Mathematics 325 Exam III Fall 2000 Name: Dr. Grow (1 pt.)

1. (33 pts.) Consider the eigenvalue problem

(*) $-X''(x) = \lambda X(x)$ for $0 < x < 1$, $X'(0) = 0 = X'(1) - X(1)$.

(a) Find an algebraic equation that characterizes the positive eigenvalues of (*).

(b) Show graphically from the equation in part (a) that there is an infinite sequence of positive eigenvalues of (*).

(c) Is zero an eigenvalue of (*)? Support your answer.

(d) Are there any negative eigenvalues of (*)? Support your answer.

(e) Write the eigenfunction(s) coresponding to each eigenvalue of (*).

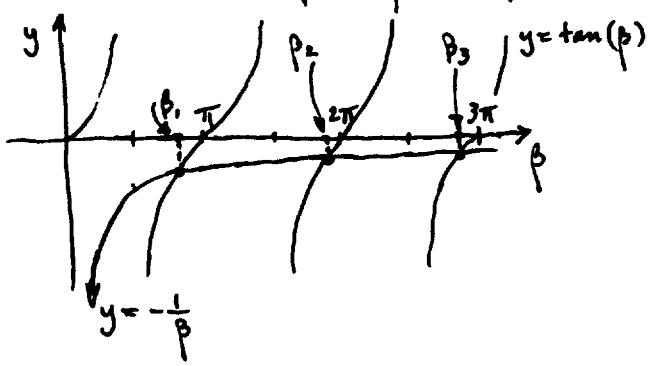
(a) $\underline{\lambda = \beta^2 \ (\beta > 0)}$: $X'' + \beta^2 X = 0 \Rightarrow X(x) = c_1\cos(\beta x) + c_2 \sin(\beta x)$

$X'(x) = -\beta c_1 \sin(\beta x) + \beta c_2 \cos(\beta x)$

$0 = X'(0) = -\beta c_1 \cdot 0 + \beta c_2 \cdot 1 \Rightarrow c_2 = 0$

$0 = X'(1) - X(1) = -\beta \cancel{c_1} \sin\beta - \cancel{c_1}\cos(\beta) \Rightarrow \boxed{\tan(\beta) = -\frac{1}{\beta}}$

(b)



$\therefore$ there exists an infinite sequence of positive eigenvalues $\lambda_n = \beta_n^2$ $(n = 1, 2, 3, \ldots)$ satisfying $\lim_{n\to\infty} \left[\lambda_n - (n\pi)^2\right] = 0$.

(c) $\underline{\lambda = 0}$: $X'' = 0 \Rightarrow X(x) = c_1 x + c_2$, $X'(x) = c_1$

$0 = X'(0) = c_1$ and $0 = X'(1) - X(1) = c_1 - (c_1 + c_2) = c_2 \Rightarrow c_1 = c_2 = 0$

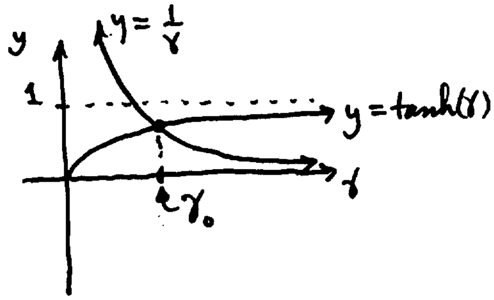
I.e. $\boxed{\text{0 is not an eigenvalue}}$.

(d) $\underline{\lambda = -\gamma^2 \ (\gamma > 0)}$: $X'' - \gamma^2 X = 0 \Rightarrow X(x) = c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x)$

$X'(x) = \gamma c_1 \sinh(\gamma x) + \gamma c_2 \cosh(\gamma x)$

$0 = X'(0) = \gamma c_1 \cdot 0 + \gamma c_2 \cdot 1 \Rightarrow c_2 = 0.$

$0 = X'(1) - X(1) = \gamma \cancel{c_1} \sinh(\gamma) - \cancel{c_1}\cosh(\gamma) \Rightarrow \boxed{\tanh(\gamma) = \frac{1}{\gamma}}$



