) - \sin^2(A)\\ &= \cos^2(A) - (1 - \cos^2(A)) = 2\cos^2(A) - 1.\end{aligned}$$

$$\tan(2A) = \tan(A+A) = \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)} = \frac{2\tan(A)}{1 - \tan^2(A)}.$$

Product Identities

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

These identities can easily be derived from the sum and difference identities and need not be memorized. For example, adding the cosine sum and difference identities gives

$$\begin{array}{rrcl} & \cos(A-B) & = & \cos(A)\cos(B) + \sin(A)\sin(B) \\ + & \cos(A+B) & = & \cos(A)\cos(B) - \sin(A)\sin(B) \\ \hline & \cos(A-B) + \cos(A+B) & = & 2\cos(A)\cos(B). \end{array}$$

Adding the sine sum and difference identities, we have

$$\begin{array}{rrcl} & \sin(A-B) & = & \sin(A)\cos(B) - \cos(A)\sin(B) \\ + & \sin(A+B) & = & \sin(A)\cos(B) + \cos(A)\sin(B) \\ \hline & \sin(A-B) + \sin(A+B) & = & 2\sin(A)\cos(B). \end{array}$$

Half-Angle Identities

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}$$

$$= \frac{1-\cos(A)}{\sin(A)}$$

$$= \frac{\sin(A)}{1+\cos(A)}$$

These identities can easily be derived from the double-angle identities. For example,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1$$

so

$$\frac{1+\cos(2\theta)}{2} = \cos^2(\theta).$$

Taking $\theta = A/2$ in this last identity and extracting roots gives

$$\pm\sqrt{\frac{1+\cos(A)}{2}} = \cos(A/2).$$

Also

$$\tan\left(\frac{A}{2}\right) = \frac{\sin(A/2)}{\cos(A/2)} = \frac{\pm\sqrt{\frac{1-\cos(A)}{2}}}{\pm\sqrt{\frac{1+\cos(A)}{2}}} = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}.$$

Further Product Identities

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos(u) - \cos(v) = 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{v-u}{2}\right)$$

These identities can be derived using the previous product identities. For example, consider

$$(*) \qquad \cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right].$$

Set $u = A - B$ and $v = A + B$. Then $u+v = 2A$ and $u-v = -2B$ so substituting in $(*)$ gives

$$2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \cos(u) + \cos(v).$$