Mathematics 325 | Final Exam Summer 2008 | Name: Dr. Grow

1.(25 pts.) (a) Solve the equation $\cos(x)u_x + y\sin(x)u_y = 0$ subject to $u(0,y) = e^{-y^2}$ for $-\infty < y < \infty$.

(b) In which region of the xy-plane is the solution uniquely determined?

20 pts. (a) Along the characteristic curves of the PDE, described by $\frac{dy}{dx} = \frac{y\sin(x)}{\cos(x)}$, 3 pts. to he[re]

the solution u is constant. Then $\frac{1}{y}\,dy = \frac{\sin(x)\,dx}{\cos(x)} \Rightarrow \ln|y| = -\ln|\cos(x)| +$

$\Rightarrow \ln|y\cos(x)| = c \Rightarrow y\cos(x) = \pm e^{c} = A \Rightarrow y = A\sec(x)$. (Characteristic curves)

9 pts. to here.

Along a characteristic curve, $u(x,y) = u(x, A\sec(x)) = u(0,A) = f(A)$. Thus

15 pts. to here.

the general solution is $u(x,y) = f(y\cos(x))$ where f is a C^1-function of a single real variable. Applying the auxiliary condition yields

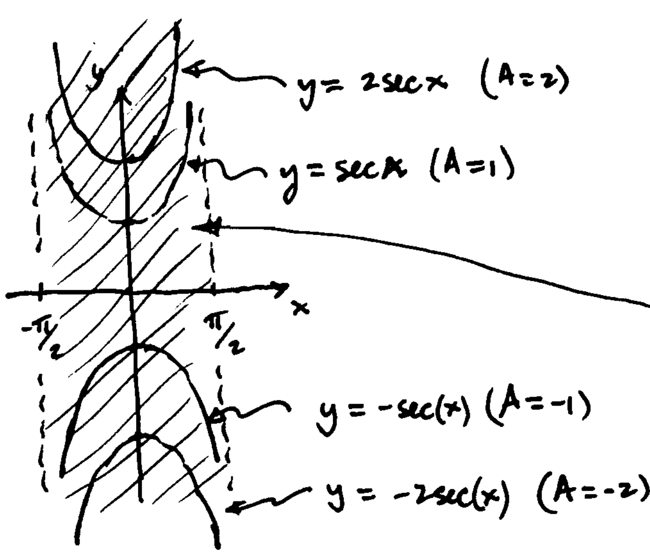
17 pts. to here

$$e^{-y^2} = u(0,y) = f(y\cos(0)) = f(y) \text{ for all real } y.$$

$$\therefore \boxed{u(x,y) = f(y\cos(x)) = e^{-y^2\cos^2(x)}}$$

20 pts. to here.

5 pts. (b)



The characteristic curves "pave" the strip,

$S: -\pi/2 < x < \pi/2,\ -\infty < y < \infty$.

Therefore the solution is uniquely determined in S.

$B^2 - 4AC = (-3)^2 - 4(1)(-4) > 0$

2.(25 pts.) Consider the partial differential equation $u_{xx} - 3u_{xy} - 4u_{yy} = 0$.

(a) Classify its order and type (linear, nonlinear, homogeneous, inhomogeneous, parabolic, etc.).

(b) Find, if possible, its general solution in the xy – plane.

(c) Find, if possible, its solution which satisfies $u(x,0) = x^3$ and $u_y(x,0) = -3x^2$ if $-\infty < x < \infty$.

5 pts. (a) The PDE is $\boxed{\text{second-order}_{(1)},\ \text{linear}_{(1)},\ \text{homogeneous}_{(1)},\ \text{and of hyperbolic type}_{(2)}}$

10 pts. (b) $\left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial y} - 4\frac{\partial^2}{\partial y^2}\right)u = 0$

2 pts. to here. $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right)u = 0$

4 pts. to here.

