Mathematics 325 Final Exam Name: Dr. Grow
Summer 2000

This final exam consists of seven problems of equal value. Please choose how much you want this exam to count by circling one of the following statements.

I want this exam to count 200 points.
I want this exam to count 300 points.

1. Classify the following partial differential equations as hyperbolic, parabolic, or elliptic, and if possible, find the general solution in the xy-plane.

(a) $u_{xx} - 3u_{xy} - 4u_{yy} = 0$ $\quad B^2-4AC = 9-4(1)(-4) > 0$ hyperbolic

(b) $u_{xx} + u_{yy} + 2u_{xy} + 36u = 0$ $\quad B^2-4AC = 4-4(1)(1) = 0$ parabolic

(a) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}\right)u = 0$

Let $\xi = 4x+y$, $\eta = x-y$.

Then $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$.

$\therefore \frac{\partial}{\partial x}+\frac{\partial}{\partial y} = 5\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial x}-4\frac{\partial}{\partial y} = 5\frac{\partial}{\partial \eta}$.

$\left(\cancel{5}\frac{\partial}{\partial \xi}\right)\left(\cancel{5}\frac{\partial}{\partial \eta}\right)u = 0$

$\frac{\partial u}{\partial \eta} = c_1(\eta)$

$u = \int c_1(\eta)\,d\eta + c_2(\xi)$

$u = f(\eta) + g(\xi)$

$\boxed{u = f(x-y) + g(4x+y)}$ where f and g are arbitrary C^2 functions of a single real variable.

(b) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u + 36u = 0$

Let $\xi = x-y$, $\eta = x+y$.

Then $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

and $\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$.

$\therefore \frac{\partial}{\partial x}+\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta}$.

$\left(2\frac{\partial}{\partial \eta}\right)^2 u + 36u = 0$

$\frac{\partial^2 u}{\partial \eta^2} + 9u = 0$

$u = c_1(\xi)\cos(3\eta) + c_2(\xi)\sin(3\eta)$

$\boxed{u = f(x-y)\cos(3(x+y)) + g(x-y)\sin(3(x+y))}$ where f and g are arbitrary C^2 functions of a single real variable.

2. (a) Write, and simplify as much as possible, the solution to $u_{tt} = u_{xx}$ in the xt-plane which satisfies

$$u(x,0) = e^{-x^2} \quad \text{and} \quad u_t(x,0) = -2xe^{-x^2} \quad \text{for } -\infty < x < \infty.$$

(b) Sketch profiles of the solution to part (a) for times t = 1, t = 2, and t = 3.

(c) Derive a general (nontrivial) relation between ϕ and ψ which will produce a solution to $u_{tt} = u_{xx}$ in the xt-plane satisfying

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty$$

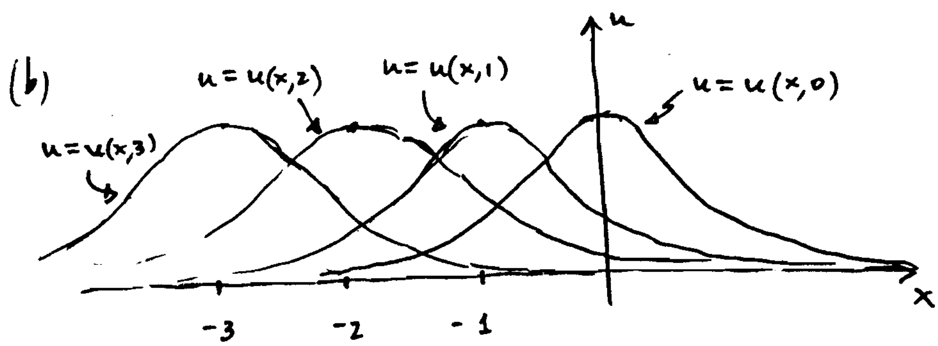
and such that u consists solely of a wave traveling to the left along the x-axis.

