Mathematics 325 Final Exam Spring 2002

This examination consists of 7 problems of equal value. You have the option of making this exam count either 200 or 300 points. Before turning in this exam paper, please indicate clearly your selection below.

I want this examination to count _____ points. Name: Dr. Grow

1. Consider the partial differential equation

$$(*) \qquad \sqrt{1+x^2}\, u_x + 2xyu_y = 0 .$$

(a) What are the order (first, second, third, etc.) and type (linear or nonlinear) of (*)?

(b) Find and sketch the graphs of three of the characteristic curves of (*).

(c) Find the general solution of (*).

(d) What is the solution to (*) satisfying $u(0,y) = y^3$ for $-\infty < y < \infty$?

2. Consider the partial differential equation

$$(*) \qquad u_{xx} - 4u_{xt} + 4u_{tt} = 0.$$

(a) Classify (*)'s order (first, second, third, etc.) and type (linear, nonlinear, hyperbolic, elliptic, etc.).

(b) Find the general solution of (*) in the xt-plane.

(c) Find the solution of (*) that satisfies

$$u(x,0) = 4x^3 \text{ and } u_t(x,0) = -4x^2$$

for $-\infty < x < \infty$.

3. Find a solution to

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty,$$

which satisfies $u(x,0) = x^4$ for $-\infty < x < \infty$. You may find the following identities useful:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-p^2}\,dp = 2\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{4}{3}\int_{-\infty}^{\infty} p^4 e^{-p^2}\,dp, \quad 0 = \int_{-\infty}^{\infty} p e^{-p^2}\,dp = \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp.$$

4. (a) If f is an absolutely integrable function and b is a real number, show that the function $g(x) = f(x-b)$ has Fourier transform

$$\hat{g}(\xi) = e^{-i\xi b}\cdot\hat{f}(\xi).$$

(b) Use Fourier transform methods to find a formula for the solution to

$$u_t - u_{xx} + u_x = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty,$$

subject to the initial condition $u(x,0) = \phi(x)$ for $-\infty < x < \infty$.

5. (a) Find a solution to ①

$$u_{tt} - u_{xx} = 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

subject to

$$②\text{-}③ \quad u_x(0,t) = u_x(1,t) = 0 \quad \text{for } t \geq 0,$$

and

$$u(x,0) \overset{④}{=} \cos(3\pi x) + 3\cos(\pi x), \quad u_t(x,0) \overset{⑤}{=} 0 \quad \text{for } 0 \leq x \leq 1.$$

(b) Show that there is only one solution to the problem in (a).

6. Consider the $2\pi$-periodic function f given on one period by $f(x) = |x|$ if $-\pi \le x < \pi$.

(a) Calculate the full Fourier series of f on $[-\pi,\pi]$.

(b) Write the sum of the first three nonzero terms of the full Fourier series of f and sketch the graph of this sum on $[-\pi,\pi]$. On the same coordinate axes, sketch the graph of f.

(c) Does the full Fourier series of f converge to f in the mean square sense on $[-\pi,\pi]$? Why?

(d) Does the full Fourier series of f converge to f pointwise on $[-\pi,\pi]$? Why?

(e) Does the full Fourier series of f converge to f uniformly on $[-\pi,\pi]$? Why?

(f) Use the results above to help find the sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

(g) Use the results above to help find the sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$.

7. Find the steady-state temperature distribution inside an annular plate with inner radius 1 and outer radius 2 if the inner edge $r = 1$ is insulated and on the outer edge $r = 2$ the temperature is maintained as $|\theta|$ for $-\pi \le \theta < \pi$. (Hint: You should find the results of problem 6 useful.)

