11 problems are of equal value. I choose to have my test be worth (circle one) 200 points / 300 points.

1. (a) Find the general solution in the xy-plane of
\[ yu_x - xu_y = 0. \]
(b) Sketch and name some characteristic curves of \((*)\).

2. (a) Classify the partial differential equation
\[ u_{xx} - 4u_{xt} + 4u_{tt} = 0 \]
as elliptic, parabolic, or hyperbolic.
(b) If it is possible, derive the general solution to \((*)\) in the xt-plane.
(c) Find the solution to \((*)\) in the upper xt-halfplane which satisfies the initial conditions
\[ u(x,0) = xe^{2x} + 4x^2 \quad \text{and} \quad u_t(x,0) = (x-2)e^{2x} + 4x \quad \text{for} \quad -\infty < x < \infty. \]

3. Let \( \phi \) be an absolutely integrable function on \((-\infty, \infty)\). Use Fourier transform methods to find a solution to
\[ u_t - u_{xx} + 3u = 0 \quad \text{in} \quad -\infty < x < \infty, \quad 0 < t < \infty, \]
which satisfies the initial condition
\[ u(x,0) = \phi(x) \quad \text{for} \quad -\infty < x < \infty. \]

4. Consider an infinite string with linear density \( \rho = 1 \) and tension \( T = 1 \), initially occupying the position of the x-axis. At time \( t = 0 \) and at general horizontal position \( x \), the string is displaced vertically by an amount \( e^{-x^2} \) and released with velocity \( 2xe^{-x^2} \).
(a) Find the vertical displacement of the string as a function of position \( x \) and time \( t \), and simplify your formula as much as possible.
(b) Sketch profiles of the vertical displacement function at times \( t = 1, t = 2, \) and \( t = 3 \).

5. The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Centigrade on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies \( u_r = \gamma \)
where \( \gamma \) is a positive constant.
(a) Find the temperature distribution function for the material.
(b) What are the hottest and coldest temperatures in the material?
(c) Is it possible to choose \( \gamma \) so that the temperature on the outer boundary is 20 degrees Centigrade? Support your answer.
6. (a) Find a solution to
\[ \nabla^2 u = 0 \]
in the cube \( C: 0 < x < 1, 0 < y < 1, 0 < z < 1, \)
subject to the boundary conditions
\[ u(x,y,1) = \sin(\pi x)\sin^3(\pi y) \quad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1, \]
and \( u = 0 \) on the other five faces of the cube \( C. \)
(b) State the maximum/minimum principle for harmonic functions, and use it to show that the problem in part (a) has only one solution.

7. (a) Find a solution to
(1) \[ u_{tt} - u_{xx} + 2u_t = 0 \]
in \( 0 < x < \pi, 0 < t < \infty, \)
satisfying
(2) \[ u_x(0,t) = 0 = u_x(\pi,t) \quad \text{for } t \geq 0, \]
(3) \[ u(x,0) = 0 \quad \text{for } 0 \leq x \leq \pi, \]
(4) \[ u_t(x,0) = x^2 \quad \text{for } 0 \leq x \leq \pi. \]
(b) If \( u = u(x,t) \) satisfies (1)-(2), show that its energy
\[ E(t) = \frac{1}{2} \int_0^\pi [u_t^2(x,t) + u_x^2(x,t)] \, dx \]
is decreasing on \( 0 \leq t < \infty. \)
(c) Is there only one solution to the problem in part (a)? Why or why not?
\#1. \[ y u_x - x u_y = 0 \] (*)

\[ D_{y,x} u = 0 \]

Characteristic curves:
\[ \frac{dy}{dx} = \frac{-x}{y} \]
\[ \Rightarrow y \, dy = -x \, dx \]
\[ \Rightarrow \frac{1}{2} y^2 = -\frac{1}{2} x^2 + c \]
\[ \Rightarrow x^2 + y^2 = k \quad (\text{circles with center at } (0,0)) \]