(a) $$u(x,t) = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} -2\xi e^{-\xi^2}\, d\xi \qquad \text{(d'Alembert)}$$

$$= \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2}\left(e^{-\xi^2}\right)\Big|_{\xi=x-t}^{x+t}$$

$$= \boxed{e^{-(x+t)^2}}$$

(b)



(c) $$u(x,t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi$$

$$= \frac{1}{2}\left[\varphi(x+t) + \int_0^{x+t} \psi(\xi)\, d\xi\right] + \underbrace{\frac{1}{2}\left[\varphi(x-t) + \int_{x-t}^{0} \psi(\xi)\, d\xi\right]}_{\text{We need this to vanish identically.}}$$

$$\therefore \quad \varphi(\xi) + \int_{\xi}^{0} \psi(\eta)\, d\eta = 0 \quad \text{for all } -\infty < \xi < \infty.$$

$$\Rightarrow \quad \boxed{\varphi'(\xi) = \psi(\xi)} \quad \text{for all } -\infty < \xi < \infty.$$

3. Find the solution to

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

which satisfies

$$u(x,0) = x^3 \quad \text{for } -\infty < x < \infty.$$

You may find the following identities useful:

$$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} p e^{-p^2}\,dp = 0, \quad \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}, \quad \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp = 0.$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} y^3\,dy = \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}}(x+p\sqrt{4t})^3\,dp$$

Let $p=(y-x)/\sqrt{4t}$.

Then $dp = dy/\sqrt{4t}$.

$$= \int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}}\left(x^3 + 3x^2 p\sqrt{4t} + 3x(p\sqrt{4t})^2 + (p\sqrt{4t})^3\right)dp$$

$$= \frac{x^3}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{3x^2\sqrt{4t}}{\sqrt{\pi}}\cancelto{0}{\int_{-\infty}^{\infty} e^{-p^2} p\,dp} + \frac{12xt}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2} p^2\,dp + \frac{4t\sqrt{4t}}{\sqrt{\pi}}\cancelto{0}{\int_{-\infty}^{\infty} e^{-p^2} p^3\,dp}$$

$$= \boxed{x^3 + 6xt}$$

Check: $u_t - u_{xx} = 6x - 6x \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^3$

4. (a) Let f and ψ be piecewise continuous, absolutely integrable functions on $(-\infty,\infty)$. Use Fourier transform methods to solve

$$u_{xx} + u_{yy} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the boundary condition

$$u(x,0) = f(x) \quad \text{for } -\infty < x < \infty$$

and the decay conditions

$$\lim_{y\to\infty} u(x,y) = 0 \quad \text{for each } x \in (-\infty,\infty)$$

and, for each $y > 0$,

$$|u(x,y)| \le |\psi(x)| \quad \text{for all } -\infty < x < \infty.$$

(b) Compute an explicit formula for the solution in part (a) if the function f is given by $f(x) = 1$ for $|x| < 1$, and $f(x) = 0$ otherwise.

(a) $$\mathcal{F}(u_{xx}+u_{yy})(\xi) = \mathcal{F}(0)(\xi)$$

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) + (i\xi)^2\mathcal{F}(u)(\xi) = 0$$

$$\mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}$$

If $\xi > 0$ then $0 = \lim_{y\to\infty}\mathcal{F}(u)(\xi)$ so $c_1(\xi) = 0$

If $\xi < 0$ then $0 = \lim_{y\to\infty}\mathcal{F}(u)(\xi)$ so $c_2(\xi) = 0$

$$\therefore \mathcal{F}(u)(\xi) = \begin{cases} c_2(\xi)e^{-\xi y} & \text{if } \xi > 0, \\ c_1(\xi)e^{\xi y} & \text{if } \xi < 0, \end{cases}$$

$$= c(\xi)e^{-|\xi|y}$$

$$\mathcal{F}(f)(\xi) = \mathcal{F}(u)(\xi)\Big|_{y=0} = c(\xi)e^{-|\xi|y}\Big|_{y=0} = c(\xi)$$

$$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)e^{-|\xi|y}$$

From Table of Fourier Transforms, entry C (with $a=y$) we have

$$e^{-|\xi|y} = \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

$$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$\therefore \mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f * \frac{\sqrt{2}\,y}{\sqrt{\pi}\,((\cdot)^2+y^2)}\right)(\xi).$$

By the Fourier inversion theorem,

$$u(x,y) = \frac{1}{\pi}\left(f * \frac{y}{(\cdot)^2+y^2}\right)(x)$$

$$= \boxed{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)\,y\,ds}{(x-s)^2+y^2}}$$

(b) $$u(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{y\,ds}{(x-s)^2+y^2}$$

