(a) Find a solution to
\[ u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1, \ 0 < t < \infty, \]
subject to
\[ u_x(0,t) = 0 = u_x(1,t) \quad \text{for} \quad t \geq 0, \]
and
\[ u(x,0) = \cos^2(\pi x), \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq 1. \]

(b) Use the energy method to show that there is only one solution to the problem in part (a).
HW 14: (a) \[ u(x,t) = X(x)T(t) \] in the homogeneous portion of the problem leads to
\[
\begin{cases}
X''(x) + \lambda X(x) = 0, & X(0) = 0 = X'(1), \\
T''(t) + \lambda T(t) = 0, & T'(0) = 0.
\end{cases}
\]
The eigenvalues are \( \lambda_n = (n\pi)^2 \) and the eigenfunctions are \( X_n(x) = \cos(n\pi x) \) \( (n=0,1,2,...) \).
The solution to the \( \tau \)-problem is \( T_\tau(t) = \cos(n\pi \tau t) \) \( (n=0,1,2,...) \). Hence
\[
u(x,t) = \sum_{n=0}^{N} a_n \cos(n\pi x) \cos(n\pi \tau t) \]
solves the homogeneous portion of the problem for any \( N \geq 1 \) and any constants \( a_0, a_1, ..., a_N \).

\( \frac{1}{2} + \frac{1}{2} \cos(2\pi x) = \cos^2(\pi x) \leq u(x,0) = \sum_{n=0}^{N} a_n \cos(n\pi x) \) for \( 0 \leq x \leq 1 \) \( \Rightarrow a_0 = \frac{1}{2}, a_1 = \frac{1}{2}, \) and all other \( a_n = 0 \).

(b) Let \( v = v(x,t) \) be any other solution to the problem in (a) and consider the energy function
\[ E(t) = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] \, dx \]
of the difference \( w(x,t) = u(x,t) - v(x,t) \).
Note that \( w \) solves \( w_{tt} - w_{xx} = 0 \) in \( 0 < x < 1, \ t \geq 0 \), \( w_x(0,t) = w_x(1,t) \) for \( t \geq 0 \).
\( w(x,0) = 0 \) \( \Rightarrow w_t(x,0) = 0 \) for \( 0 \leq x \leq 1 \).
\[ \frac{dE}{dt} = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] \, dx = \int_0^1 [w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{xt}(x,t)] \, dx = \int_0^1 [w_x(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t)] \, dx = \int_0^1 \frac{1}{2} [w_t(x,t)w_x(x,t)] \, dx = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] \, dx = 0. \]
Therefore, for all \( t \geq 0 \), \( E(t) = E(0) = \frac{1}{2} \int_0^1 [w_t^2(x,0) + w_x^2(x,0)] \, dx = 0. \) By the vanishing theorem, it follows that \( \frac{1}{2} [w_t^2(x,t) + w_x^2(x,t)] = 0 \) for all \( 0 \leq x \leq 1 \) and all \( t \geq 0 \).
Consequently \( w_t(x,t) = w_x(x,t) = 0 \) for all \( 0 \leq x \leq 1 \) and all \( t \geq 0 \). It follows that \( u(x,t) = \text{constant} \) for all \( 0 \leq x \leq 1, \ t \geq 0 \). But \( \Theta \) implies this constant is zero.

I.e. \( u(x,t) = v(x,t) \) for all \( 0 \leq x \leq 1 \) and all \( t \geq 0 \).