Mathematics 315 | Final Exam, Part II Spring 2007 | Name: Dr. Grow (1 pt.)

This portion of the 200-point final examination is open book/notes. You are to solve three problems of your choosing, subject to the constraint that **at least one problem must be chosen from Group 1** and **at least one problem must be chosen from Group 2.**

Group 1.

1.(36 pts.) Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$ denote the (infinite) sequence of prime numbers, let $\pi(x)$ denote the number of primes less than or equal to x, and let $f(x) = \frac{1}{x}$ for $x > 0$.

(a) Show that $\int_1^x f(t)\,d(\pi(t)) = \sum_{p_k \le x} \frac{1}{p_k}$ for $x > 2$.

(b) Show that $\int_1^x f(t)\,d(\pi(t)) = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\,dt$ for $x > 2$.

(c) Use (a) and (b) to verify that

$$\sum_{p_k \le x} \frac{1}{p_k} = \ln(\ln(x)) + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2}\,dt + \frac{\pi(x)}{x} \quad \text{for } x > 2.$$

Assume that to each $a > 0$ there correspond real constants $B = B(a) > 1$ and $C = C(a) > 0$ such that

$$(*) \qquad \left|\pi(x) - \frac{x}{\ln(x)}\right| \le Cxe^{-a\sqrt{\ln(x)}} \quad \text{for } x \ge B.$$

(d) Use (*) to help show that $\lim_{y > x \to \infty} \left|\int_x^y \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}\right| = 0.$

(e) Why does the improper Riemann integral $\int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}$ converge?

(f) Use (c) and (*) to help show that

$$\lim_{x \to \infty}\left(\sum_{p_k \le x} \frac{1}{p_k} - \ln(\ln(x))\right) = \int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2}\,dt.$$

2.(36 pts.) Let F be a continuous real function on the unit cube

$$\mathbf{Q} = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

in $\mathbb{R}^3$. Show that to each $\varepsilon > 0$ there corresponds a positive integer N and a finite collection $f_1, g_1, h_1, \ldots, f_N, g_N, h_N$ of real polynomials on the unit interval $[0,1]$ such that

$$\left|F(x,y,z) - \sum_{k=1}^N f_k(x)g_k(y)h_k(z)\right| < \varepsilon$$

for all (x,y,z) in $\mathbf{Q}$.

3.(36 pts.) Consider the 2π-periodic function f determined by $f(x) = \frac{\pi - x}{2}$ if $-\pi \le x < \pi$.

(a) Compute the Fourier series of f with respect to the orthogonal set of functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ on the interval $[-\pi,\pi]$.

(b) For each x in the interval $[-\pi,\pi]$, discuss the pointwise convergence (or lack thereof) of the Fourier series of f at x to $f(x)$.

(c) Discuss the uniform convergence (or lack thereof) of the Fourier series of f to the function f on $[-\pi,\pi]$.

(d) Discuss the L^2 – convergence (or lack thereof) of the Fourier series of f to the function f on $[-\pi,\pi]$.

(e) Use the preceding to help compute the sums of $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

Group 2.

4.(36 pts.) Let $\langle a_n\rangle_{n=1}^{\infty}$ be a positive divergent sequence, and for every positive integer n let

$$f_n(x)=\begin{cases} a_n & \text{if } x\in\left(\frac{1}{n+1},\frac{1}{n}\right), \\ 0 & \text{otherwise in } (0,1). \end{cases}$$

(a) If $\left\langle \frac{a_n}{n^2}\right\rangle_{n=1}^{\infty}$ is a bounded sequence, show that $\left\langle \int_0^1 f_n dx\right\rangle_{n=1}^{\infty}$ is a bounded sequence.

(b) Place an X in each blank below that would imply

$$\lim_{n\to\infty}\int_0^1 f_n dx=\int_0^1 \lim_{n\to\infty} f_n dx$$

and an O in each blank otherwise. Supply reasons for your answers.

(i) ________ $\left\langle \frac{a_n}{\ln(n)}\right\rangle_{n=2}^{\infty}$ is a bounded sequence.

(ii) ________ $\lim_{n\to\infty}\frac{a_n}{n^3\ln\left(1+\frac{1}{\sqrt{n}}\right)}=0.$

(iii) ________ $\lim_{n\to\infty}\frac{a_n}{n^2}=0.$

(iv) ________ $\left\langle \frac{a_n}{n^2\ln(n)}\right\rangle_{n=2}^{\infty}$ is a bounded sequence.