Along such curves,
\[ u(x,y) = u(x, t\sqrt{k-x^2}) = u(0, t\sqrt{k}) = f(k) \]

\[ \therefore [u(x,y) = f(x^2+y^2)] \quad \text{is the general solution to (*)}, \quad \text{where } f = f(t) \]

is a $C^1$ function on $[0,\infty)$. 

\[ x^2 + y^2 = 1 \]

\[ x^2 + y^2 = 4 \]
\[ u_{xx} - 4u_{xt} + 4u_{tt} = 0 \]

\[ \left( \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial t} \right)^2 u = 0 \]

Let \[ \begin{cases} \xi = 2x + t \\ \eta = -x + 2t \end{cases} \]

\[ \begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = 2 \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi} + 2 \frac{\partial u}{\partial \eta}
\end{align*} \]

i.e. \[ \begin{align*}
\frac{\partial}{\partial x} &= 2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi} + 2 \frac{\partial}{\partial \eta}
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial \xi} - 2 \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \xi} + 2 \frac{\partial}{\partial \eta} \right) \\
&= \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \xi} + 2 \frac{\partial}{\partial \eta} \right) \\
&= -5 \frac{\partial}{\partial \eta}
\end{align*} \]

(+) is equivalent to \[ (-5 \frac{\partial^2}{\partial \eta^2}) u = 0 \]

\[ 25 \frac{\partial u}{\partial \eta^2} = 0 \]

\[ \therefore u = a(\xi) \eta + b(\xi) \]

(b) General solution: \[ u(x,t) = f(2x+t)(-x+2t) + g(2x+t) \]

where \( f \) and \( g \) are \( C^2 \)-functions of a single real variable.

\[ u_t(x,t) = f'(2x+t)(-x+2t) + 2f(2x+t) + g'(2x+t) \]

\[ \begin{align*}
\{ & xe + 4x^2 = u(x,0) = xf(2x) + g(2x) \\
(\sqrt{x} - 2x)^2 + 4x = u_t(x,0) = -xf'(2x) + 2f(2x) + g'(2x) \}
\end{align*} \]

\[ \downarrow \]

\[ \begin{align*}
(2x) e^{2x} + 8x &= -2xf'(2x) - f(2x) + 2g'(2x) \\
(x - 2)^2 e^{2x} + 4x &= -xf'(2x) + 2f(2x) + g'(2x)
\end{align*} \]

(\( \text{multiply second equation by } 2 \text{ and subtract from first.} \)
\[ 2 \text{ (cont.)} \quad 5e^{2x} = -5f(2x) \implies f(v) = -e^v. \]

Substituting for \( f \) in \( xe^{2x} + 4x^2 = -xf(2x) + g(2x) \) gives
\[ xe^{2x} + 4x^2 = xe^{2x} + g(2x) \]
\[ \implies (2x)^2 = g(2x) \implies g(v) = v^2. \]

\[ \therefore u(x,t) = f(2x+t)(-x+2t) + g(2x+t) \]
\[ = e^{\frac{2x+t}{-2t+x} + (2x+t)^2} \quad (c) \]
\[ f(u_t - u_{xx} + 3u)(\xi) = f(a)(\xi) \]

\[ \frac{\partial}{\partial t} f(u)(\xi) - (i\xi) \frac{\partial}{\partial \xi} f(u)(\xi) + 3f(u)(\xi) = 0 \]

\[ \frac{\partial}{\partial t} f(u)(\xi) + (\xi^2 + 3)f(u)(\xi) = 0 \]

Integrating factor: \( \mu(t) = e^{\int (\xi^2 + 3) dt} = e^{\frac{(\xi^2 + 3)t}{2}} \)

\[ e^{\frac{(\xi^2 + 3)t}{2}} \frac{\partial}{\partial t} \left[ e^{-\frac{(\xi^2 + 3)t}{2}} f(u)(\xi) \right] = 0 \]

\[ e^{\frac{(\xi^2 + 3)t}{2}} f(u)(\xi) = c(\xi) \]

\[ f(u)(\xi) = c(\xi) e^{-\frac{\xi^2 + 3}{2}t} \]

\[ f(q)(\xi) = f(u(\cdot, 0))(\xi) = c(\xi) e^0 = c(\xi) \]

\[ \therefore f(u)(\xi) = f(q)(\xi) e^{-\frac{\xi^2 + 3}{2}t} \]

