This examination consists of 7 problems of equal value. You have the option of making this exam count either 200 or 300 points. Before turning in this exam paper, please indicate clearly your selection below.

I want this examination to count ____ points. Name: Dr. Grow

1. Consider the partial differential equation

\[ \sqrt{1 + x^2} u_x + 2xyu_y = 0. \]

(a) What are the order (first, second, third, etc.) and type (linear or nonlinear) of (1)?
(b) Find and sketch the graphs of three of the characteristic curves of (1).
(c) Find the general solution of (1).
(d) What is the solution to (1) satisfying \( u(0,y) = y^3 \) for \(-\infty < y < \infty\)?

2. Consider the partial differential equation

\[ u_{xx} - 4u_{xt} + 4u_{tt} = 0. \]

(a) Classify (1)'s order (first, second, third, etc.) and type (linear, nonlinear, hyperbolic, elliptic, etc.).
(b) Find the general solution of (1) in the xt-plane.
(c) Find the solution of (1) that satisfies \( u(x,0) = 4x^3 \) and \( u_t(x,0) = -4x^2 \) for \(-\infty < x < \infty\).

3. Find a solution to

\[ u_t - u_{xx} = 0 \]

for \(-\infty < x < \infty\) and \(0 < t < \infty\),

which satisfies \( u(x,0) = x^4 \) for \(-\infty < x < \infty\). You may find the following identities useful:

\[
\int_{-\infty}^{\infty} e^{-p^2} \, dp = 2 \int_{0}^{\infty} e^{-p^2} \, dp = \frac{4}{3} \int_{0}^{\infty} 4 e^{-p^2} \, dp, \quad \int_{-\infty}^{\infty} p e^{-p^2} \, dp = \int_{-\infty}^{\infty} p^3 e^{-p^2} \, dp.
\]

4. (a) If \( f \) is an absolutely integrable function and \( b \) is a real number, show that the function \( g(x) = f(x-b) \) has Fourier transform

\[ \hat{g}(\xi) = e^{-i\xi b} \hat{f}(\xi). \]

(b) Use Fourier transform methods to find a formula for the solution to

\[ u_t - u_{xx} + u_x = 0 \]

for \(-\infty < x < \infty\) and \(0 < t < \infty\),

subject to the initial condition \( u(x,0) = \phi(x) \) for \(-\infty < x < \infty\).

5. (a) Find a solution to

\[ u_{tt} - u_{xx} = 0 \]

for \(0 < x < 1, 0 < t < \infty\),

subject to

\[ \text{(1)} \quad u_x(0,t) = u_x(1,t) = 0 \quad \text{for} \ t \geq 0, \]

and

\[ u(x,0) = \cos(3\pi x) + 3\cos(\pi x), \quad u_t(x,0) = 0 \quad \text{for} \ 0 \leq x \leq 1. \]

(b) Show that there is only one solution to the problem in (a).
6. Consider the $2\pi$-periodic function $f$ given on one period by $f(x) = |x|$ if $-\pi \leq x < \pi$.

(a) Calculate the full Fourier series of $f$ on $[-\pi, \pi]$.

(b) Write the sum of the first three nonzero terms of the full Fourier series of $f$ and sketch the graph of this sum on $[-\pi, \pi]$. On the same coordinate axes, sketch the graph of $f$.

(c) Does the full Fourier series of $f$ converge to $f$ in the mean square sense on $[-\pi, \pi]$? Why?

(d) Does the full Fourier series of $f$ converge to $f$ pointwise on $[-\pi, \pi]$? Why?

(e) Does the full Fourier series of $f$ converge to $f$ uniformly on $[-\pi, \pi]$? Why?

(f) Use the results above to help find the sum $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

(g) Use the results above to help find the sum $\sum_{k=0}^{8} \frac{1}{(2k+1)^4}$.