Let $z = \frac{s-x}{y}$. Then $dz = \frac{ds}{y}$

$$u(x,y) = \frac{1}{\pi}\int_{\frac{-1-x}{y}}^{\frac{1-x}{y}}\frac{\cancel{y}\,dz}{\cancel{y}(z^2+1)}$$

$$= \frac{1}{\pi}\left[\text{Arctan}\left(\frac{1-x}{y}\right) - \text{Arctan}\left(\frac{-1-x}{y}\right)\right]$$

$$= \boxed{\frac{1}{\pi}\left[\text{Arctan}\left(\frac{1+x}{y}\right) + \text{Arctan}\left(\frac{1-x}{y}\right)\right]}$$

5. Solve $\nabla^2 u = 0$ in the unit cube C: $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, subject to the boundary conditions

$$u(x,y,0) = \sin(\pi x)\sin^3(\pi y) \quad \text{for } 0 \le x \le 1,\ 0 \le y \le 1,$$

nd $u = 0$ on the other five faces of the cube C.

$u(x,y,z) = X(x)Y(y)Z(z)$, $u_{xx}+u_{yy}+u_{zz}=0 \Rightarrow X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \text{constant} = \lambda \Rightarrow -\frac{Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda = \text{constant} = \mu$$

$\therefore$ $X''(x) + \lambda X(x) = 0$, $X(0) = X(1) = 0$

$Y''(y) + \mu Y(y) = 0$, $Y(0) = Y(1) = 0$

$Z''(z) - (\lambda+\mu)Z(z) = 0$, $Z(1) = 0$

B.C.'s on Faces $x=0, x=1 \Rightarrow$ $X(0)Y(y)Z(z) = 0$ and $X(1)Y(y)Z(z) = 0$ for $0 \le y \le 1$ and $0 \le z \le 1$.

Nontriviality of solution $u = XYZ \Rightarrow X(0) = X(1) = 0$.

B.C.'s on faces $y=0$, $y=1$, and $z=1$ similarly yield $Y(0) = Y(1) = 0 = Z(1)$.

Eigenvalues	Eigenfunctions	
$\lambda_l = (l\pi)^2$	$X_l(x) = \sin(l\pi x)$	$(l=1,2,3,\ldots)$
$\mu_m = (m\pi)^2$	$Y_m(y) = \sin(m\pi y)$	$(m=1,2,3,\ldots)$

Solution to the z-equation when $\lambda = \lambda_l$ and $\mu = \mu_m$: $Z_{l,m}(z) = A\cosh(\pi z\sqrt{l^2+m^2}) + B\sinh(\pi z\sqrt{l^2+m^2})$

$Z_{l,m}(1) = 0 \Rightarrow Z_{l,m}(z) = \sinh(\pi(1-z)\sqrt{l^2+m^2})$ (up to a constant multiple)

Formal solution to the homogeneous part of the problem:

$$u(x,y,z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{l,m}\sin(l\pi x)\sin(m\pi y)\sinh(\pi(1-z)\sqrt{l^2+m^2})$$

We want to choose the coefficients $B_{l,m}$ $(l=1,2,3,\ldots, m=1,2,3,\ldots)$ so the nonhomogeneous B.C. is met:

$$\sin(\pi x)\sin^3(\pi y) = u(x,y,0) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} B_{l,m}\sin(l\pi x)\sin(m\pi y)\sinh(\pi\sqrt{l^2+m^2}) \quad \begin{pmatrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{pmatrix}$$

$$\text{But } \sin^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3 = \frac{e^{3i\theta} - 3e^{2i\theta}\cdot e^{i\theta} + 3e^{i\theta}\cdot e^{-2i\theta} - e^{-3i\theta}}{-8i} = -\frac{1}{4}\underbrace{\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)}_{\sin(3\theta)} + \frac{3}{4}\underbrace{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}_{\sin(\theta)}$$

$$\therefore \quad \sin(\pi x)\sin^3(\pi y) = \frac{3}{4}\sin(\pi x)\sin(\pi y) - \frac{1}{4}\sin(\pi x)\sin(3\pi y).$$

Choose $B_{1,1}\sinh(\pi\sqrt{2}) = \frac{3}{4}$, $B_{1,3}\sinh(\pi\sqrt{10}) = -\frac{1}{4}$, and all other $B_{l,m} = 0$.

$$\therefore \boxed{u(x,y,z) = \frac{3}{4\sinh(\pi\sqrt{2})}\sin(\pi x)\sin(\pi y)\sinh(\pi(1-z)\sqrt{2}) - \frac{1}{4\sinh(\pi\sqrt{10})}\sin(\pi x)\sin(3\pi y)\sinh(\pi(1-z)\sqrt{10})}$$

6. (a) Solve $\nabla^2 u = 0$ in the unit disk $0 \le r < 1$ subject to the boundary condition $u(1;\theta) = 1$ if $0 < \theta < \pi$ and $u(1;\theta) = 0$ if $-\pi < \theta < 0$.