5.(36 pts.) In this problem you may assume that the Riemann-Lebesgue Lemma holds for functions in $L^1(-\pi,\pi)$: If $f\in L^1(-\pi,\pi)$ then $\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)dx=0=\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$

Let $\langle n_k \rangle_{k=1}^{\infty}$ be an increasing sequence of positive integers, let E be the set of all x in $(-\pi, \pi)$ for which $\langle \sin(n_k x) \rangle_{k=1}^{\infty}$ is a convergent sequence, and let A be any measurable subset of E.

(a) Show that $\lim\limits_{k\to\infty} \int_A \sin(n_k x)dx = 0$.

(b) Show that $\lim\limits_{k\to\infty} 2\int_A (\sin(n_k x))^2\, dx = \lim\limits_{k\to\infty} \int_A (1-\cos(2n_k x))\,dx = m(A)$.

(c) Use (a) and (b) to help show that $m(E) = 0$.

6.(36 pts.) Let f be a bounded measurable function on $[0,1]$ and define

$$F(x) = \int_0^x f(t)dt \quad \text{for } x \text{ in } [0,1].$$

(a) Show that F is continuous on $[0,1]$.

(b) Show that F is of bounded variation on $[0,1]$.

Assume that Lebesgue's theorem for differentiation of monotone functions holds: If g is increasing on (a,b) then $g'(x)$ exists a.e. in (a,b).

(c) Why does $F'(x)$ exist a.e. in $(0,1)$?

(d) Use (c) to help show that $\int_0^y \{F'(t) - f(t)\}\, dt = 0$ for all y in $[0,1]$.

(e) Use (c) and (d) to help show that $F'(x) = f(x)$ a.e. in $[0,1]$.

6 pts.

#1. (a) Let $H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$ denote the Heaviside unit step function.

Then $\pi(t) = \sum_{p_k \leq t} H(t - p_k)$ for $t > 0$. Since $f(t) = \frac{1}{t}$ is continuous on $(0, \infty)$, Theorem 6.16 in Rudin implies that

$$\int_1^x f(t)\,d(\pi(t)) = \sum_{p_k \leq x} f(p_k) = \sum_{p_k \leq x} \frac{1}{p_k} \quad \text{for } x > 0.$$

6 pts. (b) Note that f is continuous and of bounded variation on each closed, bounded subinterval of $(0, \infty)$ and π is increasing, and hence of bounded variation, on each closed, bounded subinterval of $(0, \infty)$. Therefore integration-by-parts for Riemann-Stieltjes integrals and Theorem 6.17 in Rudin imply

$$\int_1^x f(t)\,d(\pi(t)) = f(x)\pi(x) - f(1)\pi(1) - \int_1^x \pi(t)\,d(f(t))$$

$$= \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\,dt \qquad \text{for } x > 0.$$

6 pts (c) By parts (a) and (b),

$$(\dagger) \quad \sum_{p_k \leq x} \frac{1}{p_k} = \int_1^x f(t)\,d(\pi(t)) = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\,dt \qquad \text{for } x > 0.$$

For $x > 1$, we have

$$(\dagger\dagger) \quad \int_1^x \frac{\pi(t)}{t^2}\,dt = \int_1^e \frac{\pi(t)}{t^2}\,dt + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_e^x \frac{dt}{t\ln(t)} \quad \left\{ \begin{array}{l} \text{Let } u = \ln(t). \\ \text{Then} \\ du = \frac{1}{t}dt \end{array} \right.$$

$$= \int_1^e \frac{\pi(t)}{t^2}\,dt + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^{\ln(x)} \frac{1}{u}\,du$$

$$= \int_1^e \frac{\pi(t)}{t^2}\,dt + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \ln(\ln(x)).$$

Substituting from (††) into (†) yields the desired identity for $x > 1$:

$$\sum_{p_k \le x} \frac{1}{p_k} = \ln(\ln(x)) + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2}\,dt + \frac{\pi(x)}{x}.$$