$\boxed{\text{There is one negative eigenvalue } \lambda_0 = -\gamma_0^2.}$

(e) $\lambda_n = \beta_n^2 \Rightarrow \boxed{X_n(x) = \cos(\beta_n x)}$ $(n = 1, 2, 3, \ldots)$

$\lambda_0 = -\gamma_0^2 \Rightarrow \boxed{X_0(x) = \cosh(\gamma_0 x)}$

2. (33 pts.) <u>ASSUME</u> that the Fourier series of the function

$$\phi(x) = x^3 - x \qquad \text{for } 0 \le x \le 1$$

with respect to the orthogonal set $\{\sin(n\pi x)\}_{n=1}^{\infty}$ on $[0,1]$ is given by

$$\sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3}.$$

(a) Write the partial sum of the Fourier sine series of ϕ corresponding to the first three nonzero terms, and sketch its graph on the same set of coordinate axes as the graph of ϕ.

(b) Does the Fourier sine series of ϕ converge to ϕ uniformly on $[0,1]$? Justify your answer.

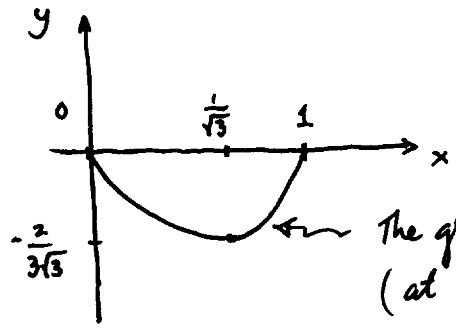
(c) Does the Fourier sine series of ϕ converge pointwise to ϕ on $[0,1]$? Justify your answer.

(d) Does the Fourier sine series of ϕ converge to ϕ in the L^2-sense on $[0,1]$? Justify your answer.

(e) Use the preceding parts of this problem to help compute the sums of the two series

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{m^6}.$$

(a) $$S_3(x) = \sum_{n=1}^{3} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3} = \boxed{\frac{12}{\pi^3}\left(-\sin(\pi x) + \frac{1}{8}\sin(2\pi x) - \frac{1}{27}\sin(3\pi x)\right)}$$



The graphs of $y = S_3(x)$ and $y = \varphi(x)$ are indistinguishable (at this resolution) on $[0,1]$.

(b) Yes, the convergence of the sine series of φ to φ is uniform on $[0,1]$ by Theorem 2 of Sec. 5.4. (Note that $\{\sin(n\pi x)\}_{n=1}^{\infty}$ is the complete set of (orthogonal) eigenfunctions of $-X'' = \lambda X$, $X(0) = X(1) = 0$ (hermitian B.C.'s) and $\varphi(x) = x^3 - x$ is C^2 on $[0,1]$ and satisfies the B.C.'s: $\varphi(0) = 0 = \varphi(1)$.)

(c) Yes, the sine series of φ converges pointwise to φ on $[0,1]$ since uniform convergence (on bounded intervals) implies pointwise convergence.

(cont.)

(d) Yes, the sine series of φ converges in the L^2-sense to φ on $[0,1]$ since uniform convergence (on bounded intervals) implies L^2-convergence.

(e) By (c), $x^3 - x = \varphi(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3}$ for all $0 \le x \le 1$.

Take $x = \frac{1}{2}$: $-\frac{3}{8} = \varphi\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi/2)}{(n\pi)^3}$.

n	$\sin(n\pi/2)$
1	1
2	0
3	-1
4	0
$\vdots$	
(even) 2m	0
(odd) 2m+1	$(-1)^m$

$$\therefore \quad -\frac{3}{8} = \frac{12}{\pi^3} \sum_{m=0}^{\infty} \frac{(-1)^{2m+1} \sin((2m+1)\pi/2)}{(2m+1)^3}$$

$$\Rightarrow \boxed{\frac{\pi^3}{32} = \left(-\frac{\pi^3}{12}\right)\left(-\frac{3}{8}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}}$$