# A Brief Table of Fourier Transforms

| | $f(x)$ | $\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$ |
|---|---|---|
| A. | $\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$ | $\sqrt{\dfrac{2}{\pi}} \dfrac{\sin(b\xi)}{\xi}$ |
| B. | $\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$ | $\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$ |
| C. | $\dfrac{1}{x^2 + a^2} \quad (a > 0)$ | $\sqrt{\dfrac{\pi}{2}} \dfrac{e^{-a\lvert\xi\rvert}}{a}$ |
| D. | $\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$ | $\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$ |
| E. | $\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$ | $\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$ |
| . . | $\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$ | $\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$ |
| G. | $\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$ | $\sqrt{\dfrac{2}{\pi}} \dfrac{\sin(b(\xi - a))}{\xi - a}$ |
| H. | $\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$ | $\dfrac{i}{\sqrt{2\pi}} \dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$ |
| I. | $e^{-ax^2} \quad (a > 0)$ | $\dfrac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$ |
| J. | $\dfrac{\sin(ax)}{x} \quad (a > 0)$ | $\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$ |

# Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

**Theorem 2.** Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

**Theorem 3.** $L^2$ Convergence The Fourier series converges to $f(x)$ in the mean-square sense in $(a, b)$ provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

**Theorem 4.** Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on $(a, b)$, provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point $x$ $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

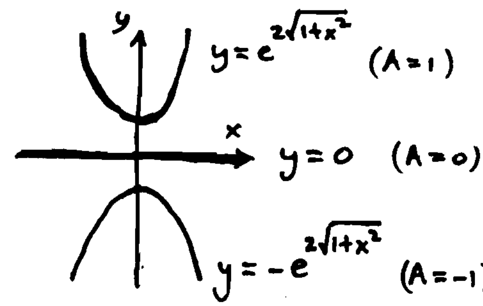
The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

**Theorem 4$\infty$.** If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

4 $\boxed{\#1.}$ $\sqrt{1+x^2}\, u_x + 2xyu_y = 0$  (a) first-order, linear (homogeneous)

8 (b) Characteristic curves: $\dfrac{dy}{dx} = \dfrac{2xy}{\sqrt{1+x^2}} \Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{2x}{\sqrt{1+x^2}}\,dx$  $\left(\begin{array}{l} w = 1+x^2 \\ dw = 2x\,dx \end{array}\right)$

$$\ln(y) = 2\sqrt{1+x^2} + c \Rightarrow \boxed{y = Ae^{2\sqrt{1+x^2}}}$$




8 (c) Along each characteristic curve the solution $u = u(x,y)$ is constant:

$$u(x,y) = u(x, Ae^{2\sqrt{1+x^2}}) = u(0, Ae^2) = f(A).$$

Therefore $\boxed{u(x,y) = f(ye^{-2\sqrt{1+x^2}})}$ where $f$ is any $C^1$-function of a single real variable.

8 (d) $y^3 = u(0,y) = f(ye^{-2})$ for all $-\infty < y < \infty$. Let $z = ye^{-2}$; then $f(z) = (e^2 z)^3$ $= e^6 z^3$ for all $-\infty < z < \infty$. Consequently, the solution to the I.V.P. is

$$u(x,y) = f(ye^{-2\sqrt{1+x^2}}) = e^6 (ye^{-2\sqrt{1+x^2}})^3 = \boxed{e^6 e^{-6\sqrt{1+x^2}} y^3}$$

$\boxed{\#2.}$ (*) $u_{xx} - 4u_{xt} + 4u_{tt} = 0$  $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$

4 (a) $\boxed{\text{Second-order, linear, parabolic}}$

12 (b)
$$0 = \left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial t} + 4\frac{\partial^2}{\partial t^2}\right)u$$
$$= \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u$$

Let $\begin{cases} \xi = 2x + t, \\ \eta = x - 2t. \end{cases}$  Then $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial x} = 2\dfrac{\partial v}{\partial \xi} + \dfrac{\partial v}{\partial \eta}$

and $\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial t} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial t} = \dfrac{\partial v}{\partial \xi} - 2\dfrac{\partial v}{\partial \eta}$.

I.e. as operators, $\dfrac{\partial}{\partial x} = 2\dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$ and $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \xi} - 2\dfrac{\partial}{\partial \eta}$.

Therefore $\dfrac{\partial}{\partial x} - 2\dfrac{\partial}{\partial t} = \left(\cancel{2\dfrac{\partial}{\partial \xi}} + \dfrac{\partial}{\partial \eta}\right) - 2\left(\cancel{\dfrac{\partial}{\partial \xi}} - 2\dfrac{\partial}{\partial \eta}\right) = 5\dfrac{\partial}{\partial \eta}$, so (*) is equivalent to $\quad 25\dfrac{\partial^2 u}{\partial \eta^2} = 0 \quad \Rightarrow \quad \dfrac{\partial u}{\partial \eta} = f(\xi) \quad \Rightarrow \quad u = \eta f(\xi) + g(\xi)$ .

