Mathematics 325
Homework Assignment 6

Due Date: ____________________

Name: __________________________

Consider the differential equation

(*) $$u_{tt} - u_{xx} = 0 \quad \text{for} \quad -\infty < x < \infty,\ 0 < t < \infty.$$

(a) Solve (*) subject to the initial conditions

$$u(x,0) = e^{-x^2} \quad \text{and} \quad u_t(x,0) = -2xe^{-x^2} \quad \text{for} \quad -\infty < x < \infty.$$

(b) Sketch profiles of the solution $u = u(x,t)$ at $t = 1,\ 2,$ and 3 in order to show that the solution is a wave traveling to the left along the $x-$axis. What is its speed?

(c) Derive the general nontrivial relation between ϕ and ψ which will produce a solution to $u_{tt} - u_{xx} = 0$ in the $xt-$plane satisfying

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for} \quad -\infty < x < \infty$$

and such that u consists solely of a wave traveling to the left along the $x-$axis.

$$u_{tt} - u_{xx} = 0 \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$

40% (a) $\quad u(x,0) = e^{-x^2}$ and $u_t(x,0) = -2xe^{-x^2} \qquad (-\infty < x < \infty).$

By d'Alembert,

$$u(x,t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\,d\xi$$

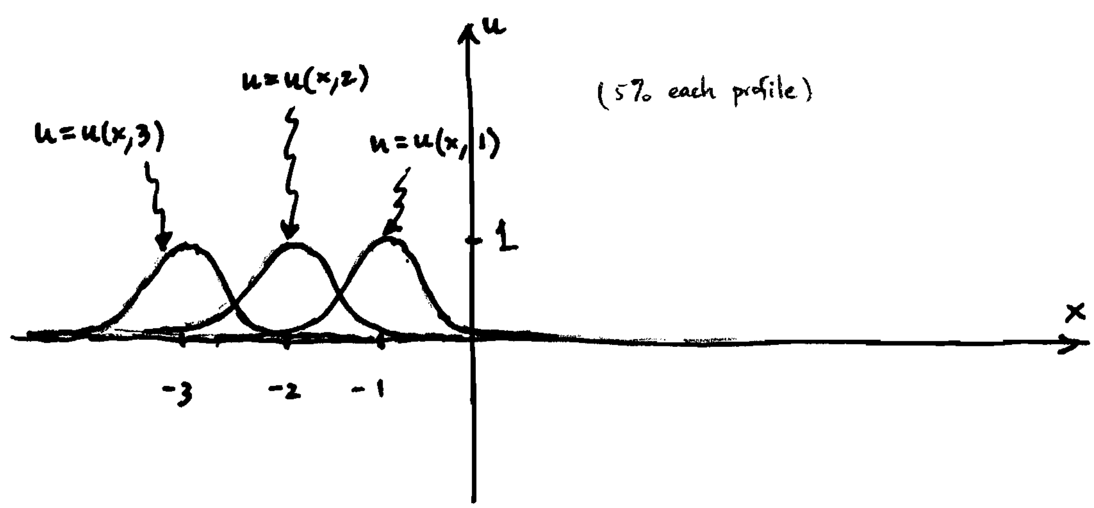
$$= \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} -2\xi e^{-\xi^2}\,d\xi$$

$$= \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \frac{1}{2} e^{-\xi^2}\Big|_{x-t}^{x+t}$$

$$= \frac{1}{2}\left[e^{-(x+t)^2} + \cancel{e^{-(x-t)^2}}\right] + \frac{1}{2}\left[e^{-(x+t)^2} - \cancel{e^{-(x-t)^2}}\right]$$

$$\boxed{u(x,t) = e^{-(x+t)^2}}.$$

20% (b)



The solution is a wave with $\boxed{\text{speed } 1}$ traveling to the left along the x-axis.

(5%)

40 % (c) By d'Alembert, the solution to the wave initial-value problem is

$$u(x,t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\,d\xi$$

$$= \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{0}^{x+t} \psi(\xi)\,d\xi - \frac{1}{2}\int_{0}^{x-t} \psi(\xi)\,d\xi$$

$$= \underbrace{\frac{1}{2}\varphi(x+t) + \frac{1}{2}\int_{0}^{x+t} \psi(\xi)\,d\xi}_{\text{wave moving to left along x-axis}} + \underbrace{\frac{1}{2}\varphi(x-t) - \frac{1}{2}\int_{0}^{x-t} \psi(\xi)\,d\xi}_{\text{wave moving to right along x-axis}} .$$

(20% to here)

The portion of $u(x,t)$ that moves to the right along the x-axis must be constant; i.e.

$$\frac{1}{2}\varphi(z) - \frac{1}{2}\int_{0}^{z} \psi(\xi)\,d\xi \quad = \quad \text{constant}$$

for all $-\infty < z < \infty$. We can obtain an equivalent relationship between φ and ψ by differentiation:

$$\frac{1}{2}\varphi'(z) - \frac{1}{2}\psi(z) = 0$$

(40% to here) $\rightarrow$ $\boxed{\psi(z) = \varphi'(z)}$ for all $-\infty < z < \infty$.