Consider the partial differential equation

\[ u_{xx} - 3u_{xt} - 4u_t = 0. \]

(a) Classify the order and type (linear, nonlinear, parabolic, etc.) of (*)
(b) Find the general solution of (*) in the \(xt\) - plane, if possible.
(c) Find the solution of (*) that satisfies

\[ u(x, 0) = x^3 \quad \text{and} \quad u_t(x, 0) = -3x^2 \]

for \(-\infty < x < \infty\).
\[ u_{xx} - 3u_{xt} + 4u_{tt} = 0 \]

**linear homogeneous**

\[ B^2 - 4AC = (-3)^2 - 4(1)(-4) = 25 \]

**hyperbolic**

**second-order**

\[ \left( \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial t} \right) u = 0 \]

\[ \text{Let} \quad \begin{cases} y = 4x + t \\ \eta = x - t \end{cases} \]

\[ \text{Then} \quad \frac{\partial y}{\partial x} = \frac{\partial \eta}{\partial x} = \frac{1}{3} \]

\[ \frac{\partial y}{\partial t} = \frac{\partial \eta}{\partial t} = -\frac{2}{3} \]

\[ i.e. \quad \frac{\partial}{\partial x} = \frac{1}{3} \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial t} = -\frac{2}{3} \frac{\partial}{\partial y} \]

\[ \therefore \quad \frac{\partial^2}{\partial x^2} - 4 \frac{\partial^2}{\partial t^2} = \frac{4}{3} \frac{\partial^2}{\partial y^2} + \frac{2}{3} \frac{\partial^2}{\partial y^2} - 4 \left( \frac{\partial^2}{\partial y^2} - \frac{2}{3} \right) = \frac{5}{3} \frac{\partial^2}{\partial y^2} \]

\[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} = \frac{4}{3} \frac{\partial^2}{\partial y^2} + \frac{2}{3} \frac{\partial^2}{\partial y^2} + \frac{2}{3} - \frac{2}{3} = \frac{5}{3} \frac{\partial^2}{\partial y^2} \]

Thus, the pde is equivalent to \( \left( \frac{5}{3} \frac{\partial}{\partial y} \right) \left( \frac{5}{3} \frac{\partial}{\partial y} \right) u = 0 \Rightarrow \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial y} \right) = 0 \).

The general solution is \( u = f(y) + g(y) \), i.e. \[ u(x,t) = f(4x+t) + g(x-t) \]

where \( f \) and \( g \) are arbitrary \( C^2 \)-functions of a single real variable.

\[ x^3 = u(x,0) = f(4x) + g(x) \Rightarrow 3x^2 = 4f'(4x) + g'(x) \]

\[ u_t(x,t) = f'(4x+t) - g'(x-t) \Rightarrow -3x^2 = u_t(x,0) = f'(4x) - g'(x) \]

Adding equations (1) and (2) gives \( 0 = 5f'(4x) \Rightarrow f'(x) = 0 \Rightarrow f(x) = c_1 \).

Substituting this result in equation (1) gives \( 3x^2 = g'(x) \Rightarrow g(x) = x^3 + c_2 \).

But \( x^3 = f(4x) + g(x) = c_1 + x^3 + c_2 \) so \( c_1 + c_2 = 0 \). Thus

\[ u(x,t) = f(4x+t) + g(x-t) = c_1 + (x-t)^3 + c_2 \Rightarrow u(x,t) = (x-t)^3 \]