7. Find the steady-state temperature distribution inside an annular plate with inner radius 1 and outer radius 2 if the inner edge $r = 1$ is insulated and on the outer edge $r = 2$ the temperature is maintained as $|\theta|$ for $-\pi \leq \theta < \pi$. (Hint: You should find the results of problem 6 useful.)
A Brief Table of Fourier Transforms

\[ \mathcal{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, dx \]

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( \mathcal{F}(\xi) )</th>
</tr>
</thead>
</table>
| \( A. \) \[
\begin{cases} 
1 & \text{if } -b < x < b, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ \frac{2 \sin(b\xi)}{\pi \xi} \] |
| \( B. \) \[
\begin{cases} 
1 & \text{if } c < x < d, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ \frac{e^{-ic\xi} - e^{-id\xi}}{i\xi 2\pi} \] |
| \( C. \) \[
\frac{1}{x^2 + a^2} \quad (a > 0)
\] | \[ \frac{\pi e^{-a|\xi|}}{2a} \] |
| \( D. \) \[
\begin{cases} 
x & \text{if } 0 < x \leq b, \\
2b - x & \text{if } b < x < 2b, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ -1 + 2e^{-ib\xi} - e^{-2ib\xi} \]
| \( E. \) \[
\begin{cases} 
e^{-ax} & \text{if } x > 0, \\
0 & \text{otherwise.}
\end{cases}
\] \( (a > 0) \) | \[ \frac{1}{(a + i\xi) \sqrt{2\pi}} \] |
| \( \cdots \) \[
\begin{cases} 
e^{ax} & \text{if } b < x < c, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ \frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi) \sqrt{2\pi}} \] |
| \( G. \) \[
\begin{cases} 
e^{iax} & \text{if } -b < x < b, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ \frac{2 \sin(b(\xi - a))}{\pi \xi - a} \] |
| \( H. \) \[
\begin{cases} 
e^{iax} & \text{if } c < x < d, \\
0 & \text{otherwise.}
\end{cases}
\] | \[ \frac{i e^{i(a-\xi)} - e^{i(a-\xi)}}{\sqrt{2\pi} a - \xi} \] |
| \( I. \) \[
e^{-ax^2} \quad (a > 0) \] | \[ \frac{1}{\sqrt{2\pi} a} e^{-\xi^2/(4a)} \] |
| \( J. \) \[
\frac{\sin(ax)}{x} \quad (a > 0) \] | \[
\begin{cases} 
0 & \text{if } |\xi| \geq a, \\
\sqrt{\pi/2} & \text{if } |\xi| < a.
\end{cases}
\] |
Convergence Theorems

\[ X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any symmetric BC.} \quad (1) \]

Now let \( f(x) \) be any function defined on \( a \leq x \leq b \). Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

**Theorem 2. Uniform Convergence**  The Fourier series \( \sum A_n X_n(x) \) converges to \( f(x) \) uniformly on \( [a, b] \) provided that

(i) \( f(x), f'(x), \text{ and } f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f(x) \) satisfies the given boundary conditions.

**Theorem 3. \( L^2 \) Convergence**  The Fourier series converges to \( f(x) \) in the mean-square sense in \( (a, b) \) provided only that \( f(x) \) is any function for which

\[
\int_a^b |f(x)|^2 \, dx \text{ is finite.} \quad (8)
\]

**Theorem 4. Pointwise Convergence of Classical Fourier Series**

(i) The classical Fourier series (full or sine or cosine) converges to \( f(x) \) pointwise on \( (a, b) \), provided that \( f(x) \) is a continuous function on \( a \leq x \leq b \) and \( f'(x) \) is piecewise continuous on \( a \leq x \leq b \).

(ii) More generally, if \( f(x) \) itself is only piecewise continuous on \( a \leq x \leq b \) and \( f'(x) \) is also piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series converges at every point \( x \) \((-\infty < x < \infty)\). The sum is

\[
\sum_n A_n X_n(x) = \frac{1}{2} [f(x+) + f(x-)] \quad \text{for all } a < x < b. \quad (9)
\]

The sum is \( \frac{1}{2} [f_{\text{ext}}(x+) + f_{\text{ext}}(x-)] \) for all \(-\infty < x < \infty\), where \( f_{\text{ext}}(x) \) is the extended function (periodic, odd periodic, or even periodic).