Let $\xi = \beta x - \alpha y = x - y$

$\eta = -(\delta x - \gamma y) = -(-4x - y)$
$= 4x + y$

$\frac{\partial}{\partial x} = \overset{1}{\cancel{\frac{\partial \xi}{\partial x}}}\frac{\partial}{\partial \xi} + \overset{4}{\cancel{\frac{\partial \eta}{\partial x}}}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + 4\frac{\partial}{\partial \eta}$

$\frac{\partial}{\partial y} = \overset{-1}{\cancel{\frac{\partial \xi}{\partial y}}}\frac{\partial}{\partial \xi} + \overset{1}{\cancel{\frac{\partial \eta}{\partial y}}}\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

$\therefore \frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 5\frac{\partial}{\partial \eta}$

and $\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y} = 5\frac{\partial}{\partial \xi}$

$\therefore \quad 5\frac{\partial}{\partial \eta}\left(5\frac{\partial}{\partial \xi}\right)u = 0$

$\Rightarrow \quad \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$ 6 pts. to here.

$\Rightarrow \quad \frac{\partial u}{\partial \xi} = \int 0\, d\eta + c_1(\xi)$

$\Rightarrow \quad u = \int c_1(\xi)\, d\xi + c_2(\eta)$

$\therefore \quad u = f(\xi) + g(\eta)$. 8 pts. to here.

$\therefore$ $\boxed{u(x,y) = f(x-y) + g(4x+y) \text{ where } f \text{ and } g \text{ are } C^2\text{-functions of a single real variable.}}$ 10 pts. to here

10 pts. (c) $u_y(x,y) = -f'(x-y) + g'(4x+y)$ so

(*) $-3x^2 = u_y(x,0) = -f'(x) + g'(4x)$ for all real x. 2 pts. to here

(†) $x^3 = u(x,0) = f(x) + g(4x)$ " " " "

(**) $\Rightarrow 3x^2 = f'(x) + 4g'(4x)$ " " " "

(OVER)

4 pts. to here.

Adding (*) and (**) yields $0 = 5g'(4x)$ for all real x $\Rightarrow$ $g(x) = c$ 6 pts. to here

for all real x. Substituting in (†) gives $x^3 = f(x) + c$. 8 pts. to here.

$\therefore \quad u(x,y) = f(x-y) + g(4x+y) = (x-y)^3 - c + c$

$$\boxed{u(x,y) = (x-y)^3}$$

10 pts. to here.

3.(25 pts.) A homogeneous solid material occupying $D=\{(x,y,z)\in\mathbb{R}^3:\ 4\le x^2+y^2+z^2\le 100\}$ is completely insulated and its initial temperature at position (x,y,z) in D is $200/\sqrt{x^2+y^2+z^2}$.

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t\ge 0$.

(b) Use Gauss' divergence theorem to help show that the heat energy $H(t)=\iiint_D c\rho u(x,y,z,t)\,dV$ of the material in D at time t is a constant function of time. Here c and ρ denote the (constant) specific heat and mass density, respectively, of the material in D.

Bonus (10 pts.). Compute the (constant) steady-state temperature that the material in D reaches after a long time.

15pts. (a)
$$\begin{cases} u_t - k\nabla^2 u = 0 & \text{if } 4< x^2+y^2+z^2<100 \text{ and } 0<t<\infty, & (5)\\ \dfrac{\partial u}{\partial n} = 0 & \text{if } t\ge 0 \text{ and } x^2+y^2+z^2=4 \text{ or } x^2+y^2+z^2=100, & (5)\\ u(x,y,z,0)=\dfrac{2}{\sqrt{x^2+y^2+z^2}} & \text{if } 4\le x^2+y^2+z^2\le 100. & (5)\end{cases}$$

5pts. to here.

10 pts. (b)
$$\frac{dH}{dt}=\iiint_D \frac{\partial}{\partial t}\left(c\rho u(x,y,z,t)\right)dV=\iiint_D c\rho u_t\,dV=\iiint_D c\rho k\nabla^2 u\,dV=\iiint_D c\rho k\,\nabla\cdot(\nabla u)\,dV$$
$$=c\rho k\iint_{\partial D}\nabla u\cdot\vec{n}\,dS = c\rho k\iint_{\partial D}\cancel{\frac{\partial u}{\partial n}}^{\,0}dV=0.$$
Therefore $H(t)=$ constant for all $t\ge 0$

10 pts. to here.