By Table entry I, \( F(e^{-\frac{\xi^2}{2a}})(\xi) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{\xi^2}{4a}} \). Taking \( \frac{1}{\sqrt{2\pi}} = t \) (i.e. \( a = \frac{1}{4t} \)), we have \( F\left( \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4t}} \right)(\xi) = e^{-\frac{\xi^2}{4t}} \).

\[ \therefore f(u)(\xi) = f(q)(\xi) F\left( \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4t}} \right)(\xi) e^{-\frac{3}{2}t} \]

Using the convolution fact, \( \hat{f}(f \ast g)(\xi) = \sqrt{2\pi} \hat{f}(f)(\xi) \hat{g}(g)(\xi) \) gives
#3 (cont.)

\[ \mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}} \mathcal{F} \left( \varphi \ast \frac{1}{\sqrt{4\pi t}} e^{-\frac{\xi^2}{4t}} \right)(\xi) \cdot e^{-3t} \]

\[ = \mathcal{F} \left( e^{-3t} \varphi \ast \frac{1}{\sqrt{4\pi t}} e^{-\frac{\xi^2}{4t}} \right)(\xi). \]

By the Fourier inversion theorem,

\[ u(x,t) = e^{-3t} \left( \varphi \ast \frac{1}{\sqrt{4\pi t}} e^{-\frac{\xi^2}{4t}} \right)(x) \]

\[ = e^{-3t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \varphi(y) e^{-\frac{(x-y)^2}{4t}} dy \]

for all \(-\infty < x < \infty\) and all \(0 < t < \infty\).
\[
\begin{aligned}
\rho u_{tt} - T u_{xx} &= 0, \quad u(x,0) = e^{-x^2}, \quad u_t(x,0) = 2x e^{-x^2} \\
\end{aligned}
\]

\[
\begin{aligned}
u_{tt} - u_{xx} &= 0, \\
u(x,t) &= \frac{1}{2} [\varphi(x-t) + \varphi(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} \varphi(z) dz \\
\text{where } \varphi(x) &= e^{-x^2} \text{ and } \varphi(x) = 2x e^{-x^2}. \\
\therefore u(x,t) &= \frac{1}{2} \left[ e^{-(x-t)^2} + e^{-(x+t)^2} \right] + \frac{1}{2} \int_{x-t}^{x+t} 2x e^{-z^2} dz \\
&= \frac{1}{2} \left[ e^{-(x-t)^2} + e^{-(x+t)^2} \right] + \frac{1}{2} \left[ -e^{-x^2} \right] \\
&= \frac{1}{2} \left[ e^{-(x-t)^2} + e^{-(x+t)^2} \right] - \frac{1}{2} \left[ e^{-(x-t)^2} - e^{-(x+t)^2} \right] \\
\hline
\end{aligned}
\]

\[
\begin{aligned}
u(x,t) &= e^{-(x-t)^2}
\end{aligned}
\]

(a)

(b)
(b) Initial temperature: \( u(2) = \frac{4}{3} + 100 \cdot \frac{1}{2} = 100 \) deg C.

(c) \( u(2) = 100 - 2(20) = 20 \) deg C.

Assume \( u \) is a periodic function (i.e., \( u = u(x) \) independent of \( \theta \) and \( \phi \)).

\[ u(x, \theta, \phi) = 100 \quad \text{for all} \quad -\pi < \theta < \pi, \quad 0 < \phi < \pi. \]

\( u(x, \theta, \phi) = 0 \quad \text{for all} \quad -\pi < \theta < \pi, \quad 0 < \phi < \pi. \]
\#6 \quad u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in} \quad 0 < x < 1, \ 0 < y < 1, \ 0 < z < 1,

u(0,y,z) = u(1,y,z) = 0 \quad \text{for all} \quad 0 \leq y \leq 1, \ 0 \leq z \leq 1

u(x,0,z) = u(x,1,z) = 0 \quad \text{for} \quad 0 \leq x \leq 1, \ 0 \leq z \leq 1

\quad u(x,y,0) = 0 \quad \text{and} \quad u(x,y,1) = \sin(\pi x) \sin^2(\pi y) \quad \text{for} \quad 0 \leq x \leq 1, \ 0 \leq y \leq 1.

