Mathematics 325 Exam I Name: Dr. Grow

Summer 2008

1.(25 pts.) Find the general solution of $\overbrace{u_x+2u_y}^{\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)u}+(2x-y)u=\overbrace{2x^2+3xy-2y^2}^{(2x-y)(x+2y)}$ in the xy-plane.

Bonus (10 pts.) Find the solution of this partial differential equation that satisfies the auxiliary condition $u(x,0)=x+1-\dfrac{5}{2x}$ for $x>0$.

Let $\begin{cases} \xi = x+2y, \\ \eta = 2x-y. \end{cases}$ If v is a C^1-function of two real variables then

4 pts. to here.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial x}}^{1} + \frac{\partial v}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial x}}^{2} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial y}}^{2} + \frac{\partial v}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial y}}^{-1} \Rightarrow \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$$

8 pts to here

Therefore the pde is equivalent to $\left[\frac{\partial}{\partial \xi} + \cancel{2\frac{\partial}{\partial \eta}} + 2\left(2\frac{\partial}{\partial \xi} - \cancel{\frac{\partial}{\partial \eta}}\right)\right]u + \eta u = \eta\xi$

12 pts. to here. $5\dfrac{\partial u}{\partial \xi} + \eta u = \eta \xi$ ← (This is a linear 1st-order ODE in the variable ξ with parameter η.)

$$\frac{\partial u}{\partial \xi} + \frac{\eta}{5}u = \frac{\eta\xi}{5}$$

Integrating factor: $e^{\int \frac{\eta}{5}d\xi} = e^{\frac{\eta\xi}{5}}$ 16 pts. to here.

$$\underbrace{e^{\frac{\eta\xi}{5}}\frac{\partial u}{\partial \xi} + \frac{\eta}{5}e^{\frac{\eta\xi}{5}}u}_{\text{Exact expression!}} = \frac{\eta\xi}{5}e^{\frac{\eta\xi}{5}} \Rightarrow \frac{\partial}{\partial \xi}\left(e^{\frac{\eta\xi}{5}}u\right) = \frac{\eta\xi}{5}e^{\frac{\eta\xi}{5}}$$

Integrate both sides with respect to ξ holding η fixed: $e^{\frac{\eta\xi}{5}}u = \int \overbrace{\frac{\eta\xi}{5}}^{v}\underbrace{e^{\frac{\eta\xi}{5}}d\xi}_{dV}$

$$\Rightarrow e^{\frac{\eta\xi}{5}}u = \frac{\eta\xi}{5}\cdot\frac{5}{\eta}e^{\frac{\eta\xi}{5}} - \int \frac{5}{\eta}e^{\frac{\eta\xi}{5}}\frac{\eta}{5}d\xi = \xi e^{\frac{\eta\xi}{5}} - \frac{5}{\eta}e^{\frac{\eta\xi}{5}} + c_1(\eta)$$

$$\therefore\ u = \xi - \frac{5}{\eta} + f(\eta)e^{-\frac{\eta\xi}{5}} \Rightarrow \boxed{u(x,y) = x+2y - \frac{5}{2x-y} + f(2x-y)e^{\frac{2y^2-3xy-2x^2}{5}}}$$

20 pts. to here. 25 pts.

where f is an arbitrary C^1-function of a single real variable.

Bonus: $x+1-\dfrac{5}{2x} = u(x,0) = x - \dfrac{5}{2x} + f(2x)e^{\frac{-2x^2}{5}}$ for all $x>0$

$$\Rightarrow e^{\frac{(2x)^2}{10}} = e^{\frac{2x^2}{5}} = f(2x) \Rightarrow f(z) = e^{\frac{z^2}{10}} \text{ for } z>0.$$

Therefore

$$u(x,y) = x + 2y - \frac{5}{2x-y} + e^{\frac{(2x-y)^2}{10}} \cdot e^{\frac{2y^2-3xy-2x^2}{5}} .$$

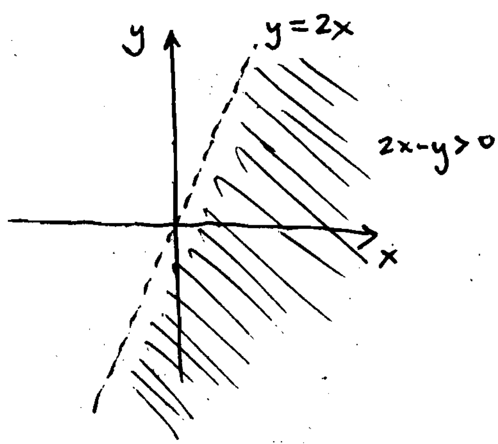
$$\frac{(2x-y)^2}{10} + \frac{2y^2-3xy-2x^2}{5} = \frac{\cancel{4x^2} - 4xy + y^2 + 4y^2 - 6xy - \cancel{4x^2}}{10}$$

$$= -xy + \frac{y^2}{2}$$

$$\therefore \boxed{u(x,y) = x + 2y - \frac{5}{2x-y} + e^{-xy+\frac{y^2}{2}}}$$

10 pts. to here.