10 pts. **Bonus**: Let $U_0=\lim_{t\to\infty}u(x,y,z,t)$ be the steady-state temperature of the material in D

Then $H(0)=\lim_{t\to\infty}H(t)=\lim_{t\to\infty}\iiint_D c\rho u(x,y,z,t)\,dV=\iiint_D c\rho U_0\,dV=c\rho U_0\cdot \mathrm{vol}(D)$.

1 pt. to here.

4 pts. to here

But $\mathrm{vol}(D)=\frac{4}{3}\pi(R_2^3-R_1^3)=\frac{4}{3}\pi(10^3-2^3)=\frac{4\cdot 992\pi}{3}$ and $H(0)=\iiint_D c\rho u(x,y,z,0)\,dV$

$$=\iiint_D c\rho\frac{200}{\sqrt{x^2+y^2+z^2}}dV=\int_0^{2\pi}\int_0^{\pi}\int_2^{10}c\rho\frac{200}{r}r^2\sin\varphi\,dr\,d\varphi\,d\theta=\int_0^{2\pi}\int_0^{\pi}c\rho 100r^2\Big|_{r=2}^{10}\sin\varphi\,d\varphi\,d\theta$$

7 pts. to here

$$=9600c\rho(-\cos\varphi)\Big|_{\varphi=0}^{\pi}\cdot 2\pi = 38{,}400c\rho\pi.$$ Thus $38{,}400\cancel{c}\cancel{\rho}\cancel{\pi}=\cancel{c}\cancel{\rho}U_0\frac{4}{3}\cdot 992\cancel{\pi}$

$$\Rightarrow U_0=\frac{3}{4\cdot 992}\cdot 38{,}400=\boxed{\frac{900}{31}\cong 29}.$$ 10pts. to here.

4.(25 pts.) Use Fourier transform methods to find a solution to

$$u_{xx} + u_{yy} \overset{(1)}{=} 0 \quad \text{for} \quad -\infty < x < \infty,\ 0 < y < \infty,$$

which satisfies

$$u(x,0) \overset{(2)}{=} \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\lim_{y\to\infty} u(x,y) \overset{(3)}{=} 0 \quad \text{for each } x \text{ in } (-\infty,\infty).$$

We take the Fourier transform of (1) with respect to the variable x:

$$\mathcal{F}(u_{xx} + u_{yy})(\xi) = \mathcal{F}(0)(\xi)$$

$$(i\xi)^2\mathcal{F}(u)(\xi) + \frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) = 0$$

4 pts. to here.

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) - \xi^2\mathcal{F}(u)(\xi) = 0 .$$

The general solution to this second-order linear homogeneous ODE in the variable y, with ξ as a parameter, is $\mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}$. We apply (3) to get

8 pts. here.

$$\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}\right) = \lim_{y\to\infty}\mathcal{F}(u)(\xi) = \lim_{y\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,y)e^{-i\xi x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\underbrace{\lim_{y\to\infty}u(x,y)}_{0}\Big)e^{-i\xi x}\,dx$$

$= 0$. Consider first the case when $\xi > 0$. If $c_1(\xi) \neq 0$ then $\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}\right)$ $= \lim_{y\to\infty}\left(c_1(\xi)e^{\xi y}\right) = \pm\infty$, depending on the algebraic sign of $c_1(\xi)$. Since this contradicts the fact that the limit must be 0, it follows that $c_1(\xi) = 0$ if $\xi > 0$.

A similar argument shows that $c_2(\xi) = 0$ if $\xi < 0$. Consequently,

12 pts. to here.

$$(*) \qquad \mathcal{F}(u)(\xi) = \begin{cases} c_2(\xi)e^{-\xi y} & \text{if } \xi > 0 \\ c_1(\xi)e^{\xi y} & \text{if } \xi < 0 \end{cases} = c(\xi)e^{-|\xi|y} .$$

Evaluating $(*)$ at $y=0$ and applying (2) gives

15 pts. to here.

$$(**) \qquad c(\xi) = \mathcal{F}(u)(\xi)\Big|_{y=0} = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(\chi_{(-1,1)})(\xi). \qquad \text{(OVER)}$$

By entry C in the table of Fourier transforms accompanying this exam,

17 pts. to here.

$$(***) \quad \mathcal{F}\left(\frac{1}{(\cdot)^2+y^2}\right)(\xi) = \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-|\xi|y}}{y} \quad \text{so} \quad \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi) = e^{-|\xi|y}.$$

Substituting from (**) and (***) into (*) gives

$$\mathcal{F}(u)(\xi) = \mathcal{F}(\chi_{(-1,1)})(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\chi_{(-1,1)} * \sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$= \mathcal{F}\left(\frac{1}{\pi}\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

20 pts. to here.