(d) Fix $a > 0$ and choose constants $B = B(a) > 1$ and $C = C(a) > 0$ such that (∗) holds. For all $y > x \ge B$ we have

$$\left|\int_x^y \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}\right| \le \int_x^y \left|\pi(t) - \frac{t}{\ln(t)}\right|\frac{dt}{t^2}$$

$$\le \int_x^y \frac{C e^{-a\sqrt{\ln(t)}}}{t}\,dt \longleftarrow \left\{\begin{array}{l} \text{Let } v^2 = \ln(t) \\ \text{Then } 2v\,dv = \frac{1}{t}dt \end{array}\right.$$

$$= \int_{\sqrt{\ln(x)}}^{\sqrt{\ln(y)}} 2Cve^{-av}\,dv \longleftarrow \left\{\begin{array}{l} \text{Let } U = 2Cv \text{ and } dV = e^{-av}dv \\ \text{Then } dU = 2C\,dv \text{ and } V = \frac{e^{-av}}{-a} \end{array}\right.$$

$$= -\frac{2Cv}{a}e^{-av}\Big|_{\sqrt{\ln(x)}}^{\sqrt{\ln(y)}} + \int_{\sqrt{\ln(x)}}^{\sqrt{\ln(y)}} \frac{2C}{a}e^{-av}\,dv$$

$$= \frac{2C}{a}\left(\sqrt{\ln(x)}\,e^{-a\sqrt{\ln(x)}} - \sqrt{\ln(y)}\,e^{-a\sqrt{\ln(y)}}\right) + \frac{2C}{a^2}\left(e^{-a\sqrt{\ln(x)}} - e^{-a\sqrt{\ln(y)}}\right).$$

Since $te^{-at} \to 0$ and $e^{-at} \to 0$ as $t \to \infty$, it follows from the above estimate that

$$\lim_{y > x \to \infty} \left|\int_x^y \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}\right| = 0.$$

6 pts (e) Let $\langle x_n \rangle_{n=1}^{\infty}$ be any sequence of real numbers that tends to ∞ with n. Then part (d) shows that the sequence of numbers

$\left\langle \int_e^{x_n} \left(\pi(t) - \frac{t}{\ln(t)} \right) \frac{dt}{t^2} \right\rangle_{n=1}^{\infty}$ is Cauchy, and hence is convergent.

It follows that the improper Riemann integral

$$\int_e^{\infty} \left(\pi(t) - \frac{t}{\ln(t)} \right) \frac{dt}{t^2} = \lim_{x \to \infty} \int_e^{x} \left(\pi(t) - \frac{t}{\ln(t)} \right) \frac{dt}{t^2}$$

is convergent.

6 pts (f) From (*) we see that

$$\left| \frac{\pi(x)}{x} \right| \leq \left| \frac{\pi(x)}{x} - \frac{1}{\ln(x)} \right| + \left| \frac{1}{\ln(x)} \right|$$

$$\leq C e^{-a\sqrt{\ln(x)}} + \frac{1}{\ln(x)} \longrightarrow 0 \quad \text{as } x \to \infty.$$

Therefore, (c) and (e) imply that

$$\lim_{x \to \infty} \left(\sum_{p_k \leq x} \frac{1}{p_k} - \ln(\ln(x)) \right) = \lim_{x \to \infty} \left(\int_e^{x} \left(\pi(t) - \frac{t}{\ln(t)} \right) \frac{dt}{t^2} + \int_1^{e} \frac{\pi(t)}{t^2} dt + \frac{\pi(x)}{x} \right)$$

$$= \int_e^{\infty} \left(\pi(t) - \frac{t}{\ln(t)} \right) \frac{dt}{t^2} + \int_1^{e} \frac{\pi(t)}{t^2} dt.$$

#2. Let $\mathcal{Q}$ be the family of all functions φ on Q of the form

$$(*) \qquad \varphi(x,y,z) = \sum_{k=1}^{N} f_k(x) g_k(y) h_k(z)$$

where N is some positive integer and $f_1, g_1, h_1, \ldots, f_N, g_N, h_N$ is a (finite) collection of real polynomials on the unit interval $[0,1]$. Clearly $\mathcal{Q}$ is an algebra of real continuous functions on the compact metric space Q. Since $1 \in \mathcal{Q}$, the algebra $\mathcal{Q}$ vanishes at no point of Q. Also, if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two distinct points of Q, then the function