Note: This is the unique solution to the IVP on the region $2x-y>0$.



2.(25 pts.) Classify the following second-order partial differential equations as hyperbolic, parabolic, or elliptic. If possible, find the general solution of each in the xy-plane.

(a) $u_{xx} + u_{xy} + 3u_{yy} + u_{yx} = 0 \longrightarrow B^2 - 4AC = 2^2 - 4(1)(3) = -8 < 0$ $\boxed{\text{elliptic}}$ 5 pts.

(b) $u_{xx} + u_{yy} - 2u_{xy} + 4u = 0 \longrightarrow B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$ $\boxed{\text{parabolic}}$ 5 pts.

(b) $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + 4u = 0$ Let $\begin{cases} \xi = x - y, \\ \eta = x + y. \end{cases}$ 5 pts. If v is a C^1-function of two real variables then

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} \Rightarrow \frac{\partial}{\partial y} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

Therefore the pde in (b) is equivalent to $\left[\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} - \left(-\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right]^2 u + 4u = 0$

5 pts. $\cancel{4}\frac{\partial^2 u}{\partial \xi^2} + \cancel{4}u = 0 \leftarrow$ (ODE in the variable ξ, const. coefficient, second order.)

$\therefore u(\xi,\eta) = c_1(\eta)\cos(\xi) + c_2(\eta)\sin(\xi)$

$\therefore \boxed{u(x,y) = f(x+y)\cos(x-y) + g(x+y)\sin(x-y)}$ 5 pts.

where f and g are C^2-functions (arbitrary) of a single real variable.

3.(25 pts.) (a) Derive the general solution of $u_{tt} - c^2 u_{xx} = 0$ in the xt-plane.

(b) Derive a formula for the solution of the partial differential equation in part (a) which satisfies the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for all real x. Here ϕ and ψ are two given "smooth" functions of a single real variable.

Bonus (10 pts.) Derive a general relation between ϕ and ψ which will produce a solution to the initial value problem in parts (a) and (b) consisting of a single wave traveling to the right along the x-axis.

4 pts. to here.

(12 pts.) (a) $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$. Let $\begin{cases} \xi = x+ct, \\ \eta = x-ct. \end{cases}$ If v is a C^1-function of two real variables then $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial v}{\partial \eta}\cancelto{1}{\frac{\partial \eta}{\partial x}} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$,

$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \xi}\cancelto{c}{\frac{\partial \xi}{\partial t}} + \frac{\partial v}{\partial \eta}\cancelto{-c}{\frac{\partial \eta}{\partial t}} \Rightarrow \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$.

Therefore the pde is equivalent to $\left[\left(\cancel{c\frac{\partial}{\partial \xi}} - c\frac{\partial}{\partial \eta}\right) - c\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right]\left[\left(c\frac{\partial}{\partial \xi} - \cancel{c\frac{\partial}{\partial \eta}}\right) + c\left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right]u = 0$

$\Rightarrow \left(-2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \xi}\right)u = 0 \Rightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} = 0$. Integrate with respect to η holding ξ fixed 8 pts. to here.

to get $\frac{\partial u}{\partial \xi} = c_1(\xi)$. Integrate with respect to ξ holding η fixed to get

$u = \int c_1(\xi)\,d(\xi) + c_2(\eta) = f(\xi) + g(\eta)$. Thus $\boxed{u(x,t) = f(x+ct) + g(x-ct)}$.

where f and g are arbitrary C^2-functions of a single real variable. 12 pts. to here.

(13 pts.) (b) $u_t(x,t) = cf'(x+ct) - cg'(x-ct)$

$\varphi(x) = u(x,0) = f(x) + g(x) \Longrightarrow \varphi'(x) \overset{①}{=} f'(x) + g'(x)$

3 pts. to here. $\psi(x) = u_t(x,0) \overset{②}{=} cf'(x) - cg'(x)$

Multiply ① by c and add to ②.

$\therefore\ c\varphi'(x) + \psi(x) = 2cf'(x) \Rightarrow f'(x) = \frac{1}{2c}\psi(x) + \frac{1}{2}\varphi'(x)$

$\Rightarrow f(x) = \frac{1}{2c}\int_0^x \psi(s)\,ds + \frac{1}{2}\varphi(x) + A$ 6 pts. to here.

Similarly $c\varphi'(x) - \psi(x) = 2cg'(x) \rightarrow g'(x) = -\frac{1}{2c}\psi(x) + \frac{1}{2}\varphi'(x)$

$\Rightarrow g(x) = \frac{1}{2c}\int_x^0 \psi(s)\,ds + \frac{1}{2}\varphi(x) + B$ 9 pts. to here.

Since $\varphi(x) = f(x) + g(x) = \cancel{\frac{1}{2c}\int_0^x \psi(s)\,ds} + \frac{1}{2}\varphi(x) + A + \cancel{\frac{1}{2c}\int_x^0 \psi(s)\,ds} + \frac{1}{2}\varphi(x) + B = \varphi(x) + A + B$, 10 pts. to here.

it follows that $A+B=0$. Therefore

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)ds + A$$

$$+ \frac{1}{2}\varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{0}\psi(s)ds + B$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds}$$ (d'Alembert's formula)

13 pts. to here.