Applying the inversion theorem,

21 pts. to here.

$$u(x,y) = \left(\frac{1}{\pi}\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-z)^2+y^2}\chi_{(-1,1)}(z)\,dz$$

23 pts. to here

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{y}{(x-z)^2+y^2}\,dz \longleftarrow \left\{\begin{array}{l}\text{Let } s = z-x.\\ \text{Then } ds = dz.\end{array}\right.$$

$$= \frac{1}{\pi}\int_{-1-x}^{1-x}\frac{y}{s^2+y^2}\,ds$$

$$= \frac{1}{\pi}\operatorname{Arctan}\left(\frac{s}{y}\right)\Bigg|_{s=-1-x}^{1-x}$$

$$= \boxed{\frac{1}{\pi}\operatorname{Arctan}\left(\frac{1-x}{y}\right) + \frac{1}{\pi}\operatorname{Arctan}\left(\frac{1+x}{y}\right)}.$$

25 pts. to here.

②-③ $u_x(0,y,z) = 0 = u_x(1,y,z)$ if $0 \le y \le 1,\ 0 \le z \le 1$ ⑥ $u_y(x,1,z) = 0$ if $0 \le x \le 1,\ 0 \le z \le 1$

④-⑤ $u_z(x,y,0) = 0 = u_z(x,y,1)$ if $0 \le x \le 1,\ 0 \le y \le 1$

5.(25 pts.) Solve $\nabla^2 u = 0$ ① in the cube $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$ given that u satisfies the inhomogeneous **Dirichlet** condition $u(x,0,z) \overset{⑦}{=} 4\sin^2(\pi x)\sin^2(\pi z)$ if $0 \le x \le 1,\ 0 \le z \le 1$, and u satisfies homogeneous **Neumann** boundary conditions on the other five faces. (Hint: You may find the identity $2\sin^2(\theta) = 1 - \cos(2\theta)$ useful.)

1 pt. to here.

We seek nontrivial solutions of the form $u(x,y,z) = X(x)Y(y)Z(z)$ to the homogeneous portion of this problem: ①-②-③-④-⑤-⑥. Substituting into ① leads to

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0 \Rightarrow -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \lambda,$$

and then to $-\dfrac{Z''(z)}{Z(z)} = \dfrac{Y''(y)}{Y(y)} - \lambda = \mu$. Substituting in ②,③,④,⑤, and ⑥ and using the assumption that $X(x)Y(y)Z(z)$ is not identically 0 on the unit cube yields $X'(0) = 0 = X'(1)$, $Z'(0) = 0 = Z'(1)$, and $Y'(1) = 0$. Thus

$$(*) \begin{cases} X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(1), & \text{4 pts. to here.} \\ Z''(z) + \mu Z(z) = 0, \quad Z'(0) = 0 = Z'(1), & \text{7 pts. to here.} \\ Y''(y) - (\lambda+\mu)Y(y) = 0, \quad Y'(1) = 0. & \text{9 pts. to here.} \end{cases}$$

The eigenvalue problems in the first two lines of (*) have solutions

$$\lambda_\ell = (\ell\pi)^2, \quad X_\ell(x) = \cos(\ell\pi x) \quad (\ell = 0,1,2,\dots) \qquad \text{12 pts. to here.}$$

$$\mu_m = (m\pi)^2, \quad Z_m(z) = \cos(m\pi z) \quad (m = 0,1,2,\dots) \qquad \text{15 pts. to here.}$$

Substituting $\lambda + \mu = \lambda_\ell + \mu_m = \pi^2(\ell^2 + m^2)$ in the DE in the third line of (*) and applying the B.C. yields $Y_{\ell,m}(y) = \cosh(\pi(y-1)\sqrt{\ell^2+m^2})$, up to a constant factor.