$$\varphi(x,y,z) = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$$

belongs to $\mathcal{Q}$ and satisfies $\varphi(x_1, y_1, z_1) = 0 \neq \varphi(x_2, y_2, z_2)$. Thus $\mathcal{Q}$ separates points on Q. The Stone-Weierstrass Theorem (7.32 in Rudin) then implies that $\mathcal{Q}$ is uniformly dense in $C(Q)$. That is, to each continuous real function F on Q and each $\varepsilon > 0$, there corresponds a function φ of the form $(*)$ in $\mathcal{Q}$ such that

$$|F(x,y,z) - \varphi(x,y,z)| < \varepsilon$$

for all (x,y,z) in Q.

#3. (a) $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\pi - x}{2}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi}{2}\,dx = \frac{\pi}{2}.$

If $n \neq 0$ then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi-x}{2}\right)\cos(nx)\,dx - \frac{i}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi-x}{2}\right)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2}\right)\cancelto{0}{\int_{-\pi}^{\pi}\cos(nx)\,dx} - \frac{1}{4\pi}\cancelto{0}{\int_{-\pi}^{\pi}\underbrace{x\cos(nx)}_{\text{odd}}\,dx} - \frac{i}{2\pi}\left(\frac{\pi}{2}\right)\cancelto{0}{\int_{-\pi}^{\pi}\sin(nx)\,dx} + \frac{i}{4\pi}\int_{-\pi}^{\pi}\overbrace{x\sin(nx)}^{\text{even}}\,dx$$

$$= \frac{i}{2\pi}\int_0^{\pi}\underbrace{x}_{u}\,\underbrace{\sin(nx)\,dx}_{dv} = \frac{i}{2\pi}\left[\frac{-x\cos(nx)}{n}\Big|_0^{\pi} - \cancelto{0}{\int_0^{\pi}\frac{-\cos(nx)}{n}\,dx}\right] = \frac{-i\cos(n\pi)}{2n} = \frac{i(-1)^{n+1}}{2n}$$

Therefore the Fourier series of f is

$$\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx} = \frac{\pi}{2} + \frac{i}{2}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}e^{inx} + \frac{(-1)^{-n+1}}{-n}e^{-inx}\right)$$

$$= \frac{\pi}{2} + \frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big(\overbrace{e^{inx} - e^{-inx}}^{2i\sin(nx)}\Big)$$

7 pts.

$$= \boxed{\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n\sin(nx)}{n}}.$$

(b) Since f is 2π-periodic and of bounded variation on $[-\pi,\pi]$, the Dirichlet-Jordan theorem (Rudin, p. 200, #17) implies

$$\lim_{N\to\infty} S_N(f;x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

for all x in $\mathbb{R}$. Since f is continuous if $-\pi < x < \pi$ and satisfies $f(\pi^+) = f(-\pi^+) = \pi$, $f(-\pi^-) = f(\pi^-) = 0$, it follows that

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n} = \begin{cases} \pi/2 & \text{if} \quad x = -\pi, \\ \frac{\pi - x}{2} & \text{if} \quad -\pi < x < \pi, \\ \pi/2 & \text{if} \quad x = \pi. \end{cases}$$

7 pts. In particular, the Fourier series of f converges to $f(x)$ for all $-\pi < x < \pi$, but the Fourier series of does not converge to $f(-\pi)$ nor $f(\pi)$ at the endpoints $x = \pm\pi$.

7 pts. (c) Since the Fourier series of f does not converge pointwise to f on $[-\pi, \pi]$, it follows that the Fourier series of f does not converge uniformly to f on $[-\pi, \pi]$.

7 pts (d) Since f is 2π-periodic and Riemann integrable on $[-\pi, \pi]$, the Fourier series of f converges in the L^2-sense to f on $[-\pi, \pi]$ (see Rudin, Theorem 8.16).

(e) By part (c),

$$\frac{\pi}{4} = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi/2)}{n}.$$

But $\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ (-1)^k & \text{if } n = 2k+1 \text{ is odd,} \end{cases}$ so

$$\frac{\pi}{4} = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} \cdot (-1)^k}{2k+1} = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and hence $\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}}$. + 4 pts.