**Theorem 4\( ^\infty \).**  If \( f(x) \) is a function of period \( 2l \) on the line for which \( f(x) \) and \( f'(x) \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{1}{2} [f(x+) + f(x-)] \) for \(-\infty < x < \infty\).
#1. \[ \sqrt{1+x^2} \, u_x + 2xy \, u_y = 0 \] (a) first-order, linear (homogeneous)

(b) Characteristic curves: \[ \frac{dy}{dx} = \frac{2xy}{\sqrt{1+x^2}} \] \[ \Rightarrow \int \frac{dy}{y} = \int \frac{2x}{\sqrt{1+x^2}} \, dx \quad \left( \frac{W = \sqrt{1+x^2}}{\frac{dw}{dx} = 2x \, dx} \right) \]

\[ \ln(y) = 2\sqrt{1+x^2} + C \quad \Rightarrow \quad y = Ae^{2\sqrt{1+x^2}} \quad (A=1) \]

\[ y = 0 \quad (A=0) \]

\[ y = -e^{2\sqrt{1+x^2}} \quad (A=-1) \]

(c) Along each characteristic curve the solution \( u = u(x,y) \) is constant:

\[ u(x,y) = u(x, Ae^{-2\sqrt{1+x^2}}) = u(x, Ae^0) = f(A) \]

Therefore \[ u(x,y) = f(ye^{-2\sqrt{1+x^2}}) \] where \( f \) is any \( C^1 \) function of a single real variable.

(d) \[ y^3 = u(x,y) = f(ye^{-2}) \quad \text{for all} -u < y < \infty. \quad \text{Let} \quad z = ye^{-2}; \quad \text{then} \quad f(z) = (e^2 z)^3 \]

\[ = e^6 z^3 \quad \text{for all} -u < z < \infty. \quad \text{Consequently, the solution to the I.V.P. is} \]

\[ u(x,y) = f(ye^{-2\sqrt{1+x^2}}) = e^6(ye^{-2\sqrt{1+x^2}})^3 = \left( \frac{6 - 6\sqrt{1+x^2}}{e^6} \right) y^3 \]

#2. \[ u_{xx} - 4u_{xt} + 4u_{tt} = 0 \]

(a) Second-order, linear, hyperbolic

(b) \[ a = \left( \frac{2^2}{\partial x^2} - 4 \frac{2^2}{\partial x \partial t} + 4 \frac{2^2}{\partial t^2} \right) u \]

\[ = \left( \frac{\partial^2}{\partial x^2} - 2 \frac{\partial}{\partial t} \right) \left( \frac{\partial^2}{\partial x^2} - 2 \frac{\partial}{\partial t} \right) u \]

Let \[ \gamma = 2x + t, \quad \text{Then} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \gamma \partial \gamma} + \frac{\partial^2 u}{\partial \eta \partial \gamma} = \frac{\partial^2 u}{\partial \gamma^2} + \frac{\partial^2 u}{\partial \eta \partial \gamma} \]

\[ \eta = x - 2t. \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \gamma \partial t} + \frac{\partial^2 u}{\partial \eta \partial t} = \frac{\partial^2 u}{\partial \gamma^2} - 2 \frac{\partial u}{\partial \eta} \]

I.e. as operators, \[ \frac{\partial}{\partial x} = \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \gamma} - 2 \frac{\partial}{\partial \eta} \]

Therefore \[ \frac{\partial^2 - 2 \frac{\partial}{\partial t}}{\partial x} = \left( \frac{\partial^2}{\partial \gamma^2} + \frac{\partial}{\partial \eta} \right) - 2 \left( \frac{\partial}{\partial \gamma} - 2 \frac{\partial}{\partial \eta} \right) = 5 \frac{\partial}{\partial \eta}, \quad \text{so} (a) \] is
equivalent to \( 25 \frac{\partial^2 u}{\partial \eta^2} = 0 \) \( \Rightarrow \) \( \frac{\partial u}{\partial \eta} = f(\eta) \) \( \Rightarrow \) \( u = \eta f(\eta) + g(\eta) \).