17 pts. to here.

Thus the superposition principle implies that

$$u(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell\pi x)\cos(m\pi z)\cosh(\pi(y-1)\sqrt{\ell^2+m^2})$$

19 pts. to here.

is a formal solution to ①-②-③-④-⑤-⑥ for any choice of constants $A_{\ell,m}$. We need to choose these constants so that ⑦ is satisfied:

(OVER)

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} A_{l,m}\cosh\left(-\pi\sqrt{l^2+m^2}\right)\cos(l\pi x)\cos(m\pi z) = u(x,0,z)$$

$$= 4\sin^2(\pi x)\sin^2(\pi z)$$

$$= \left[1-\cos(2\pi x)\right]\left[1-\cos(2\pi z)\right]$$

$$= 1-\cos(2\pi x)-\cos(2\pi z)+\cos(2\pi x)\cos(2\pi z)$$

must hold for all $0 \le x \le 1$ and $0 \le z \le 1$. Consequently, equating "like" coefficients produces

$$\left.\begin{aligned} A_{0,0} &= 1 \\ A_{2,0}\cosh(-2\pi) &= -1 \\ A_{0,2}\cosh(-2\pi) &= -1 \\ A_{2,2}\cosh\left(-2\pi\sqrt{2}\right) &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A_{0,0} &= 1 \\ A_{2,0} &= \frac{-1}{\cosh(2\pi)} = A_{0,2} \\ A_{22} &= \frac{1}{\cosh(2\pi\sqrt{2})} \end{aligned}$$

22 pts. to ...re.

and all other $A_{l,m} = 0$. That is,

25 pts. to here.

$$\boxed{u(x,y,z) = 1 - \frac{\cos(2\pi x)\cosh(2\pi(y-1))}{\cosh(2\pi)} - \frac{\cos(2\pi z)\cosh(2\pi(y-1))}{\cosh(2\pi)} + \frac{\cos(2\pi x)\cos(2\pi z)\cosh(2\pi(y-1)\sqrt{2})}{\cosh(2\pi\sqrt{2})}}$$

solves 1-2-3-4-5-6-7.

6.(25 pts.) 8 (a) Show that the full Fourier series of $f(x)=x^3-x$ on [-1,1] is $\sum_{n=1}^{\infty}\frac{12(-1)^n\sin(n\pi x)}{(\pi n)^3}$.

7 (b) Show that the full Fourier series of f converges uniformly to f on [-1,1].

5 (c) Use the results above to help compute the sum of the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

5 (d) Use Parseval's identity and the results above to help compute the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^6}$.

(a) The full Fourier series of f is $a_0+\sum_{n=1}^{\infty}\left[a_n\cos(n\pi x)+b_n\sin(n\pi x)\right]$ where

1 pt. to here. $$a_0=\frac{\langle f,1\rangle}{\langle 1,1\rangle}=\frac{1}{2}\int_{-1}^{1}\underbrace{(x^3-x)}_{\text{odd}}dx=0,$$

2 pts. to here. $$a_n=\frac{\langle f,\cos(n\pi\cdot)\rangle}{\langle\cos(n\pi\cdot),\cos(n\pi\cdot)\rangle}=\int_{-1}^{1}\overbrace{\underbrace{(x^3-x)}_{\text{odd}}\underbrace{\cos(n\pi x)}_{\text{even}}}^{\text{odd}}dx=0 \quad \text{for } n=1,2,3,\dots,$$

3 pts. to here. $$b_n=\frac{\langle f,\sin(n\pi\cdot)\rangle}{\langle\sin(n\pi\cdot),\sin(n\pi\cdot)\rangle}=\int_{-1}^{1}\underbrace{(x^3-x)\sin(n\pi x)}_{\text{even}}dx=2\int_{0}^{1}\overbrace{(x^3-x)}^{v}\overbrace{\sin(n\pi x)dx}^{dv}=$$