By Parseval's Theorem (Rudin, Theorem 8.16),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi-x}{2}\right)^2 dx = \left|\frac{\pi}{2}\right|^2 + \sum_{n=1}^{\infty}\left(\left|\frac{i(-1)^{n+1}}{2n}\right|^2 + \left|\frac{i(-1)^{-n+1}}{2(-n)}\right|^2\right)$$

$$\Rightarrow \quad \frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \quad \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}} \, .$$

+4 pts.

7 pts.

#4. (a) Suppose $0 \le \dfrac{a_n}{n^2} \le M$ for some real number M and all integers $n \ge 1$. Then

$$0 \le \int_0^1 f_n\,dx = \int_{1/(n+1)}^{1/n} a_n\,dx = a_n\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{a_n}{n(n+1)} \le \frac{a_n}{n^2} \le M$$

for all $n \ge 1$ so $\left\langle \int_0^1 f_n\,dx \right\rangle_{n=1}^{\infty}$ is a bounded sequence.

(b) Clearly $\lim\limits_{n\to\infty} f_n(x) = 0$ for each x in $(0,1)$ so $\int_0^1 \lim\limits_{n\to\infty} f_n\,dx = \int_0^1 0\,dx = 0$

Note also that the computation in part (a) shows that $\int_0^1 f_n\,dx = \dfrac{a_n}{n(n+1)}$ for all $n \ge 1$. Therefore $\lim\limits_{n\to\infty} \int_0^1 f_n\,dx = \int_0^1 (\lim\limits_{n\to\infty} f_n)\,dx$ if and only if $\lim\limits_{n\to\infty} \dfrac{a_n}{n(n+1)} = 0$

(i) Suppose that there is a real number M such that $0 \le \dfrac{a_n}{\ln(n)} \le M$ for all $n \ge 2$. Then

$$0 \le \frac{a_n}{n(n+1)} = \frac{a_n}{\ln(n)} \cdot \frac{\ln(n)}{n(n+1)} \le \frac{M\ln(n)}{n(n+1)} \to 0 \quad \text{as} \quad n \to \infty,$$

so the "X" in the blank is justified.

8 pts. (2+6)

(ii) It is easy to see (using concavity of $x \mapsto \ln(1+x)$) that $\frac{x}{2} \le \ln(1+x) \le x$ for all $0 \le x \le 2$, so $\dfrac{1}{2\sqrt{n}} \le \ln\left(1 + \dfrac{1}{\sqrt{n}}\right) \le \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$. Therefore we take $a_n = n^2$ for $n \ge 1$ and note that

$$0 \le \frac{a_n}{n^3 \ln\left(1 + \frac{1}{\sqrt{n}}\right)} = \frac{1}{n \ln\left(1 + \frac{1}{\sqrt{n}}\right)} \le \frac{2}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

However $\dfrac{a_n}{n(n+1)} = \dfrac{n}{n+1} \to 1$ as $n \to \infty$, so the "O" in the blank is justified.

7 pts (iii) Suppose $\lim_{n\to\infty} \frac{a_n}{n^2} = 0$. Then
(2+5)

$$0 \le \frac{a_n}{n(n+1)} \le \frac{a_n}{n^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

so the "X" in the blank is justified.

7 pts (iv) We may take $a_n = n^2 \ln(n)$ for $n \ge 1$ and note that
(2+5)

$$\frac{a_n}{n^2 \ln(n)} = 1 \quad \text{is bounded and} \quad \frac{a_n}{n(n+1)} = \frac{n \ln(n)}{n+1} \longrightarrow \infty$$

as $n \to \infty$. Therefore the "O" in the blank is justified.

10 pts.

#5. (a) Note that $f(x) = \chi_A(x)$ is a measurable function on $(-\pi,\pi)$ with $\int_{-\pi}^{\pi} |f(x)|\,dx = \int_{-\pi}^{\pi} \chi_A(x)\,dx = m(A) < \infty$ so $f \in L^1(-\pi,\pi)$. By the Riemann-Lebesgue Lemma,

$$0 = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\sin(n_k x)\,dx = \lim_{k\to\infty} \int_A \sin(n_k x)\,dx .$$

10 pts. → (b) By the Riemann-Lebesgue Lemma,

$$(\dagger) \qquad 0 = \lim_{k\to\infty} \int_{-\pi}^{\pi} \chi_A(x)\cos(2n_k x)\,dx = \lim_{k\to\infty} \int_A \cos(2n_k x)\,dx .$$