By Parseval's identity $\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |\varphi(x)|^2 dx$, we have

$$\sum_{n=1}^{\infty} \left|\frac{12(-1)^n}{(n\pi)^3}\right|^2 \underbrace{\int_0^1 \sin^2(n\pi x)\,dx}_{1/2 \text{ (for all } n)} = \int_0^1 (x^3 - x)^2 dx = \int_0^1 (x^6 - 2x^4 + x^2)\,dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{144}{n^6\pi^6} \cdot \frac{1}{2} = \left(\frac{x^7}{7} - \frac{2x^5}{5} + \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{7} - \frac{2}{5} + \frac{1}{3} = \frac{8}{105}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{8}{105} \cdot \frac{\pi^6}{72} = \frac{\pi^6}{945}}$$

3.(33 pts.) (a) Find a solution to

$$u_{xx} + u_{yy} = 0 \quad \text{in } 0 < x < 1,\ 0 < y < 1,$$

which satisfies

$$u(0,y) = 0 = u(1,y) \quad \text{for } 0 \le y \le 1,$$

and

$$u(x,0) = 0, \quad u(x,1) = x^3 - x \quad \text{for } 0 \le x \le 1.$$

(Hint: You may find some of the results of problem 2 useful.)

(b) State the maximum/minimum principle for harmonic functions, and use it to show that your solution in part (a) is the only one possible.

(a) $u(x,y) = X(x)Y(y) \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow -\dfrac{X''(x)}{X(x)} = \dfrac{Y''(y)}{Y(y)} = \lambda$

$$\begin{cases} X'' + \lambda X = 0, & X(0) = X(1) = 0 \\ Y'' - \lambda Y = 0, & Y(0) \end{cases} \Rightarrow \begin{array}{l} \lambda_n = (n\pi),\ X_n(x) = \sin(n\pi x) \quad (n=1,2,3,\ldots) \\ Y_n(y) = \sinh(n\pi y) \end{array}$$

$\therefore\ u(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sinh(n\pi y)$ solves the homogeneous part of the problem.

By #2(d) Want this equality to hold for $0 \le x \le 1$

$$\sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3} = x^3 - x = u(x,1) = \sum_{n=1}^{\infty} [c_n \sinh(n\pi)] \sin(n\pi x)$$

$\therefore\ c_n \sinh(n\pi) = \dfrac{12(-1)^n}{(n\pi)^3}$ for all $n = 1,2,3,\ldots$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)\sinh(n\pi y)}{(n\pi)^3 \sinh(n\pi)}}\ .$$

(b) (Weak) Max/min Principle: Let $u = u(x,y)$ be a solution to $u_{xx} + u_{yy} = 0$ in a bounded open set D in the plane and let u be continuous on $\bar{D} = D \cup \partial D$. Then $\max_{(x,y)\in\bar{D}} u(x,y) = \max_{(x,y)\in\partial D} u(x,y)$ and $\min_{(x,y)\in\bar{D}} u(x,y) = \min_{(x,y)\in\partial D} u(x,y)$.

The solution to the problem 3(a) is unique. To see this, suppose that $u = v(x,y)$ were another solution. Then $w(x,y) = u(x,y) - v(x,y)$ would solve $w_{xx} + w_{yy} = 0$ in $D: 0<x<1, 0<y<1$, w would be continuous

(cont.)

on $\overline{D}: 0 \le x \le 1, 0 \le y \le 1$, and $w(x,y) = 0$ for all $(x,y) \in \partial D$. Hence, the max/min principle for harmonic functions would imply that, for all $(x_0, y_0) \in \overline{D}$,

$$0 = \min_{(x,y)\in\partial D} w(x,y) \le w(x_0,y_0) \le \max_{(x,y)\in\partial D} w(x,y) = 0 .$$

That is, $w(x_0,y_0) = 0$ for all $(x_0,y_0) \in \overline{D}$, and hence $u(x_0,y_0) = v(x_0,y_0)$ for all $(x_0,y_0) \in \overline{D}$.