(b) What is the value of the solution to (a) at the center of the disk? Justify your answer.

(a) Poisson's Formula: $$u(r;\theta) = \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi} \frac{h(\varphi)\,d\varphi}{a^2 - 2ar\cos(\theta-\varphi) + r^2} \qquad \begin{pmatrix} 0 \le r < a \\ -\pi \le \theta < \pi \end{pmatrix}$$

In our case, $a = 1$ and $h(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } -\pi < \theta < 0. \end{cases}$

$$\therefore \boxed{u(r;\theta) = \frac{1-r^2}{2\pi}\int_0^{\pi} \frac{d\varphi}{1 - 2r\cos(\theta-\varphi) + r^2}} \qquad \begin{pmatrix} 0 \le r < 1 \\ -\pi \le \theta < \pi \end{pmatrix}$$

Using the integral formula $$\int \frac{d\psi}{A + B\cos(\psi)} = \frac{2}{\sqrt{A^2 - B^2}}\,\text{Arctan}\left(\frac{\tan(\psi/2)\sqrt{A^2-B^2}}{A+B}\right)$$

(which can be derived using the Weierstrass substitution $z = \tan(\frac{\psi}{2})$), we have

$$u(r;\theta) = \frac{1-r^2}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{d\psi}{1 + r^2 - 2r\cos(\psi)}$$

$$\begin{aligned} A+B &= 1 + r^2 - 2r = (1-r)^2 \\ A^2 - B^2 &= (1+r^2)^2 - 4r^2 \\ &= 1 - 2r^2 + r^4 \\ &= (1-r^2)^2 \end{aligned}$$

$$= \frac{1-r^2}{2\pi}\cdot\frac{2}{\sqrt{(1-r^2)^2}}\,\text{Arctan}\left[\frac{\tan(\psi/2)\sqrt{(1-r^2)^2}}{(1-r)^2}\right]\Bigg|_{\psi=-\theta}^{\pi-\theta}$$

$$= \frac{1}{\pi}\left\{\text{Arctan}\left[\frac{\tan(\frac{\pi-\theta}{2})\cancel{(1-r)}(1+r)}{(1-r)^{\cancel{2}}}\right] - \text{Arctan}\left[\frac{\tan(-\theta/2)\cancel{(1-r)}(1+r)}{(1-r)^{\cancel{2}}}\right]\right\}$$

$$= \boxed{\frac{1}{\pi}\left\{\text{Arctan}\left[\frac{\cot(\theta/2)(1+r)}{1-r}\right] + \text{Arctan}\left[\frac{\tan(\theta/2)(1+r)}{1-r}\right]\right\}}$$

(b) By the mean-value property for harmonic functions,

$$u(0;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(1;\varphi)\,d\varphi = \frac{1}{2\pi}\int_0^{\pi} 1\cdot d\varphi = \boxed{\frac{1}{2}}.$$

Alternate solution to #6.

(a) $$u(r;\theta) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{in\theta} \left(\frac{r}{a}\right)^{|n|}$$

In our case, $a=1$ and $h(\theta) = \begin{cases} 1 & \text{if } 0<\theta<\pi, \\ 0 & \text{if } -\pi<\theta<0. \end{cases}$

$$\hat{h}(n) = \frac{\langle h, e^{in(\cdot)}\rangle}{\langle e^{in(\cdot)}, e^{in(\cdot)}\rangle} = \frac{\int_{-\pi}^{\pi} h(\theta) e^{-in\theta}\, d\theta}{\int_{-\pi}^{\pi} |e^{in\theta}|^2\, d\theta} = \frac{1}{2\pi}\int_0^{\pi} 1\cdot e^{-in\theta}\, d\theta$$

$$\overset{(n\neq 0)}{=} \frac{1}{2\pi}\left(\frac{e^{-in\theta}}{-in}\right)\Bigg|_0^{\pi} = \frac{i}{2\pi n}\Big[\underbrace{e^{-in\pi}}_{(-1)^n} - 1\Big] = \begin{cases} 0 & \text{if } n=2k \text{ is even} \\ \dfrac{-i}{\pi(2k-1)} & \text{if } n=2k-1 \text{ is odd} \end{cases}$$

$$\hat{h}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{\pi} 1\, d\theta = \frac{1}{2}$$

$$u(r;\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \left[\hat{h}(n) e^{in\theta} + \hat{h}(-n) e^{-in\theta}\right]$$

$$= \boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} r^{2k-1} \frac{\sin((2k-1)\theta)}{2k-1}}$$