(10 pts.) Bonus: $$u(x,t) = \underbrace{\frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)ds}_{\text{Wave traveling to left}} + \underbrace{\frac{1}{2}\varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{0}\psi(s)ds}_{\text{Wave traveling to right}}$$

3 pts. to here.

In order to have a solution that is a single wave traveling to the right, we must have

$$\frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)ds = \text{constant}$$

for all real x and t. Therefore

7 pts. to here. $$\frac{1}{2}\varphi(z) + \frac{1}{2c}\int_0^{z}\psi(s)ds = \text{constant}$$

for all real z. Differentiating yields

$$\frac{1}{2}\varphi'(z) + \frac{1}{2c}\psi(z) = 0 \quad \text{for all real } z$$

10 pts. to here. $$\Rightarrow \boxed{\varphi'(z) = -\frac{1}{c}\psi(z) \text{ for all real } z}$$

4.(25 pts.) A homogeneous solid material occupying $D = \{(x,y,z) \in \mathbb{R}^3 : 4 \le x^2 + y^2 + z^2 \le 100\}$ is completely insulated and its initial temperature at position (x,y,z) in D is $200/\sqrt{x^2+y^2+z^2}$.

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t \ge 0$.

(b) Use Gauss' divergence theorem to help show that the heat energy $H(t) = \iiint_D c\rho u(x,y,z,t)\,dV$ of the material in D at time t is a constant function of time. Here c and ρ denote the (constant) specific heat and mass density, respectively, of the material in D.

Bonus (10 pts.) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

15 pts. (a)
$$\begin{cases} u_t - k\nabla^2 u = 0 & \text{if } 4 < x^2+y^2+z^2 < 100 \text{ and } 0 < t < \infty, \quad (5) \\ \dfrac{\partial u}{\partial n} = 0 & \text{if } 4 = x^2+y^2+z^2 \text{ or } 100 = x^2+y^2+z^2 \text{ and } t \ge 0, \quad (5) \\ u(x,y,z,0) = \dfrac{2}{\sqrt{x^2+y^2+z^2}} & \text{if } 4 \le x^2+y^2+z^2 \le 100. \quad (5) \end{cases}$$

15 pts. (b)
$$\frac{dH}{dt} = \frac{d}{dt}\iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho \frac{\partial u}{\partial t}(x,y,z,t)\,dV = \iiint_D c\rho(k\nabla^2 u)\,dV \quad (5)$$
$$= c\rho k \iint_{\partial D} \nabla u \cdot \vec{n}\,dV = c\rho k \iint_{\partial D} \overset{0}{\cancel{\frac{\partial u}{\partial n}}}\,dV = 0.$$

Therefore $H(t) = $ constant for all $t \ge 0$. (5)

10 pts. **Bonus**: Let $U_0 = \lim_{t\to\infty} u(x,y,z,t)$ be the steady-state temperature of the material in D.

Then $H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty} \iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho U_0\,dV = c\rho U_0 \,\mathrm{vol}(D)$ (1)

But
$$H(0) = \iiint_D c\rho u(x,y,z,0)\,dV = \iiint_D c\rho \frac{200}{\sqrt{x^2+y^2+z^2}}\,dV \overset{\text{spherical coordinates}}{=} \int_0^{2\pi}\int_0^{\pi}\int_2^{10} c\rho \frac{200}{\cancel{r}} r^{\cancel{2}} \sin\varphi\,dr\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \overset{100}{\cancel{200}} c\rho \cdot \frac{r^2}{\cancel{2}}\Big|_{r=2}^{10} \sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} 100c\rho(100-4)\sin\varphi\,d\varphi\,d\theta$$
$$= 9600c\rho(-\cos\varphi)\Big|_0^{\pi} \cdot 2\pi = 38{,}400c\rho\pi \quad (3) \qquad \text{(OVER)}$$

and $\quad vol(D) = \frac{4}{3}\pi(R_1^3 - R_2^3) = \frac{4}{3}\pi(10^3 - 2^3) = \frac{4}{3}\pi \cdot 992 .$ (3)

Thus

$$38{,}400\, \cancel{c}\cancel{\rho}\cancel{\pi} = \cancel{c}\cancel{\rho}\, U_0 \frac{4}{3}\cancel{\pi} \cdot 992$$

$$U_0 = \frac{3}{4 \cdot 992} \cdot 38{,}400 = \boxed{\frac{900}{31}} \cong 29 \qquad (3)$$

Math 325

Exam I

Summer 2008

n : 20

μ : 65.6

σ : 27.3

Distribution of Scores:

87 – 100	6
73 – 86	3
60 – 72	1
50 – 59	2
0 – 49	8

Distribution of Letter Grades:

A	6
B	3
C	3
D	8
F	