4 pts. to here $$\cancel{2(x^3-x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1}^{\,0}+\frac{2}{n\pi}\int_0^1\underbrace{(3x^2-1)}_{v}\underbrace{(+\cos(n\pi x))dx}_{dv}=\frac{2}{n\pi}\left[\cancel{(3x^2-1)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1}^{\,0}-\int_0^1 6x\frac{\sin(n\pi x)}{n\pi}dx\right]$$ 5 pts. to here

$$=-\frac{12}{(n\pi)^2}\int_0^1\overbrace{x}^{v}\overbrace{\sin(n\pi x)dx}^{dv}=\frac{-12}{(n\pi)^2}x\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1+\cancel{\frac{12}{(n\pi)^2}\int_0^1-\frac{\cos(n\pi x)}{n\pi}dx}^{\,0}=\frac{12(-1)^n}{(n\pi)^3}.$$ 6 pts. to here. 7 pts. to her

$$\therefore\ \boxed{f(x)\sim\sum_{n=1}^{\infty}\frac{12(-1)^n\sin(n\pi x)}{(n\pi)^3}}.$$ 8 pts. to here

(b) The set of orthogonal functions for the full Fourier series of f on $[-1,1]$ is $\Phi=\{1,\cos(\pi x),\sin(\pi x),\cos(2\pi x),\sin(2\pi x),\dots\}$, and this is the complete set of eigenfunctions for the problem $-X''(x)=\lambda X(x)$ with periodic boundary conditions $\underbrace{X(1)=X(-1)\text{ and }X'(1)=X'(-1)}_{\text{symmetric (or hermitian) B.C.'s}}$. The function $f(x)=x^3-x$ is

1 pt. to here

(OVER)

continuous on $[-1,1]$ and has derivatives $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$

3 pts. to here. which are also continuous on $[-1,1]$. Note that f satisfies the periodic

5 pts. to here. B.C.'s on $[-1,1]$: $f(1) = 0 = f(-1)$ and $f'(1) = 2 = f'(-1)$. Therefore

7 pts. to here. Theorem 2 (on the "Convergence Theorems" attached to this exam) implies that

$$\boxed{\text{the Fourier series of } f \text{ converges to } f \text{ uniformly on } [-1,1].}$$

1 pt. to here. (c) $x^3 - x = f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3}$ for all $-1 \le x \le 1$. Set

2 pts. to here. $x = 1/2$ in this identity to obtain

n	$\sin(n\pi/2)$
1	1
2	0
3	-1
4	0
$\vdots$	
$2k$	0
$2k+1$	$(-1)^k$

$$-\frac{3}{8} = \left(\frac{1}{2}\right)^3 - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi/2)}{(n\pi)^3}$$

$$= \sum_{k=0}^{\infty} \frac{12(-1)^{2k+1} \sin((2k+1)\pi/2)}{((2k+1)\pi)^3}$$

$$\Rightarrow \quad -\frac{3}{8} = -\sum_{k=0}^{\infty} \frac{12(-1)^k}{[(2k+1)\pi]^3}$$

5 pts. to here. $$\Rightarrow \quad \boxed{\frac{\pi^3}{32}} = \left(-\frac{\pi^3}{12}\right)\left(-\frac{3}{8}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

(d) By Parseval's identity, $|a_0|^2 \int_a^b 1^2 dx + \sum_{n=1}^{\infty} \left(|a_n|^2 \int_a^b \cos^2(n\pi x) dx + |b_n|^2 \int_a^b \sin^2(n\pi x) dx \right) =$

2 pts. to here. $\int_a^b |f(x)|^2 dx$. Applying this to our case produces $\sum_{n=1}^{\infty} \left| \frac{12(-1)^n}{(n\pi)^3} \right|^2 \int_{-1}^{1} \sin^2(n\pi x) dx = \int_{-1}^{1} |x^3 - x|^2 d$

But $\int_{-1}^{1} \sin^2(n\pi x) dx = \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos(2\pi n x)\right) dx = 1$ and $\int_{-1}^{1} |x^3 - x|^2 dx = 2\int_0^1 (x - x^3)^2 dx$

$= 2\int_0^1 (x^2 - 2x^4 + x^6) dx = 2\left(\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{16}{105}$. Thus

$$\frac{(12)^2}{\pi^6} \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{16}{105} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\cancel{16}}{105} \cdot \frac{\pi^6}{\cancel{16} \cdot 9} = \boxed{\frac{\pi^6}{945}}.$$ 5 pts. to here.