\text{(Nontrivial solution to homogeneous part of the problem)}

\[ u(x,y,z) = X(x)Y(y)Z(z) \]

\[ X^{\prime\prime}(x) + \lambda X(x) = 0 \quad \text{for} \quad 0 < x < 1 \]

\[ -\frac{X^{\prime\prime}(x)}{X(x)} = \frac{Y^{\prime\prime}(y)}{Y(y)} + \frac{Z^{\prime\prime}(z)}{Z(z)} = \lambda \]

\[ \Rightarrow -\frac{Y^{\prime\prime}(y)}{Y(y)} = \frac{Z^{\prime\prime}(z)}{Z(z)} - \lambda = \mu \]

\[ X^{\prime\prime}(x) + \lambda X(x) = 0, \quad X(0) = X(1) = 0 \]

\[ Y^{\prime\prime}(y) + \mu Y(y) = 0, \quad Y(0) = Y(1) = 0 \]

\[ Z^{\prime\prime}(z) - (\lambda + \mu)Z(z) = 0, \quad Z(0) = 0 \]

\[ \Rightarrow \quad X(x) = \sinh(l \pi x), \quad \lambda_l = (l \pi)^2 \quad (l = 1, 2, 3, \ldots) \]

\[ Y_m(y) = \sin(m \pi y), \quad \mu_m = (m \pi)^2 \quad (m = 1, 2, 3, \ldots) \]

\[ Z_{l,m}(z) = \sinh(\pi \sqrt{l^2 + m^2} z) \]

\[ u(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{l,m} \sinh(l \pi x) \sin(m \pi y) \sinh(\pi \sqrt{l^2 + m^2} z) \]
#6 (cont.)

\[ \sin(\pi x) \sin^3(\pi y) = u(x, y, z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{c_{l,m}}{l^2 + m^2} \sin(\pi x \sqrt{l^2 + m^2}) \sin(\pi y \sqrt{l^2 + m^2}) \]

\[
\sin(\pi x) \left[ \frac{3}{4} \sin(\pi y) - \frac{1}{4} \sin(3\pi y) \right] = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{l,m} \sinh(\pi \sqrt{l^2 + m^2}) \sin(\pi x \sqrt{l^2 + m^2}) \sin^2(\pi y \sqrt{l^2 + m^2})
\]

\[ c_{1,1} \sinh(\pi \sqrt{10}) = \frac{3}{4}, \quad c_{1,3} \sinh(\pi \sqrt{10}) = -\frac{1}{4}, \quad \text{and all other } c_{l,m} = 0. \]

\[ u(x, y, z) = \frac{3 \sin(\pi x) \sin(\pi y) \sinh(\pi \sqrt{10})}{4 \sinh(\pi \sqrt{10})} - \frac{\sin(\pi x) \sin(3\pi y) \sinh(\pi \sqrt{10})}{4 \sinh(\pi \sqrt{10})} \]

(Weak) Maximum Principle for harmonic functions:

(b) If \( \nabla^2 u = 0 \) in an open bounded region \( \Omega \) and \( u \) is continuous on \( \overline{\Omega} = \Omega \cup \partial \Omega \), then \( \max_{\overline{\Omega}} u(x) = \max_{\partial \Omega} u(x) \) and \( \min_{\overline{\Omega}} u(x) = \min_{\partial \Omega} u(x) \).