$\therefore \quad \boxed{u(x,t) = (x-2t)f(2x+t) + g(2x+t)}$ where $f$ and $g$ are $C^2$-functions of a single real variable.

12 (c) $\quad u_t(x,t) = -2f(2x+t) + (x-2t)f'(2x+t) + g'(2x+t)$

$$\begin{cases} 4x^3 = u(x,0) = x f(2x) + g(2x) & \text{for all real } x. \\ -4x^2 = u_t(x,0) = -2f(2x) + xf'(2x) + g'(2x) & \text{for all real } x. \end{cases}$$

Differenting the first identity in the system above gives

$$12x^2 = 1\cdot f(2x) + 2xf'(2x) + 2g'(2x) .$$

Multiplying the second identity in the system by 2 gives

$$-8x^2 = -4f(2x) + 2xf'(2x) + 2g'(2x) ,$$

Subtracting the last equation from the preceding one yields

$20x^2 = 5f(2x)$ for all real $x$; i.e. $4x^2 = (2x)^2 = f(2x)$ so $f(z) = z^2$ for all $z$.

Substituting this for $f$ in the first identity of the system produces

$$4x^3 = x f(2x) + g(2x) = x(2x)^2 + g(2x) \Rightarrow g(z) = 0 \text{ for all real } z.$$

$$\therefore \quad u(x,t) = (x-2t)f(2x+t) + g(2x+t) = \boxed{(x-2t)(2x+t)^2} .$$

+12 $\boxed{\#3.}$ $\quad u(x,t) = \dfrac{1}{\sqrt{4\pi t}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^4\, dy$ . Let $p = \dfrac{y-x}{\sqrt{4t}}$. Then $dp = \dfrac{dy}{\sqrt{4t}}$.

+18 $\quad \therefore \quad u(x,t) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-p^2}(x + p\sqrt{4t})^4 dp$

22 $$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left\{ x^4 + 4x^3 p\sqrt{4t} + 6x^2p^2(4t) + 4xp^3 4t\sqrt{4t} + p^4(4t)^2 \right\}dp$$

24 $$= \frac{x^4}{\sqrt{\pi}}\cancelto{\sqrt{\pi}}{\int_{-\infty}^{\infty} e^{-p^2}dp} + \frac{4x^3\sqrt{4t}}{\sqrt{\pi}}\cancelto{0}{\int_{-\infty}^{\infty} pe^{-p^2}dp} + \frac{24x^2t}{\sqrt{\pi}}\cancelto{\frac{\sqrt{\pi}}{2}}{\int_{-\infty}^{\infty} p^2e^{-p^2}dp} + \frac{16xt\sqrt{4t}}{\sqrt{\pi}}\cancelto{0}{\int_{-\infty}^{\infty} p^3e^{-p^2}dp}$$

$$+ \frac{16t^2}{\sqrt{\pi}}\cancelto{\frac{3\sqrt{\pi}}{4}}{\int_{-\infty}^{\infty} p^4e^{-p^2}dp}$$

+28 #3 (cont.) $u(x,t) = \boxed{x^4 + 12x^2 t + 12t^2}$

Check: $u_t - u_{xx} = (12x^2 + 24t) - (12x^2 + 24t) \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^4$

$\boxed{\#4}$ (a) $\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-b)e^{-i\xi x}\,dx$ Let $y = x-b$. Then $dy = dx$.

+4 $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi(y+b)}\,dy = e^{-i\xi b}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}\,dy = \boxed{e^{-i\xi b}\hat{f}(\xi)}$.