\[
\therefore \quad u(x,t) = (x-2t)f(2x+t) + g(2x+t)
\]

where \( f \) and \( g \) are \( C^2 \)-functions of a single real variable.

\[ (c) \quad u_t(x,t) = -2f(2x+t) + (x-2t)f'(2x+t) + g'(2x+t) \]

\[
\begin{align*}
4x^3 &= u(x,0) = xf(2x) + g(2x) & \text{for all real } x . \\
-4x^2 &= u_t(x,0) = -2f(2x) + xf(2x) + g'(2x) & \text{for all real } x .
\end{align*}
\]

Differentiating the first identity in the system above gives

\[
12x^2 = 4f(2x) + 2xf'(2x) + 2g'(2x).
\]

Multiplying the second identity in the system by 2 gives

\[
-8x^2 = -4f(2x) + 2xf'(2x) + 2g'(2x),
\]

Subtracting the last equation from the preceding one yields

\[
4x^3 = 5f(2x) \quad \text{for all real } x; \quad \text{i.e., } 4x^3 = (2x)^2 = f(2x) \quad \text{so } f(x) = \frac{x^2}{4} \text{ for all } x .
\]

Substituting this for \( f \) in the first identity of the system produces

\[
4x^3 = xf(2x) + g(2x) = x(2x)^2 + g(2x) \quad \Rightarrow \quad g(2x) = 0 \quad \text{for all real } x .
\]

\[
\therefore \quad u(x,t) = (x-2t)f(2x+t) + g(2x+t) = \frac{(x-2t)(2x+t)^2}{4t}.
\]

\[ \#3. \quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^4 \, dy . \quad \text{Let } p = y - x \sqrt{t} . \quad \text{Then } dp = \frac{dy}{\sqrt{4t}} .
\]

\[
\therefore \quad u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x + p \sqrt{4t})^4 dp
\]

\[
\begin{align*}
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left\{ x^4 + 4xp \sqrt{4t} + 6x^2 p^2 (4t) + 4xp^3 \sqrt{4t} + p^4 (4t) \right\} dp \\
&= \frac{x^4}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp + 4x \int_{-\infty}^{\infty} e^{-p^2} p dp + 24x^2 \int_{-\infty}^{\infty} e^{-p^2} p^2 dp + 16x^3 \int_{-\infty}^{\infty} e^{-p^2} p^2 dp + 16x^4 \int_{-\infty}^{\infty} e^{-p^2} p^4 dp
\end{align*}
\]
\( u(x,t) = x^4 + 12x^2t + 12t^2 \)

Check: \( u_t - u_{xx} = (12x^2 + 24t) - (12x^2 + 24t) = 0 \)

\( u(x,0) = x^4 \)

\( \#4 \)

(a) \( \hat{g}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-b) e^{-isx} \, dx \) \quad \text{Let } y = x-b.

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-is(y+b)} \, dy = e^{-isb} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-isy} \, dy = e^{-isb} \hat{f}(s). \]

(b) \( \mathcal{F}(u_t - u_{xx} + u_x)(s) = \mathcal{F}(0)(s) = 0 \)

\[ \frac{\partial}{\partial t} \mathcal{F}(u)(s) - \mathcal{F}(u_{xx})(s) + \mathcal{F}(u_x)(s) = 0 \]

Using the property \( \mathcal{F}'(s) = is\mathcal{F}(s) \) we have

\[ \frac{\partial}{\partial t} \mathcal{F}(u)(s) - (is) \mathcal{F}(u)(s) + is \mathcal{F}(u)(s) = 0 \]

\[ \frac{\partial}{\partial t} \mathcal{F}(u)(s) + (is + s^2) \mathcal{F}(u)(s) = 0. \] (Linear, first-order ODE in the variable \( t \))