Suppose \( u = v(x, y, z) \) were another solution to the problem in part (a). Then \( w = u(x, y, z) - v(x, y, z) \) solves \( \nabla^2 w = 0 \) in \( C \), \( w \) is continuous in \( \overline{C} = C \cup \partial C \), and \( w = 0 \) on \( \partial C \). Hence

\[
0 = \min_{(x, y) \in \partial C} w(x, y, z) \leq w(x_0, y_0, z_0) \leq \max_{(x, y) \in \partial C} w(x, y, z) = 0
\]

for all \((x_0, y_0, z_0) \in \overline{C} \). That is, \( 0 = w(x_0, y_0, z_0) = u(x_0, y_0, z_0) - v(x_0, y_0, z_0) \) for all \((x_0, y_0, z_0) \in \overline{C} \), and hence \( u = v \).
\#7 \quad u(x,t) = \mathcal{X}(x)T(t) \quad \rightarrow \quad \left\{ \begin{array}{l}
u_{tt} - \nu_{xx} + 2u_t = 0 \\
u_x(0,t) = u_x(\pi,t) = 0 \\
u(x,0) = 0 \end{array} \right.

\mathcal{X}(x)T''(t) - \mathcal{X}''(x)T(t) + 2\mathcal{X}(x)T'(t) = 0

\Rightarrow \quad \frac{T''(t) + 2T'(t)}{T(t)} = \frac{\mathcal{X}''(x)}{\mathcal{X}(x)} = -\lambda

\Rightarrow \left\{ \begin{array}{l} \mathcal{X}''(x) + \lambda\mathcal{X}(x) = 0, \quad \mathcal{X}''(0) = \mathcal{X}''(\pi) = 0 \\
T''(t) + 2T'(t) + \lambda T(t) = 0, \quad T(0) = 0 \end{array} \right.

\lambda_n = \frac{n^2}{a}, \quad \mathcal{X}_n(x) = \cos(nx), \quad (n = 0, 1, 2, \ldots)

\Rightarrow \quad T_n''(t) + 2T_n'(t) + n^2T_n(t) = 0, \quad T_n(t) = 0.

T_n(t) = e^{\kappa_n t} \quad \text{leads to} \quad \kappa_n^2 + 2\kappa_n + n^2 = 0 \quad \Rightarrow \quad \kappa_n = -1 \pm \sqrt{4 - 4n^2}

\therefore \quad \kappa_0 = -1, \quad \kappa_1 = -1, \quad \kappa_n = -1 \pm i\sqrt{n^2 - 1} \quad \text{for} \quad n \geq 2.

\therefore \quad T_0(t) = a_0 e^{-t}, \quad T_1(t) = a_1 e^{-t} + b_1 e^t, \quad T_n(t) = e^{-t} \left[ a_n \cos(t\sqrt{n^2 - 1}) + b_n \sin(t\sqrt{n^2-1}) \right] \quad (n \geq 2)

T_n(0) = 0 \quad \Rightarrow \quad a_0 + b_0 = 0 \quad \text{and} \quad a_n = 0 \quad (n \geq 1). \quad \text{Up to a constant factor},

\therefore \quad u_0(x,t) = 1 - e^{-2t}, \quad u_1(x,t) = te \cos(x), \quad u_n(x,t) = e^{-t} \sin(t\sqrt{n^2-1}) \cos(nx). \quad (n \geq 2)

\therefore \quad u(x,t) = a_0 (1 - e^{-2t}) + \sum_{n=2}^{\infty} a_n e^{-t} \sin(t\sqrt{n^2-1}) \cos(nx)

\therefore \quad u_t(x,t) = 2a_0 e^{-t} + b_1 (1-t)e \cos(x) + \sum_{n=2}^{\infty} \left[ b_n e^{-t} \sin(t\sqrt{n^2-1}) \cos(nx) \right]
\[
\begin{align*}
\# 7 \text{ (cont.)} & \quad x^2 = \sum_{n=2}^{\infty} b_n \sin(nx) \quad \text{for all } 0 \leq x \leq \pi. \\
\therefore 2b_0 &= \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^{\pi} x^2 \, dx}{\int_0^{\pi} 1 \, dx} = \frac{\pi}{3}, \\
b_1 &= \frac{\langle x \sin(x) \rangle}{\langle \sin(x), \sin(x) \rangle}, \quad \text{and} \quad \sqrt{n-1} b_n = \frac{\langle x \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle} (n \geq 2).
\end{align*}
\]