(b) $\mathcal{F}\left(u_t - u_{xx} + u_x\right)(\xi) = \mathcal{F}(0)(\xi) = 0$

$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - \mathcal{F}(u_{xx})(\xi) + \mathcal{F}(u_x)(\xi) = 0$

Using the property $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$ we have

$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + i\xi\mathcal{F}(u)(\xi) = 0$

+6 $\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (i\xi + \xi^2)\mathcal{F}(u)(\xi) = 0$. (Linear, first-order ODE in the variable $t$)

Integrating factor: $e^{\int(i\xi+\xi^2)\,dt} = e^{i\xi t}\cdot e^{\xi^2 t}$

$e^{(i\xi+\xi^2)t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (i\xi+\xi^2)e^{(i\xi+\xi^2 t)}\mathcal{F}(u)(\xi) = 0$

$\frac{\partial}{\partial t}\left[e^{(i\xi+\xi^2)t}\mathcal{F}(u)(\xi)\right] = 0$

$\therefore\ \mathcal{F}(u)(\xi) = c_1(\xi)e^{-(i\xi+\xi^2)t}$

Applying the initial condition $u(x,0) = \varphi(x)$ for all real $x$ gives

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}\big(u(\cdot,0)\big)(\xi) = c_1(\xi)e^{-(i\xi+\xi^2)0} = c_1(\xi)$$

+15 $$\therefore \quad \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\, e^{-\xi^2 t}\cdot e^{-i\xi t}$$

Using table entry I. with $a = \frac{1}{4t}$ yields

$$\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi) = e^{-\xi^2 t}.$$

Substituting this identity in the next-to-last equation produces

$$\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\cdot e^{-i\xi t}.$$

Since $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\,\mathcal{F}(g)(\xi)$,

$$\mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\cdot e^{-i\xi t}.$$

+21 $$= \mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}}\right)(\xi)\cdot e^{-i\xi t}$$

Using part (a) of this problem plus the inversion theorem, we find that

$$u(x,t) = \left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}}\right)(x-t)$$

+24 $$= \boxed{\int_{-\infty}^{\infty} \frac{e^{\frac{(x-t-y)^2}{4t}}}{\sqrt{4\pi t}}\,\varphi(y)\,dy}$$

$\boxed{\#5.}$ (Separation of variables) We seek nontrivial solutions to the homogeneous part of the problem (①-②-③-⑤) of the form $u(x,t) = X(x)T(t)$. Substituting in ① yields $X(x)T''(t) - X''(x)T(t) = 0 \Rightarrow -\dfrac{X''(x)}{X(x)} = -\dfrac{T''(t)}{T(t)} = \text{constant} = \lambda.$

Substituting in ②-③-⑤ we have

$$X'(0)T(t) = 0 = X'(1)T(t) \quad \text{for } t \geq 0 \quad \text{and} \quad X(x)T'(0) = 0 \quad \text{for } 0 \leq x \leq 1.$$

+12 Thus $\begin{cases} \boxed{X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(1)} \leftarrow \text{Eigenvalue problem.} \\ T''(t) + \lambda T(t) = 0, \quad T'(0) = 0. \end{cases}$

The eigenvalues / eigenfunctions are

+16 $$\lambda_n = (n\pi)^2, \quad X_n(x) = \cos(n\pi x) \qquad (n = 0, 1, 2, \dots)$$

+18 and the solutions to the $t$-problem with $\lambda = \lambda_n = (n\pi)^2$ are $T_n(t) = \cos(n\pi t)$ $(n = 0, 1, 2, \dots)$

Consequently $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\cos(n\pi t)$ is a formal solution to ①-②-③-⑤ for any constants $a_0, a_1, a_2, \dots$ We want to choose the constants to satisfy ④:

+21 $$\cos(3\pi x) + 3\cos(\pi x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \qquad \text{for } 0 \leq x \leq 1.$$

By inspection, $a_1 = 3$, $a_3 = 1$, and other coefficients should be zero. Thus

+24 (a) $$\boxed{u(x,t) = 3\cos(\pi x)\cos(\pi t) + \cos(3\pi x)\cos(3\pi t)}$$

+4 (b) Suppose there is another solution to the problem in (a), say $u = v(x,t)$. Consider $w(x,t) = u(x,t) - v(x,t)$ and its energy function

$$E(t) = \int_0^1 \left\{ \tfrac{1}{2}[w_t(x,t)]^2 + \tfrac{1}{2}[w_x(x,t)]^2 \right\} dx \qquad (t \geq 0).$$