Using the identity $2\sin^2(\theta) = 1-\cos(2\theta)$ and $(\dagger)$, we have

$$\lim_{k\to\infty} 2\int_A [\sin(n_k x)]^2\,dx = \lim_{k\to\infty} \int_A [1-\cos(2n_k x)]\,dx = \int_A 1\,dx - \lim_{k\to\infty}\int_A \cos(2n_k x)\,dx = m(A).$$

16 pts. → (c) The assumptions of this problem imply that $f(x) = \lim_{k\to\infty} \sin(n_k x)$ exists for all x in E. Let $E^+ = \{x \in E : f(x) \geq 0\}$ and $E^- = E \setminus E^+$. Then E^+ and E^- are measurable subsets of the measurable set $E \subseteq (-\pi,\pi)$ with $E^+ \cap E^- = \phi$ and $E^+ \cup E^- = E$. By part (b) and the Dominated Convergence Theorem,

$$(*) \quad \frac{1}{2}m(E^+) = \lim_{k\to\infty} \int_{E^+} [\sin(n_k x)]^2\,dx = \int_{E^+} \lim_{k\to\infty} [\sin(n_k x)]^2\,dx = \int_{E^+} f^2(x)\,dx .$$

But $f : E \to [-1,1]$ so $f^2(x) \leq |f(x)|$ on E. Also $|f(x)| = f(x)$ on E^+, so

$$(**) \quad \int_{E^+} f^2(x)\,dx \le \int_{E^+} f(x)\,dx .$$

On the other hand, the Dominated Convergence Theorem and part (a) imply

$$(***) \quad \int_{E^+} f(x)\,dx = \lim_{k\to\infty} \int_{E^+} \sin(n_k x)\,dx = 0 .$$

Combining (*), (**), and (***) yields $\frac{1}{2}m(E^+) = 0$.

A similar argument shows that $\frac{1}{2}m(E^-) = 0$. But then,

$\frac{1}{2}m(E) = \frac{1}{2}m(E^+) + \frac{1}{2}m(E^-) = 0$. Q.E.D.

#6. 7 pts (a) Let $x \in [0,1]$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of points in $[0,1]$ converging to x. Then the Dominated Convergence Theorem implies

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_0^1 \chi_{[0,x_n]}(t) f(t)\,dt = \int_0^1 \chi_{[0,x]}(t) f(t)\,dt = F(x).$$

Thus F is continuous on $[0,1]$.

7 pts → (b) Let $0 = x_0 < x_1 < \dots < x_n = 1$ be a partition of $[0,1]$. Then

$$\begin{aligned}
\sum_{k=1}^{n} \left| F(x_k) - F(x_{k-1}) \right| &= \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f(t)\,dt \right| \\
&\le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t)|\,dt \\
&= \int_0^1 |f(t)|\,dt .
\end{aligned}$$

Therefore $\operatorname{Var}(F;0,1) \le \int_0^1 |f(t)|\,dt < \infty$ so $F \in BV[0,1]$.

7 pts → (c) Because $F \in BV[0,1]$, Jordan's Theorem guarantees the existence of real increasing functions F_1 and F_2 on $[0,1]$ such that $F = F_1 - F_2$. By Lebesgue's Theorem, $F_1'(x)$ and $F_2'(x)$ exist a.e. on $[0,1]$, and consequently $F'(x) = F_1'(x) - F_2'(x)$ exists a.e. on $[0,1]$.

5 pts → (d) Suppose that $|f(t)| \le M < \infty$ for all t in $[0,1]$. Fix y in $(0,1)$ and let $\langle h_n \rangle_{n=1}^{\infty}$ be a sequence of nonzero real numbers which converges

to zero. Then $\left|F(t+h_n)-F(t)\right| = \left|\int_t^{t+h_n} f(s)\,ds\right| \le M|h_n|$ so

$$\left|\frac{F(t+h_n)-F(t)}{h_n}\right| \le M$$ for all $t\in(0,1)$ and $n\ge 1$. Furthermore,

$$F'(t) = \lim_{n\to\infty} \frac{F(t+h_n)-F(t)}{h_n}$$ exists a.e. in $(0,1)$, so the