(b) $$\boxed{u(0;\theta) = \frac{1}{2}}$$

$\varphi'(0) = \varphi'(1) = 0$

7. (a) Show that the Fourier cosine series of the function ϕ given on $[0,1]$ by $\phi(x) = x^2(1-x)^2$ is

$$\frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)}{(m\pi)^4} .$$

$$\varphi(x) = x^2 - 2x^3 + x^4$$
$$\varphi'(x) = 2x - 6x^2 + 4x^3$$
$$\varphi''(x) = 2 - 12x + 12x^2$$

(b) Does the Fourier cosine series of ϕ converge to ϕ uniformly on $[0,1]$? Justify your answer.

$$\varphi'''(x) = -12 + 24x$$

(c) Solve

$$u_{tt} = u_{xx} \quad \text{for } 0 < x < 1,\ 0 < t < \infty, \qquad \varphi^{(4)}(x) = 24$$

subject to the homogeneous boundary/initial conditions

$$u_x(0,t) = 0 = u_x(1,t) \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \le t,\ 0 \le x \le 1,$$

and the nonhomogeneous initial condition

$$u(x,0) = x^2(1-x)^2 \quad \text{for } 0 \le x \le 1.$$

(d) Is your solution to (c) the only possible one? Give a complete justification of your answer.

(a)
$$\underset{n\ge 1}{a_n} = \frac{\langle \varphi, \cos(n\pi\cdot)\rangle}{\langle \cos(n\pi\cdot), \cos(n\pi\cdot)\rangle} = 2\int_0^1 \overbrace{\varphi(x)}^{v}\,\overbrace{\cos(n\pi x)dx}^{dV} = \cancelto{0}{2\varphi(x)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1} - \frac{2}{n\pi}\int_0^1 \overbrace{\varphi'(x)}^{v}\,\overbrace{\sin(n\pi x)dx}^{dV}$$

$$= \underbrace{\cancelto{0}{\frac{-2}{n\pi}\varphi'(x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1}}_{\text{Since } \varphi'(0)=\varphi'(1)=0} + \frac{-2}{(n\pi)^2}\int_0^1 \overbrace{\varphi''(x)}^{v}\,\overbrace{\cos(n\pi x)dx}^{dV} = \cancelto{0}{\frac{-2}{(n\pi)^2}\varphi''(x)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1} + \frac{2}{(n\pi)^3}\int_0^1 \overbrace{\varphi'''(x)}^{v}\,\overbrace{\sin(n\pi x)dx}^{dV}$$

$$= \frac{2}{(n\pi)^3}(24x-12)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1 + \cancelto{0}{\frac{2}{(n\pi)^4}\int_0^1 24\cos(n\pi x)dx} = \frac{24}{(n\pi)^4}\left[(-1)^{n+1}-1\right] = \begin{cases} \dfrac{-48}{(n\pi)^4} & \text{if } n=2m \text{ is even} \\ 0 & \text{if } n=2m+1 \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{-3}{(m\pi)^4} & \text{if } n=2m \text{ is even,} \\ 0 & \text{if } n=2m+1 \text{ is odd.} \end{cases} \qquad a_0 = \frac{\langle\varphi,1\rangle}{\langle 1,1\rangle} = \int_0^1 \varphi(x)dx = \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\Big|_0^1 = \frac{1}{30}$$

The Fourier cosine series of φ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \boxed{\frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)}{(m\pi)^4}} .$$

(b) Since φ, φ', and φ'' are continuous on $[0,1]$, and φ satisfies the Neumann B.C.'s that generate the orthogonal set $\{\cos(n\pi x)\}_{n=0}^{\infty}$ (i.e. $\varphi'(0) = \varphi'(1) = 0$), it follows that the Fourier cosine series of φ converges uniformly to φ on $[0,1]$.