Observe that $w$ solves the problem

$$\begin{cases} w_{tt} - w_{xx} \overset{⑥}{=} 0 & \text{for } 0<x<1,\ 0<t<\infty, \\ w_x(0,t) \overset{⑦}{=} 0 \overset{⑧}{=} w_x(1,t) & \text{for } t \geq 0 \\ w(x,0) \overset{⑨}{=} 0 \overset{⑩}{=} w_t(x,0) & \text{for } 0 \leq x \leq 1. \end{cases}$$

Differentiating the energy function, we find

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left\{ \frac{1}{2}\left[w_t(x,t)\right]^2 + \frac{1}{2}\left[w_x(x,t)\right]^2 \right\} dx$$

$$= \int_0^1 \left\{ w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right\} dx$$

$$= \int_0^1 \left\{ w_t(x,t) w_{xx}(x,t) + w_x(x,t) w_{xt}(x,t) \right\} dx \qquad \text{(from 6)}$$

$$= \int_0^1 \frac{\partial}{\partial x}\left\{ w_t(x,t) w_x(x,t) \right\} dx$$

$$= w_t(x,t)\, w_x(x,t) \Big|_{x=0}^{x=1}$$

$$= w_x(1,t) w_t(1,t) - w_x(0,t) w_t(0,t)$$

$$= 0 . \qquad \text{(from 7 and 8)}$$

Therefore the energy is constant for all $t \geq 0$:

$$0 \leq E(t) = E(0) = \int_0^1 \left\{ \frac{1}{2}\left[w_t(x,0)\right]^2 + \frac{1}{2}\left[w_x(x,0)\right]^2 \right\} dx = 0 \qquad \text{(from 9 and 10)}$$

By the vanishing theorem, $w_t(x,t) = w_x(x,t) = 0$ for all $0 \leq x \leq 1$ and all $t \geq 0$, and hence $w(x,t) = \text{constant}$. From 9 we see that this constant must be zero; i.e. $u(x,t) - v(x,t) = w(x,t) = 0$ for all $0 \leq x \leq 1$, $0 \leq t < \infty$. This shows that there is only one solution to the problem in (a).

**#6** (a) Since $f(x)=|x|$ is an even function, all its sine coefficients are zero.

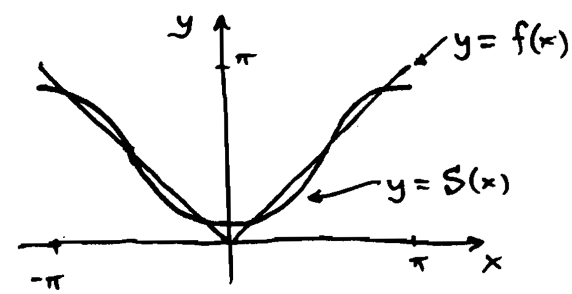
$$a_0 = \frac{\langle f,1\rangle}{\langle 1,1\rangle} = \frac{\int_{-\pi}^{\pi} f(x)\,dx}{\int_{-\pi}^{\pi} 1^2\,dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{2\pi}\Big|_0^{\pi} = \frac{\pi}{2}.$$

$(n\geq 1)$
$$a_n = \frac{\langle f,\cos(n\cdot)\rangle}{\langle \cos(n\cdot),\cos(n\cdot)\rangle} = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}{\int_{-\pi}^{\pi}\cos^2(nx)\,dx} = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \overset{u}{x}\,\overset{dv}{\cos(nx)\,dx}$$

$$= \cancel{\frac{2}{\pi}(x)\left(\frac{\sin(nx)}{n}\right)\Big|_0^{\pi}} \xrightarrow{} 0 \; - \frac{2}{\pi}\int_0^{\pi}\frac{\sin(nx)}{n}\,dx = \frac{-2}{\pi n}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi} = \frac{2\overset{(-1)^n}{\cos(n\pi)} - 2}{\pi n^2}$$

$$= \begin{cases} 0 & \text{if } n=2k \text{ is even,} \\ \dfrac{-4}{\pi(2k+1)^2} & \text{if } n=2k+1 \text{ is odd.} \end{cases} \qquad \therefore \boxed{f(x) \sim \frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{-4\cos((2k+1)x)}{\pi(2k+1)^2}}$$