Integrating factor: \( e \int (is + s^2) \, dt = e^{ist} \cdot e^{ist} \)

\[ e^{(is + s^2)t} \frac{\partial}{\partial t} \mathcal{F}(u)(s) + (is + s^2)e^{(is + s^2)t} \mathcal{F}(u)(s) = 0 \]

\[ \frac{\partial}{\partial t} \left[ e^{(is + s^2)t} \mathcal{F}(u)(s) \right] = 0 \]

\[ \therefore \mathcal{F}(u)(s) = c(s)e^{-(is + s^2)t} \]

Applying the initial condition \( u(x,0) = \varphi(x) \) for all real \( x \) gives
\[ F(q)(\lambda) = F(u(\cdot,0))(\lambda) = e^{\lambda} = e^{-i\lambda t} \]

\[ \therefore \quad F(u)(\lambda) = F(q)(\lambda) e^{-i\lambda t} \]

Using table entry I. with \( a = \frac{1}{4\pi t} \) yields

\[ F\left( e^{rac{-\lambda^2}{4\pi t}} \right)(\lambda) = e^{-\frac{\lambda^2}{4\pi t}} \]

Substituting this identity in the next-to-last equation produces

\[ F(u)(\lambda) = F(q)(\lambda) F\left( e^{rac{-\lambda^2}{4\pi t}} \right)(\lambda) e^{-i\lambda t} \]

Since \( F(f \ast g)(\lambda) = \sqrt{2\pi} F(f)(\lambda) F(g)(\lambda) \),

\[ F(u)(\lambda) = \frac{1}{\sqrt{2\pi}} F\left( q \ast e^{rac{-\lambda^2}{4\pi t}} \right)(\lambda) e^{-i\lambda t} \]

\[ = F\left( q \ast e^{rac{-\lambda^2}{4\pi t}} \right)(\lambda) e^{-i\lambda t} \]

Using part (a) of this problem plus the inversion theorem, we find that

\[ u(x,t) = (q \ast e^{rac{-\lambda^2}{4\pi t}})(x-t) \]

\[ = \int_{-\infty}^{\infty} \frac{e^{\frac{(x-t-y)^2}{4\pi t}}}{\sqrt{4\pi t}} q(y) dy \]
Separation of variables: We seek nontrivial solutions to the homogeneous part of the problem (1-5) of the form \( u(x,t) = X(x)T(t) \). Substituting in (1), yields \( X(t)T''(t) - X''(t)T(t) = 0 \) \( \Rightarrow \frac{-X''(t)}{X(t)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda \).

Substituting in (3-5) we have
\[
X(t)T(t) = 0 = X(1)T(1) \quad \text{for} \ t \geq 0 \quad \text{and} \quad X(t)T'(t) = 0 \quad \text{for} \ 0 \leq x \leq 1.
\]
Thus
\[
\begin{cases}
X''(x) + \lambda X(x) = 0, & X(0) = X(1) \\
T''(t) + \lambda T(t) = 0, & T'(0) = 0
\end{cases}
\]
--- Eigenvalue problem.

The eigenvalues/eigenfunctions are
\[
\lambda_n = (n\pi)^2, \quad X_n(x) = \cos(n\pi x) \quad (n = 1, 2, \ldots)
\]
and the solutions to the t-problem with \( \lambda = \lambda_n = (n\pi)^2 \) are \( T_n(t) = \cos(n\pi t) (n = 0, 1, 2, \ldots) \).

Consequently, \( u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \cos(n\pi t) \) is a formal solution to (1-6-3-5) for any constants \( a_0, a_1, a_2, \ldots \). We want to choose the constants to satisfy (7):
\[
\cos(3\pi x) + 3\cos(\pi x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{for} \ 0 \leq x \leq 1.
\]

By inspection, \( a_1 = 3, a_3 = 1, \) and other coefficients should be zero. Thus

(a)
\[
u(x,t) = 3\cos(\pi x)\cos(\pi t) + \cos(3\pi x)\cos(3\pi t)
\]

(b) Suppose there is another solution to the problem in (a), say \( v(x,t) \).