7.(25 pts.). Solve

$$u_t - u_{xx} \overset{①}{=} 0 \text{ in } -1<x<1,\ 0<t<\infty,$$

subject to the boundary conditions

$$u(1,t) \overset{②}{=} u(-1,t) \text{ and } u_x(1,t) \overset{③}{=} u_x(-1,t) \text{ if } t \ge 0,$$

and the initial condition

$$u(x,0) \overset{④}{=} x^3 - x \text{ if } -1 \le x \le 1.$$

(Hint: You may find the results of problem 6 useful.)
Bonus (10 pts.). Show that the solution to the problem above is unique.

2 pts. to here.

We seek nontrivial solutions of ①-②-③ of the form $u(x,t) = X(x)T(t)$. Substituting into ① yields $T'(t)X(x) - T(t)X''(x) = 0 \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$. Substituting into ②-③ yields $X(1)T(t) = X(-1)T(t)$ and $X'(1)T(t) = X'(-1)T(t)$ for all $t \ge 0$. But $u(x,t) = X(x)T(t)$ is not identically zero so

$$(*) \quad \begin{cases} X''(x) + \lambda X(x) = 0, \quad X(1) = X(-1),\ X'(1) = X'(-1), \\ T'(t) + \lambda T(t) = 0. \end{cases}$$

8 pts. to here.

10 pts. to here.

12 pts. to here. The eigenvalue problem in the first line of (*) has solutions $\lambda = \lambda_n = (n\pi)^2$ and

14 pts. to here. $X_n(x) = a_n \cos(n\pi x) + b_n \sin(n\pi x)$ ($n = 0,1,2,\ldots$ and a_n, b_n arbitrary constants). Setting

16 pts. to here. $\lambda = \lambda_n = (n\pi)^2$ in the second line of (*) and solving yields $T_n(t) = e^{-(n\pi)^2 t}$, up to a a constant factor. Thus

18 pts. to here.

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x) \right] e^{-(n\pi)^2 t}$$

is a formal solution to ①-②-③ for any choice of constants $a_0, a_1, b_1, a_2, b_2, \ldots$ We want to choose these constants so ④ is met. Using the full Fourier series for $f(x) = x^3 - x$ in $[-1,1]$ from problem 6 yields

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x) \right] = u(x,0) = x^3 - x = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3} \quad \text{for } -1 \le x \le 1$$

23 pts. to here.

Therefore choose $b_n = \frac{12(-1)^n}{(\pi n)^3}$ for $n = 1,2,3,\ldots$ and all other coefficients 0. Therefore a solution of ①-②-③-④ is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x) e^{-(n\pi)^2 t}}{(\pi n)^3}}.$$

(OVER for bonus)

25 pts. to here.

Bonus

To show that this solution is unique, suppose that $v = v(x,t)$ is any other solution of ①-②-③-④ and consider $w(x,t) = u(x,t) - v(x,t)$. Then w satisfies

1 pt. to ..

$$(1') \qquad w_t - w_{xx} = 0 \quad \text{if} \quad -1<x<1,\ 0<t<\infty,$$

$$(2')-(3') \qquad w(1,t) = w(-1,t) \quad \text{and} \quad w_x(1,t) = w_x(-1,t) \quad \text{if} \quad t \geq 0,$$

3 pts. to here.

$$(4') \qquad w(x,0) = 0 \quad \text{if} \quad -1 \leq x \leq 1.$$

6 pts. to here. Consider the energy $E(t) = \int_{-1}^{1} w^2(x,t)\,dx$ of this solution at t in $[0,\infty)$. Then

$$\frac{dE}{dt} = \int_{-1}^{1} \frac{\partial}{\partial t}\left[w^2(x,t)\right]dx = \int_{-1}^{1} 2w(x,t)w_t(x,t)\,dx = \int_{-1}^{1} \overbrace{2w(x,t)}^{U}\,\overbrace{w_{xx}(x,t)\,dx}^{dV}$$

$$= \underbrace{2w(1,t)w_x(1,t) - 2w(-1,t)w_x(-1,t)}_{0 \text{ by } (2')-(3')} - 2\int_{-1}^{1} w_x^2(x,t)\,dx$$

$$= -2\int_{-1}^{1} w_x^2(x,t)\,dx$$

$$\leq 0.$$

8 pts to here. Therefore $E = E(t)$ is a decreasing function on $[0,\infty)$, so for $t \geq 0$,

$$0 \leq E(t) \leq E(0) = \int_{-1}^{1} w^2(0,x)\,dx = \int_{-1}^{1} 0\,dx = 0 \quad (\text{by } (4')).$$