+2 (b) $S(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x).$




+4 (c) $\boxed{\text{Yes}}$, the full Fourier series of $f$ converges to $f$ in the mean square sense on $[-\pi,\pi]$ because $\int_{-\pi}^{\pi}|f(x)|^2\,dx = \int_{-\pi}^{\pi}x^2\,dx = \frac{2\pi^3}{3} < \infty$. (Cf. Theorem 3)

+4 (d) $\boxed{\text{Yes}}$, the full Fourier series of $f$ converges pointwise to $f$ on $[-\pi,\pi]$ since $f$ is continuous and $2\pi$-periodic on $-\infty < x < \infty$, and $f'$ is piecewise continuous on $[-\pi,\pi]$. By Theorem 4(ii), for all real $x$ we have

$$\frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{-4\cos((2k+1)x)}{\pi(2k+1)^2} = \frac{1}{2}\left[f_{ext}(x^+) + f_{ext}(x^-)\right]$$

$$= \frac{1}{2}\left[f(x^+) + f(x^-)\right] \qquad \text{(since } f \text{ is } 2\pi\text{-periodic}$$

$$= f(x). \qquad \text{(since } f \text{ is continuous}$$

+4 (e) $\boxed{\text{Yes}}$, the full Fourier series of $f$ converges uniformly to $f$ on $[-\pi,\pi]$, although

Theorem 2 cannot be used to justify this. (Note that $f(x) = |x|$ does not satisfy the periodic boundary conditions $\varphi(-\pi) = \varphi(\pi)$ and $\varphi'(-\pi) = \varphi'(\pi)$ that generate the orthogonal system $\{1, \cos(nx), \sin(nx)\}_{n=1}^{\infty}$ used for full Fourier series.)

To show uniform convergence, observe that

$$\max_{-\pi \le x \le \pi} \left| f(x) - \left( \frac{\pi}{2} + \sum_{k=0}^{N} \frac{-4\cos((2k+1)x)}{\pi(2k+1)^2} \right) \right| \overset{\text{from part(d)}}{=} \max_{-\pi \le x \le \pi} \left| \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4\cos((2k+1)x)}{\pi(2k+1)^2} - \frac{\pi}{2} + \sum_{k=0}^{N} \frac{4\cos((2k}{\pi(2k+} \right.$$

$$= \max_{-\pi \le x \le \pi} \left| \sum_{k=N+1}^{\infty} \frac{-4\cos((2k+1)x)}{\pi(2k+1)^2} \right| \le \max_{-\pi \le x \le \pi} \sum_{k=N+1}^{\infty} \left| \frac{-4\cos((2k+1)x)}{\pi(2k+1)^2} \right|$$

$$\le \frac{4}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{(2k+1)^2} \longrightarrow 0 \quad \text{as } N \to \infty$$

(since it is the "tail" of the series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$, which is convergent by comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 2$).

+4 (f) Take $x = 0$ in the identity obtained in part (d):

$$0 = f(0) = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4\overbrace{\cos(0)}^{1}}{\pi(2k+1)^2} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \left(\frac{-\pi}{2}\right)\left(\frac{\pi}{-4}\right) = \boxed{\frac{\pi^2}{8}}$$

+4 (g) By the Parseval identity $\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$ and the results of part (a), we have

$$\left|\frac{\pi}{2}\right|^2 \int_{-\pi}^{\pi} 1^2 dx + \sum_{k=0}^{\infty} \left| \frac{-4}{\pi(2k+1)^2} \right|^2 \int_{-\pi}^{\pi} \cos^2((2k+1)x)\,dx = \int_{-\pi}^{\pi} |x|^2 dx.$$

Evaluating the integrals gives

$$\frac{\pi^2}{4} \cdot (2\pi) + \sum_{k=0}^{\infty} \frac{16}{\pi^2(2k+1)^4} \cdot (\pi) = \frac{2\pi^3}{3} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \boxed{\frac{\pi^4}{96}}.$$

$\boxed{\#7}$ $u_t - k\nabla^2 u = 0$ models heat flow. For steady-state $\overbrace{\text{distributions}}^{\text{temperature}}$, $u_t = 0$ for all $x$ and $t$. Therefore we need to solve