Consider \( w(x,t) = u(x,t) - v(x,t) \) and its energy function
\[
E(t) = \int_0^1 \left( \frac{1}{2} w_x(x,t)^2 + \frac{1}{2} w_t(x,t)^2 \right) \, dx \quad (t \geq 0).
\]

Observe that \( w \) solves the problem
\[
\begin{cases}
w_{tt} - w_{xx} = 0 \quad \text{for} \ 0 < x < 1, 0 < t < \infty, \\
w_x(0,t) = w_x(1,t) = 0 \quad \text{for} \ t \geq 0, \\
w(x,0) = w_t(x,0) = 0 \quad \text{for} \ 0 \leq x \leq 1.
\end{cases}
\]
Differentiating the energy function, we find

\[
\frac{dE}{dt} = \int_0^1 \frac{3}{2} \left\{ \frac{1}{2} \left[ w_x(x,t) \right]^2 + \frac{1}{2} \left[ w_x(x,t) \right]^2 \right\} dx
\]

\[
= \int_0^1 \left\{ w_t(x,t) w_x(x,t) + w_x(x,t) w_{xx}(x,t) \right\} dx
\]

\[
= \int_0^1 \left\{ \frac{2}{3} \left\{ w_t(x,t) w_t(x,t) \right\} dx
\]

\[
= w_t(1,t) w_x(1,t) \bigg|_{x=0} - w_x(0,t) w_t(0,t)
\]

\[
= 0.
\]  

(from 6 and 8)

Therefore the energy is constant for all \( t \geq 0 \):

\[
0 \leq E(t) = E(0) = \int_0^1 \left\{ \frac{1}{2} \left[ w_x(x,0) \right]^2 + \frac{1}{2} \left[ w_x(x,0) \right]^2 \right\} dx = 0
\]  

(from 9 and 10)

By the vanishing theorem, \( w_t(x,t) = w_x(x,t) = 0 \) for all \( 0 \leq x \leq 1 \) and all \( t \geq 0 \), and hence \( w(x,t) = \text{constant} \). From 7 we see that this constant must be zero; i.e. \( u(x,t) - v(t,x) = w(x,t) = 0 \) for all \( 0 \leq x \leq 1 \), \( 0 \leq t < 0 \). This shows that there is only one solution to the problem in (a).
(a) Since \( f(x) = |x| \) is an even function, all its sine coefficients are zero.

\[
a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-\pi}^{\pi} f(x) \, dx}{\int_{-\pi}^{\pi} 1^2 \, dx} = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{x^2}{2\pi} \bigg|_{0}^{\pi} = \frac{\pi}{2}.
\]

(b) \( S_n(x) = \frac{\pi}{2} - \frac{4}{\pi^2} \cos(nx) - \frac{4}{\pi^4} \cos(3nx) \).

(c) \[ \text{Yes} \], the full Fourier series of \( f \) converges to \( f \) in the mean square sense on \([-\pi, \pi]\) because

\[ \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^3}{3} < \infty. \]

(d) \[ \text{Yes} \], the full Fourier series of \( f \) converges pointwise to \( f \) on \([-\pi, \pi]\) since

\( f \) is continuous and 2\( \pi \)-periodic on \(-\infty < x < \infty\), and \( f \) is piecewise continuous on \([-\pi, \pi]\). By Theorem 4(ii), for all real \( x \) we have

\[
\frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4\cos(k\pi x)}{\pi(k\pi)^2} = \frac{1}{2} \left[ f(x^+) + f(x^-) \right] \]

\[
= \frac{1}{2} \left[ f(x^+) + f(x^-) \right] \quad \text{(since} \ f \ \text{is} \ 2\pi - \text{periodic)}
\]

\[
= f(x). \quad \text{(since} \ f \ \text{is} \ \text{continuous)}
\]

(e) \[ \text{Yes} \], the full Fourier series of \( f \) converges uniformly to \( f \) on \([-\pi, \pi]\), although
Theorem 2 cannot be used to justify this. (Note that \( f(x) = |x| \) does not satisfy the periodic boundary conditions \( \phi(-\pi) = \phi(\pi) \) and \( \phi'(-\pi) = \phi'(\pi) \) that generate the orthogonal system \( \{1, \cos(mx), \sin(mx)\}_{m=1}^{\infty} \) used for full Fourier series.)