By the vanishing theorem, the nonnegative continuous integrand of $E(t)$ must be identically zero on $-1 \leq x \leq 1$ for each $t \geq 0$. That is,

$$0 = w(x,t) = u(x,t) - v(x,t) \quad \text{for } -1 \leq x \leq 1 \text{ and } 0 \leq t < \infty.$$

10 pts. to here. Hence $\boxed{\text{the solution to ①-②-③-④ is unique}}$.

8.(25 pts.) 20 (a) Solve $\nabla^2 u = 0$ inside the unit disk, subject to the boundary condition

$$u(1;\theta) = \begin{cases} 1 & \text{if} \quad 0 < \theta < \pi, \\ -1 & \text{if} \quad -\pi < \theta < 0, \\ 0 & \text{if} \quad \theta = 0 \text{ or } \pi. \end{cases}$$

5 (b) What is the value of the solution to part (a) at the center of the disk? Support your answer.

4 pts. to here. (a) Poisson's formula, $u(r;\theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{h(\varphi)\, d\varphi}{a^2 - 2ar\cos(\theta - \varphi) + r^2}$, applied to this case yields

10 pts. to here.

$$u(r;\theta) = \frac{1-r^2}{2\pi}\left[\int_{-\pi}^{0} \frac{-1\, d\varphi}{1 - 2r\cos(\varphi-\theta) + r^2} + \int_{0}^{\pi} \frac{1\, d\varphi}{1 - 2r\cos(\varphi - \theta) + r^2}\right].$$

12 pts. to here. Making use of the integration fact $\int \frac{d\psi}{A + B\cos(\psi)} = \frac{2}{\sqrt{A^2 - B^2}} \operatorname{Arctan}\left(\tan(\psi/2)\sqrt{\frac{A-B}{A+B}}\right)$

if $A > |B|$ (OVER for details), we have

16 pts. to here.

$$u(r;\theta) = \frac{\cancel{1-r^2}}{\cancel{2}\pi}\left[\frac{\cancel{2}}{\cancel{\sqrt{(1+r^2)^2 - 4r^2}}} \operatorname{Arctan}\left(\tan\left(\frac{\varphi - \theta}{2}\right)\sqrt{\frac{1+r^2+2r}{1+r^2-2r}}\right)\Bigg|_0^{\pi}\right.$$

$$\left. - \frac{\cancel{2}}{\cancel{\sqrt{(1+r^2)^2 - 4r^2}}} \operatorname{Arctan}\left(\tan\left(\frac{\varphi-\theta}{2}\right)\sqrt{\frac{1+r^2+2r}{1+r^2-2r}}\right)\Bigg|_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi}\left[\operatorname{Arctan}\left(\tan\left(\frac{\pi - \theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right) - \cancel{\operatorname{Arctan}\left(\tan\left(\frac{\theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right)}\right.$$

$$\left. - \cancel{\operatorname{Arctan}\left(\tan\left(\frac{\theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right)} + \operatorname{Arctan}\left(\tan\left(\frac{\pi - \theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right)\right]$$

20 pts. to here.

$$= \boxed{\frac{2}{\pi} \operatorname{Arctan}\left(\cot\left(\frac{\theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right)}.$$

(b) By the mean value property for harmonic functions,

5 pts.

$$u(0;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)\, d\varphi = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\, d\varphi + \int_{0}^{\pi} 1\, d\varphi\right] = \boxed{0}.$$

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (a > 0)$	$\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\,e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

Math 325

Final Exam

Summer 2008

n: 20

μ: 119.8

σ: 51.5

Distribution of Scores:

174 – 200	2
146 – 173	6
120 – 145	3
100 – 119	3
0 – 99	6

Distribution of Letter Grades

A	2
B	7
C	5
D	6
F	0