+6
$$\begin{cases} \nabla^2 u \overset{①}{=} 0 & \text{in } A = \{(r;\theta) : 1 < r < 2,\ -\pi \le \theta < \pi\}, \\ u_r(1;\theta) \overset{②}{=} 0 & \text{for } -\pi \le \theta < \pi, \\ u(2;\theta) \overset{③}{=} |\theta| & \text{for } -\pi \le \theta < \pi. \end{cases}$$

+7 (We also have the implied boundary conditions $u(r;\pi) \overset{④}{=} u(r;-\pi)$ and $u_\theta(r;\pi) \overset{⑤}{=} u_\theta(r;-\pi)$ for all $1<r<2$.) We use separation of variables; i.e. we seek nontrivial solutions to the homogeneous portion of the problem (①-②-④-⑤) of the form $u(r;\theta) = R(r)\Theta(\theta)$.

Substituting in ① gives

$$0 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta),$$

$$\Rightarrow \quad \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \text{constant} = \lambda.$$

Substituting in ②-④-⑤ gives

$$R'(1)\Theta(\theta) = 0 \text{ for } -\pi \le \theta < \pi \quad\text{and}\quad R(r)\Theta(\pi) = R(r)\Theta(-\pi),\ R(r)\Theta'(\pi) = R(r)\Theta'(-\pi)$$
for $1<r<2$.

Consequently,

+14
$$\begin{cases} r^2R''(r) + rR'(r) - \lambda R(r) = 0, \quad R'(1) = 0 \\ \boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(\pi) = \Theta(-\pi),\ \Theta'(\pi) = \Theta'(-\pi).} \leftarrow \text{Eigenvalue Problem} \end{cases}$$

The eigenvalues / eigenfunctions are

+18
$$\lambda_n = n^2, \quad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta) \qquad (n = 0,1,2,\cdots).$$

Returning to the radial problem with $\lambda = \lambda_n = n^2$ we have

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0, \quad R_n'(1) = 0.$$

The general solution to the ODE is $R_n(r) = c_1r^n + c_2r^{-n}$ when $n \ge 1$ and

$R_0(r) = c_1 + c_2 \ln(r)$ when $n=0$. If $n \ge 1$ then $R_n'(r) = nc_1 r^{n-1} - nc_2 r^{-n-1}$ so

$0 = R_n'(1) = nc_1 - nc_2 \Rightarrow c_1 = c_2$. If $n=0$, then $R_0'(r) = \frac{c_2}{r}$ so $0 = R_0'(1) = \frac{c_2}{1}$

+20 $\Rightarrow c_2 = 0$. Thus, up to a constant multiple, $R_n(r) = r^n + r^{-n}$ $(n \ge 1)$ and $R_0(r) = 1$.

Consequently

+22 $$u(r;\theta) = a_0 + \sum_{n=1}^{\infty} (r^n + r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta))$$

is a formal solution to ①-②-④-⑤ for arbitrary constants $a_0, a_1, b_1, a_2, b_2, \ldots$ We need to choose the constants so ③ is satisfied:

$$|\theta| = u(2;\theta) = a_0 + \sum_{n=1}^{\infty} (2^n + 2^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta)) \quad \text{for } -\pi \le \theta < \pi.$$

From problem 6(d) we know that

$$|\theta| = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4\cos((2k+1)\theta)}{\pi(2k+1)^2} \qquad \text{for } -\pi \le \theta \le \pi.$$

Therefore we need to choose

+26 $$a_0 = \frac{\pi}{2}, \qquad \left[2^{2k+1} + 2^{-(2k+1)}\right] a_{2k+1} = \frac{-4}{\pi(2k+1)^2} \quad \text{for } k = 0, 1, 2, \ldots,$$

and all other $a_n$ and $b_n$ equal to zero. Thus

+28 $$\boxed{u(r;\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\left(r^{2k+1} + r^{-(2k+1)}\right)\cos((2k+1)\theta)}{\left(2^{2k+1} + 2^{-(2k+1)}\right)(2k+1)^2}}.$$

This can also be expressed as

$$u(r;\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cosh((2k+1)\ln(r))\cos((2k+1)\theta)}{\cosh((2k+1)\ln(2))(2k+1)^2}.$$