To show uniform convergence, observe that

\[
\max_{-\pi \leq x \leq \pi} \left| f(x) - \left( \frac{\pi}{2} + \sum_{k=0}^{N} \frac{-4 \cos((k+1)x)}{\pi (2k+1)^2} \right) \right| = \max_{-\pi \leq x \leq \pi} \left| \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4 \cos((2k+1)x)}{\pi (2k+1)^2} - \frac{\pi}{2} - \sum_{k=0}^{N} \frac{4 \cos((k+1)x)}{\pi (2k+1)^2} \right|
\]

\[
= \max_{-\pi \leq x \leq \pi} \left| \sum_{k=N+1}^{\infty} \frac{-4 \cos((2k+1)x)}{\pi (2k+1)^2} \right| \leq \max_{-\pi \leq x \leq \pi} \sum_{k=N+1}^{\infty} \left| \frac{-4 \cos((2k+1)x)}{\pi (2k+1)^2} \right|
\]

\[
\leq \frac{4}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{(2k+1)^2} \longrightarrow 0 \text{ as } N \to \infty \text{ (since it is the "tail" of the series } \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}) \text{, which is convergent by comparison with the p-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ with } p = 2.)
\]

(f) Take \( x = 0 \) in the identity obtained in part (a):

\[
0 = f(0) = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4 \cos(0)}{\pi (2k+1)^2} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \left( -\frac{\pi}{2} \right) \left( -\frac{\pi}{4} \right) = \left( \frac{\pi^2}{8} \right)
\]

(g) By the Parseval identity \( \sum_{n=1}^{\infty} |A_n|^2 \int_{-\pi}^{\pi} |X_n(x)|^2 dx = \int_{-\pi}^{\pi} |f(x)|^2 dx \) and the results of part (a), we have

\[
\frac{\pi^2}{2} \int_{-\pi}^{\pi} 1 dx + \sum_{k=0}^{\infty} \left| \frac{-4 \cos((2k+1)x)}{\pi (2k+1)^2} \right|^2 \int_{-\pi}^{\pi} \cos^2((2k+1)x) dx = \int_{-\pi}^{\pi} |x|^2 dx.
\]

Evaluating the integrals gives

\[
\frac{\pi^2}{4} (2\pi) + \sum_{k=0}^{\infty} \frac{16}{\pi^4 (2k+1)^4} \cdot (\pi) = \frac{2\pi^3}{3} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{\pi^4 (2k+1)^4} = \frac{\pi^4}{96}.
\]
The equation $u_t - k \nabla^2 u = 0$ models heat flow. For steady-state distributions, $u_t = 0$ for all $x$ and $t$. Therefore we need to solve

$$
\begin{align*}
\nabla^2 u &= 0 \quad \text{in} \quad A = \{(r, \theta) : 1 < r < 2, -\pi \leq \theta < \pi\}, \\
\frac{\partial u}{\partial r}(r, \theta) &= 0 \quad \text{for} \quad -\pi \leq \theta < \pi, \\
u(2, \theta) &= \theta \quad \text{for} \quad -\pi \leq \theta < \pi.
\end{align*}
$$

(We also have the implied boundary conditions $u_r(r, \pi) = u_r(r, -\pi)$ and $u_\theta(r, \pi) = u_\theta(r, -\pi)$ for all $1 < r < 2$.) We use separation of variables, i.e., we seek nontrivial solutions to the homogeneous portion of the problem $(1-3)$ of the form $u(r, \theta) = R(r) \Theta(\theta)$. Substituting in $(1)$ gives

$$
0 = u_r + \frac{1}{r} u_r + \frac{1}{r^2} u_\theta = R''(r) \Theta(\theta) + \frac{1}{r} R'(r) \Theta(\theta) + \frac{1}{r^2} R(r) \Theta''(\theta),
$$

$$
\therefore \quad \frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \text{constant} = \lambda.
$$

