Mathematics 325
Homework 7

Due Date: ____________________

Name: ____________________________

Work exercise 5 on page 40 in Strauss.

#5, p. 40.

Let $u = u(x,t)$ be a solution to the damped wave equation

10% to here $$(*) \qquad u_{tt} - c^2 u_{xx} + r u_t = 0$$

in the xt-plane. (Here $c$ and $r$ are positive constants.) Consider the total energy function

20% to here $$E(t) = \int_{-\infty}^{\infty} \left[ \tfrac{1}{2} u_t^2(x,t) + \tfrac{1}{2} c^2 u_x^2(x,t) \right] dx$$

of the solution at time $t$. Then, assuming that $u$ obeys the decay conditions of the lecture on conservation of energy in waves, we have

30% to here $$\frac{dE}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \left[ \tfrac{1}{2} u_t^2(x,t) + \tfrac{1}{2} c^2 u_x^2(x,t) \right] dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left[ \tfrac{1}{2} u_t^2(x,t) + \tfrac{1}{2} c^2 u_x^2(x,t) \right] dx$$

40% to here $$= \int_{-\infty}^{\infty} \left[ u_t(x,t) u_{tt}(x,t) + c^2 u_x(x,t) u_{xt}(x,t) \right] dx$$

50% to here $$= \int_{-\infty}^{\infty} u_t(x,t) u_{tt}(x,t)\,dx + \lim_{\substack{M \to \infty \\ N \to -\infty}} \int_N^M \overbrace{c^2 u_x(x,t)}^{v} \overbrace{u_{xt}(x,t)\,dx}^{dv}$$

60% to here. $$= \int_{-\infty}^{\infty} u_t(x,t) u_{tt}(x,t)\,dx + \lim_{\substack{M \to \infty \\ N \to -\infty}} \left( c^2 u_x(x,t) u_t(x,t) \Big|_{x=N}^{M} - \int_N^M c^2 u_t(x,t) u_{xx}(x,t)\,d\right.$$

70% to here. $$= \int_{-\infty}^{\infty} u_t(x,t) u_{tt}(x,t)\,dx - \int_{-\infty}^{\infty} c^2 u_t(x,t) u_{xx}(x,t)\,dx .$$

Therefore

80% to here.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t(x,t)\Big[\overbrace{u_{tt}(x,t) - c^2 u_{xx}(x,t)}^{-ru_t(x,t) \text{ from } (**)}\Big]dx$$

$$= -r\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$$

90% to here

$$\leq 0.$$

100% to here.

It follows that the total energy of the solution is a decreasing function of time $t$.