1. (16 pts.) For each of the following equations, state the order and tell whether it is nonlinear, linear inhomogeneous, or linear homogeneous. In the case of a nonlinear equation, circle the term(s) which make it nonlinear.

(a) \( u_t - u_{xxt} + uu_x = 0 \)  
   3rd order, nonlinear

(b) \( u_{tt} - u_{xx} + \sqrt{1 + x^2} = 0 \)  
   2nd order, linear inhomogeneous

(c) \( u_x + e^t u_t = 0 \)  
   first order, linear homogeneous

(d) \( \left( \frac{\partial u}{\partial t} \right)^6 + \frac{\partial^4 u}{\partial x^4} = 0 \)  
   4th order, nonlinear

2. (16 pts.) Find the solution to \( 3u_t + 2u_x = 0 \) in the \( xt \)-plane which satisfies the auxiliary condition \( u(x, 0) = \sin(x) \) for \( -\infty < x < \infty \). Sketch some characteristic curves of the pde.

Let \( \begin{cases} \frac{\xi}{3} = 2x + 3t \\ \eta = 3x - 2t \end{cases} \). Then
\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = 2 \frac{\partial u}{\partial \xi} + \frac{3}{3} \frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = 3 \frac{\partial u}{\partial \xi} - 2 \frac{\partial u}{\partial \eta}
\]

\[ \therefore 0 = 3u_t + 2u_x = 3(3u_{\xi} - 2u_{\eta}) + 2(2u_{\xi} + 3u_{\eta}) = 13u_{\xi} \Rightarrow u = f(\eta). \]

I.e. \( u(x,t) = f(3x - 2t) \) where \( f \) is a \( C^1 \) function of a single real variable.

\[ \sin(x) = u(x,0) = f(3x) \Rightarrow f(x) = \sin \left( \frac{x}{3} \right). \]

\[ \therefore u(x,t) = \sin \left( \frac{x}{3} (3x - 2t) \right) = \begin{cases} \sin \left( x - \frac{2}{3} t \right) \end{cases} \]

Characteristic Curves:
\[ x - \frac{2}{3} t = k \]
3. (16 pts.) Find the solution to \( tu_t - xu_x = 0 \) in the \( xt \)-plane which satisfies the auxiliary condition \( u(x, x) = x^6 \) for \(-\infty < x < \infty\). Sketch some characteristic curves of the pde.

\[
(x, t) \cdot \nabla u = -xu_x + tu_t = 0 \quad \Rightarrow \quad \frac{dx}{t} = -\frac{t}{x} \quad \Rightarrow \quad \frac{dt}{x} = \frac{t}{dx} \quad \Rightarrow \quad \ln|t| = -\ln|x| + c \quad \Rightarrow \quad \ln|xt| = c
\]

or \([xt = e^c = k]\) (the characteristic curves are hyperbolas). Along such a curve \( u(x, t) = u(x, \frac{k}{x}) \equiv u(1, k) = f(k) \).

Thus, the general solution is \( u(x, t) = f(xt) \) where \( f \) is a \( C^1 \) function of a single real variable. \( x^6 = u(x, x) = f(x^2) \quad \Rightarrow \quad f(x) = x^3 \).

Therefore, the particular solution is \( u(x, t) = (xt)^3 \).

4. (18 pts.) Classify the following pde as hyperbolic, elliptic, parabolic, or none of these. Find the general solution in the \( xt \)-plane, if possible.

\[
(\frac{2}{\partial x} - 3 \frac{\partial}{\partial t})(\frac{3}{\partial x} - 3 \frac{\partial}{\partial t})u + 2(\frac{2}{\partial x} - 3 \frac{\partial}{\partial t})u = 0
\]

\[
(\star) \quad u_{xx} + 3u_{tt} - 4u_{xt} + 2u_x - 2u_t = 0
\]

\[
B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0 \quad \text{hyperbolic}
\]

Let \( \eta = x + t \). The chain rule gives

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2} - \frac{3\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \eta^2}
\]

and \( \frac{\partial^2}{\partial x \partial t} = \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \eta} \right) = 2 \frac{\partial^2}{\partial \eta^2} \). Therefore \((\star)\) is equivalent to \(-4 \frac{\partial^2}{\partial \eta^2} + 2u_x - 2u_t = 0\).

Writing \( v = \frac{2u}{\partial \eta} \) we see that \((\star)\) is equivalent to \(-\frac{\partial^2 v}{\partial \eta^2} + v = 0\). Separation of variables gives the solution \( v = c \exp\eta \). Thus \( u = \int \frac{2v}{\partial \eta} \, d\eta = \int v \, d\eta = \int c \exp\eta \, d\eta = c \exp\eta + g(\eta) \)

where \( f(\eta) \) is any particular integral of \( \int c \exp\eta \, d\eta \). Therefore the general solution of \((\star)\) in the \( xt \)-plane is

\[
u(x, t) = e^{3(x + t)} + g(x + t)
\]

where \( f \) and \( g \) are \( C^1 \) functions of a single real variable.
5. (17 pts.) Find the solution to
\[ u_{tt} - 4u_{xx} = 0 \quad \text{in} \ -\infty < x < \infty, \ 0 < t < \infty, \]
satisfying
\[ u(x,0) = \frac{1}{1 + x^2} \quad \text{and} \quad u_t(x,0) = \frac{-4x}{(1 + x^2)^2} \quad \text{for} \ -\infty < x < \infty. \]
Sketch profiles of the solution at the instants \( t = 1, 2, \) and 3.