Substituting in $(2-4)$ gives

$$
R'(r) \Theta(\theta) = 0 \quad \text{for} \quad -\pi \leq \theta < \pi \quad \text{and} \quad R(r) \Theta(\pi) = R(r) \Theta(-\pi), \quad R(r) \Theta'(\pi) = R(r) \Theta'(-\pi)
$$

for $1 < r < 2$.

Consequently,

$$
\begin{align*}
&\begin{cases}
r^2 R''(r) + r R'(r) - \lambda R(r) = 0, \quad R'(r) = 0 \\
\Theta''(\theta) + \lambda \Theta(\theta) = 0, \quad \Theta(\pi) = \Theta(-\pi), \quad \Theta'(\pi) = \Theta'(-\pi).
\end{cases}
\end{align*}
$$

The eigenvalues/eigenfunctions are

$$
\lambda_n = n^2, \quad \Theta_n(\theta) = a_n \cos(n \theta) + b_n \sin(n \theta) \quad (n = 0, 1, 2, \ldots).
$$

Returning to the radial problem with $\lambda = \lambda_n = n^2$ we have

$$
\begin{align*}
r^2 R''_n(r) + r R'_n(r) - n^2 R_n(r) = 0, \quad R'_n(r) = 0.
\end{align*}
$$

The general solution to the ODE is $R_n(r) = c_1 r^n + c_2 r^{-n}$ when $n \geq 1$ and
\[ R_0(r) = c_1 + c_2 e^{-\ln(r)} \text{ when } n = 0. \] If \( n \neq 1 \) then \( R_n'(r) = n c_1 r^{n-1} - n c_2 r^{-n-1} \) so

\[ 0 = R_n'(r) = n c_1 - n c_2 \Rightarrow c_1 = c_2. \] If \( n = 0 \), then \( R_0'(r) = \frac{c_1}{r} \) so \( 0 = R_0'(r) = \frac{c_1}{r} \)

\[ \Rightarrow c_2 = 0. \] Thus, up to a constant multiple, \( R_n(r) = r^n + r^{-n} \) (\( n \neq 1 \)) and \( R_0(r) = 1 \).

Consequently

\[ u(r; \theta) = a_0 + \sum_{k=1}^{\infty} (r^n + r^{-n})(a_k \cos(k\theta) + b_k \sin(k\theta)) \]

is a formal solution to \((1)-(5)\) for arbitrary constants \( a_0, a_1, b_1, a_2, b_2, \ldots \) We need to choose the constants so \((3)\) is satisfied:

\[ |\theta| = u(r; \theta) = a_0 + \sum_{n=1}^{\infty} (r^n + r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta)) \quad \text{for } -\pi \leq \theta \leq \pi. \]

From problem 6(d) we know that

\[ |\theta| = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4 \cos((k+1)\theta)}{\pi (2k+1)^2} \quad \text{for } -\pi \leq \theta \leq \pi. \]

Therefore we need to choose

\[ a_0 = \frac{\pi}{2}, \quad \left[ \frac{2k+1}{2} - \frac{2k+1}{2} \right] a_{2k+1} = -\frac{4}{\pi (2k+1)^2} \quad \text{for } k=0,1,2,\ldots, \]

and all other \( a_n \) and \( b_n \) equal to zero. Thus

\[ u(r; \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(r^{2k+1} + r^{-2k-1}) \cos((2k+1)\theta)}{(2k+1)^2 (2k+1)^2} \]

This can also be expressed as

\[ u(r; \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cosh((2k+1)\ln(r)) \cos((2k+1)\theta)}{\cosh((2k+1)\ln(z)) (2k+1)^2}. \]