Dominated Convergence Theorem yields

$$(\square) \qquad \lim_{n\to\infty}\int_0^y \frac{F(t+h_n)-F(t)}{h_n}\,dt = \int_0^y F'(t)\,dt.$$

On the other hand,

$$(\square\square) \qquad \lim_{n\to\infty}\int_0^y \frac{F(t+h_n)-F(t)}{h_n}\,dt = \lim_{n\to\infty}\left(\frac{1}{h_n}\int_0^y F(t+h_n)\,dt - \frac{1}{h_n}\int_0^y F(t)\,dt\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{h_n}\int_{h_n}^{y+h_n} F(x)\,dx - \frac{1}{h_n}\int_0^y F(x)\,dx\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{h_n}\int_y^{y+h_n} F(x)\,dx - \frac{1}{h_n}\int_0^{h_n} F(x)\,dx\right)$$

$$= F(y) - F(0)$$

because F is continuous on $[0,1]$ (see Rudin, Theorem 6.20). But

$$(\square\square\square) \qquad F(y)-F(0) = \int_0^y f(t)\,dt.$$

Applying (□), (□□), and (□□□) and rearranging yields the desired result:

$$\int_0^y \{F'(t) - f(t)\}\,dt = 0 \quad \text{for} \quad y \text{ in } (0,1).$$

7 pts → (e) Let $E_+ = \{t \in [0,1] : F'(t) - f(t) > 0\}$ and let $\varepsilon > 0$. By Littlewood's first principle, there exists a finite collection $I_1 = (a_1, b_1), \dots, I_N = (a_N, b_N)$ of open intervals in $[0,1]$ such that $m\left(\bigcup_{k=1}^N I_k \,\triangle\, E_+\right) < \varepsilon$. Without loss of generality we may assume $I_j \cap I_k = \phi$ for $j \neq k$. By part (d),

$$\int_{I_k} \{F'(t) - f(t)\}\,dt = \int_{a_k}^{b_k} \{F'(t) - f(t)\}\,dt = \int_0^{b_k} \{F'(t) - f(t)\}\,dt - \int_0^{a_k} \{F'(t) - f(t)\}\,dt$$

$$= 0.$$

Therefore $$\int_{\bigcup_{k=1}^N I_k} \{F'(t) - f(t)\}\,dt = \sum_{k=1}^N \int_{I_k} \{F'(t) - f(t)\}\,dt = 0.$$

Consequently,

$$\int_{E_+} \{F'(t) - f(t)\}\,dt = \int_{E_+ \setminus \bigcup_1^N I_k} \{F'(t) - f(t)\}\,dt + \int_{E_+ \cap \bigcup_1^N I_k} \{F'(t) - f(t)\}\,dt$$

$$= \int_{E_+ \setminus \bigcup_1^N I_k} \{F'(t) - f(t)\}\,dt + \int_{\bigcup_1^N I_k} \{F'(t) - f(t)\}\,dt - \int_{\bigcup_1^N I_k \setminus E_+} \{F'(t) - f(t)\}\,dt$$

$$= \int_{E_+ \setminus \bigcup_1^N I_k} \{F'(t) - f(t)\}\,dt - \int_{\bigcup_1^N I_k \setminus E_+} \{F'(t) - f(t)\}\,dt.$$

The proof of part (d) shows that if $|f(t)| \le M$ in $[0,1]$ then $|F'(t)| \le M$ a.e. in $[0,1]$. Thus

$$0 \le \int_{E_+} \{F'(t) - f(t)\}\,dt \le \int_{E_+ \setminus \bigcup_1^N I_k} |F'(t) - f(t)|\,dt + \int_{\bigcup_1^N I_k \setminus E_+} |F'(t) - f(t)|\,dt$$

$$\le 2M\,m\Big(E_+ \setminus \bigcup_1^N I_k\Big) + 2M\,m\Big(\bigcup_1^N I_k \setminus E_+\Big)$$

$$< 2M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_{E_+} \{F'(t) - f(t)\}\,dt = 0$. But $F'(t) - f(t) > 0$ on E_+ so it follows that $m(E_+) = 0$. A similar argument shows that $E_- = \{t \in [0,1] : F'(t) - f(t) < 0\}$ has measure zero. Therefore, $F'(t) - f(t) = 0$ a.e. in $[0,1]$.