(c) $u(x,t) = X(x)T(t) \Rightarrow X(x)T''(t) = X''(x)T(t) \Rightarrow -\dfrac{X''(x)}{X(x)} = -\dfrac{T''(t)}{T(t)} = \text{const.} = \lambda$

$$= \int_0^1 \overbrace{w_t(x,t)}^{v} \overbrace{w_{xx}(x,t)\,dx}^{dV} + \int_0^1 w_x(x,t)\,w_{tx}(x,t)\,dx$$

$$= \Big[w_t(1,t)\overbrace{w_x(1,t)}^{0} - w_t(0,t)\overbrace{w_x(0,t)}^{0} \Big] - \int_0^1 w_x(x,t)\cancel{w_{tx}}(x,t)\,dx + \int_0^1 w_x(x,t)\cancel{w_{tx}}(x,t)\,dx$$

$$= 0$$

so $E(t) = \text{constant}$ on $[0,\infty)$. But $w(x,0) = 0$ for all $0 \le x \le 1$ implies $w_x(x,0) = 0$.

$$\therefore \quad E(0) = \int_0^1 \Big[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0) \Big] dx = 0,$$

so $E(t) = 0$ for all $0 \le t < \infty$. But the integrand of E is continuous and nonnegative so $\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) = 0$ for all $0 \le x \le 1$ and all $0 \le t < \infty$.

Consequently $w_t(x,t) = w_x(x,t) = 0$ so $w(x,t) = \text{constant}$ on $0 \le x \le 1$, $0 \le t < \infty$.

However the I.C. $w(x,0) = 0$ $(0 \le x \le 1)$ implies $w(x,t) = 0$ on $0 \le x \le 1$, $0 \le t < \infty$,

i.e.

$$u_2(x,t) = \frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)\cos(2\pi m t)}{(m\pi)^4}$$

for all $0 \le x \le 1$ and $0 \le t < \infty$.

nontriviality of solns.

$\therefore \quad X''(x) + \lambda X(x) = 0, \; X'(0) = X'(1) = 0$

$T''(t) + \lambda T(t) = 0, \; T'(0) = 0$

$\left\{\begin{array}{l} u_x(0,t) = 0 \Rightarrow X'(0)T(t) = 0 \overset{\checkmark}{\Rightarrow} X'(0) = 0 \\ \text{similarly } u_x(1,t) = 0 \Rightarrow X'(1) = 0 \\ \text{and } \; u_t(x,0) = 0 \Rightarrow T'(0) = 0 \end{array}\right.$

Eigenvalues: Eigenfunctions

$\lambda_n = (n\pi)^2 \qquad X_n(x) = \cos(n\pi x) \qquad (n = 0, 1, 2, \ldots)$

Solution to the t-problem corresponding to $\lambda = \lambda_n$: $T_n(t) = \cos(n\pi t)$

$\therefore \; u(x,t) = \sum_{n=0}^{\infty} a_n \cos(n\pi x)\cos(n\pi t)$ is a formal solution to the homogeneous part of the problem. We want to choose the coefficients a_n $(n=0,1,2,\ldots)$ so the nonhomogeneous I.C. is met:

$$x^2(1-x)^2 = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{for } 0 \le x \le 1.$$

By part (a), $a_0 = \frac{1}{30}$ and $a_{2m} = \frac{-3}{(m\pi)^4}$, $a_{2m-1} = 0$ $(m = 1, 2, 3, \ldots)$.

$$\therefore \boxed{u(x,t) = \frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)\cos(2m\pi t)}{(m\pi)^4}}.$$

(d) The solution in (c) is $\boxed{\text{unique}}$, for suppose $u = u_2(x,t)$ is another solution. Then

$$w(x,t) = \frac{1}{30} + \sum_{m=1}^{\infty} \frac{-3\cos(2m\pi x)\cos(2m\pi t)}{(m\pi)^4} - u_2(x,t)$$

solves

$$w_{tt} = w_{xx} \qquad (0 < x < 1, \; 0 < t < \infty),$$

$$w_x(0,t) = 0 = w_x(1,t) \qquad (0 \le t < \infty),$$

$$w(x,0) = 0 = w_t(x,0) \qquad (0 \le x \le 1).$$

Let $E(t) = \int_0^1 \left[\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t)\right] dx \qquad (0 \le t < \infty).$

Then

$$E'(t) = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t)\right] dx$$

$$= \int_0^1 w_t(x,t)\, w_{tt}(x,t)\, dx + \int_0^1 w_x(x,t)\, w_{tx}(x,t)\, dx$$