Use d'Alembert's formula: 
\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x+ct)^2} + \frac{1}{1 + (x-ct)^2} \right] + \frac{1}{4} \int_{x-ct}^{x+ct} \frac{-4s}{(1 + s^2)^2} ds \]
where \( c = 2, \ \varphi(x) = \frac{1}{1 + x^2}, \) and \( \psi(x) = \frac{-4x}{(1 + x^2)^2}. \) Thus

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x+2t)^2} + \frac{1}{1 + (x-2t)^2} \right] + \frac{1}{4} \int_{x-2t}^{x+2t} \frac{-4s}{(1 + s^2)^2} ds \]

Let \( U = 1 + s^2. \) Then \( dU = 2sds \)

\[ \therefore \int \frac{-4sds}{(1 + s^2)^2} = \int \frac{-2dU}{U^2} = \frac{2}{U} = \frac{2}{1 + s^2} \]

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x+2t)^2} + \frac{1}{1 + (x-2t)^2} \right] + \frac{1}{4} \left( \frac{2}{1 + s^2} \right) \bigg|_{s = x-2t}^{s = x+2t} \]
\[ = \frac{1}{2} \left[ \frac{1}{1 + (x+2t)^2} + \frac{1}{1 + (x-2t)^2} \right] + \frac{1}{4} \left[ \frac{1}{1 + (x+2t)^2} - \frac{1}{1 + (x-2t)^2} \right] \]

\[ u(x,t) = \frac{1}{1 + (x+2t)^2} \]

The solution is a standing wave that moves to the left along the x-axis with speed 2.
A homogeneous solid material occupying the region
\[ D = \{ (x,y,z) : x^2 + y^2 + z^2 \leq 100 \} \]
is completely insulated. The initial temperature at each point in D is five times its distance from the center of D.

(a) Write (without proof) the partial differential equation and initial/boundary conditions that completely govern the temperature \( u(x,y,z,t) \) at position \((x,y,z)\) in D and time \( t \geq 0 \).

(b) Use Gauss' divergence theorem to help show that the heat energy
\[
H(t) = \iiint_D c \rho u(x,y,z,t) \, dx \, dy \, dz
\]
of the material in D at time \( t \) is actually a constant function of time. (Here \( c \) and \( \rho \) denote the (constant) specific heat and density, respectively, of the material in D.)

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

\[ (a) \quad (1) \quad u_t - \frac{k}{\rho c} \nabla^2 u = 0 \quad \text{if} \quad x^2 + y^2 + z^2 < 100 \quad \text{and} \quad t > 0, \]
\[ (2) \quad u(x,y,z,0) = 5\sqrt{x^2 + y^2 + z^2} \quad \text{if} \quad x^2 + y^2 + z^2 = 100, \]
\[ (3) \quad \vec{n} \cdot \nabla u(x,y,z,t) = 0 \quad \text{if} \quad x^2 + y^2 + z^2 = 100 \quad \text{and} \quad t \geq 0. \]
(Here \( \vec{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \) is the unit outward-pointing normal to \( \partial D \).)

\[ (b) \quad \frac{dH}{dt} = \frac{d}{dt} \iiint_D c \rho u \, dx \, dy \, dz = \iiint_D \frac{\partial}{\partial t} (c \rho u) \, dx \, dy \, dz = \iiint_D c \rho u_t \, dx \, dy \, dz = \iiint_D k \nabla^2 u \, dx \, dy \, dz \]
\[ (c) \quad \iiint_D \frac{1}{2} \vec{n} \cdot \nabla u \, dS \bigg| _{\partial D} = \iiint_D \, dS = 0. \quad \text{Therefore} \quad H(t) = \text{constant for all} \ t \geq 0. \]

\[ (c) \quad \text{For all} \ t \geq 0, \quad \iiint_D c \rho u(x,y,z,t) \, dx \, dy \, dz = H(0) = H(t) = \iiint_D c \rho u(x,y,z,t) \, dx \, dy \, dz. \quad (*) \]

Let \( U = \lim_{t \to \infty} u(x,y,z,t) \) denote the (constant) steady-state temperature that the material reaches after a long time. Taking the limit as \( t \to \infty \) of the right member of (*) and using (2) in the left member of (*) gives
\[ c \rho \iiint_D 5\sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = c \rho \iiint_D u \, dx \, dy \, dz = c \rho U \iiint_D dV = c \rho U \cdot \frac{4}{3} \pi (10)^3. \]

Using spherical coordinates to evaluate the integral on the left above gives
\[
\iiint_D 5\sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \int_0^2 \int_0^{2\pi} \int_0^\pi 5r^2 \sin(\phi) \, dr \, d\theta \, d\phi = (\frac{5}{4} \pi) \cdot (\cos \frac{\pi}{3})^4 (2\pi).
\]
\[ \therefore \quad U = \frac{c \rho 5\pi (10)^4}{c \rho \frac{4}{3} \pi (10)^3} = \frac{75}{2}. \]