\[
\begin{align*}
\langle x \sin(nx) \rangle &= \frac{1}{\int_0^{\pi} \sin(nx) \, dx} \int_0^{\pi} x \sin(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} \frac{x \sin(nx)}{x \cos(nx)} \, dx = \frac{2}{\pi} \left( \frac{x \sin(nx)}{n} \right) \bigg|_0^\pi - \frac{2}{\pi} \int_0^{\pi} 2 \sin(nx) \, dx \\
&= -\frac{4}{\pi n} \int_0^{\pi} \sin(nx) \, dx = -\frac{4}{\pi n} \left(-x \cos(nx)\right) \bigg|_0^\pi + \frac{4}{\pi n} \int_0^{\pi} \cos(nx) \, dx = \frac{4(-1)^n}{n^2}.
\end{align*}
\]

\[
\therefore b_0 = \frac{\pi^2}{6}, \quad b_1 = -4, \quad \text{and} \quad b_n = \frac{4(-1)^n}{n^2 \sqrt{n-1}} (n \geq 2).
\]

\[
(a) \quad u(x,t) = \frac{x^2}{6} (1 - e^{-2t}) - 4te^{-\sin(x)} + \sum_{n=2}^{\infty} \frac{4(-1)^n}{n^2 \sqrt{n-1}} e^{-\frac{t}{n-1}} \sin(nx).
\]

\[
(b) \quad E'(t) = \frac{d}{dt} \left\{ \frac{1}{2} \int_0^{\pi} \left[ u_t(x,t) + u_x(x,t) \right] \, dx \right\} = \int_0^{\pi} \frac{1}{2} \left[ u_t(x,t) + u_x(x,t) \right] \, dx
\]

\[
= \int_0^{\pi} \left[ u_t(x,t)u_t(x,t) + u_x(x,t)u_x(x,t) \right] \, dx = \int_0^{\pi} u_t(x,t) \left[ u_{xx}(x,t) - 2u_{xt}(x,t) \right] \, dx + \int_0^{\pi} u_x(x,t)u_x(x,t) \, dx
\]

\[
= \int_0^{\pi} u_t(x,t)u_t(x,t) \, dx - 2 \int_0^{\pi} u_{xt}(x,t) \, dx + \frac{4(-1)^n}{n^2 \sqrt{n-1}} E(t) \leq 0. \quad \text{Thus } E \text{ is non-decreasing.}
\]
There is only one solution to (1)-(2)-(3)-(4).

To see this, let $u = u(x,t)$ denote the solution found above in part (a) and let $v = v_1(x,t)$ be another solution to (1)-(2)-(3)-(4). Show $v(x,t) = u(x,t) - u_1(x,t)$ solves:

1. $\frac{v_t}{2} - \frac{v_{xx}}{2} + 2v_x = 0$ in $0 < x < \pi$, $0 < t < \infty$,
2. $v_x(0,t) = 0 = v_x(\pi,t)$ for $t \geq 0$,
3. $v(x,0) = 0$ for $0 \leq x \leq \pi$,
4. $v_t(x,0) = 0$ for $0 \leq x \leq \pi$.

By part (b), the energy function of $v$,

$$E(t) = \frac{1}{\omega} \int_0^\pi \left[ v_x^2(x,t) + v^2(x,t) \right] dx$$

is decreasing on $[0, \infty)$. Hence, for all $t > 0$,

$$0 \leq E(t) \leq E(0) = \frac{1}{\omega} \int_0^\pi \left[ v_x^2(x,0) + v^2(x,0) \right] dx = 0$$

$$E(t) = \frac{1}{\omega} \int_0^\pi \left[ v_x^2(x,t) + v^2(x,t) \right] dx = 0 \quad \text{for all } t > 0.$$

The vanishing theorem implies $v_t(x,t) = 0 = v_x(x,t)$ for all $0 \leq x \leq \pi$ and $t > 0$. Thus $v(x,t) = \text{constant}$ for $0 \leq x \leq \pi$ and $0 \leq t < \infty$, and by (3) the constant must be zero. Thus $u(x,t) = u_1(x,t)$ for all $0 \leq x \leq \pi$, $0 \leq t < \infty$. That is, $u_1 = u$. Q.E.D.