Mathematics 325 | Exam I
Summer 2001 | Name: Dr. Grow

1.(16 pts.) For each of the following equations, state the order and tell whether it is nonlinear, linear inhomogeneous, or linear homogeneous. In the case of a nonlinear equation, circle the term(s) which make it nonlinear.

(a) $u_t - u_{xxt} + uu_x = 0$ 3rd order, nonlinear

(b) $u_{tt} - u_{xx} + \sqrt{1 + x^2} = 0$ 2nd order, linear inhomogeneous

(c) $u_x + e^t u_t = 0$ first order, linear homogeneous

(d) $\left(\frac{\partial u}{\partial t}\right)^6 + \frac{\partial^4 u}{\partial x^4} = 0$ 4th order, nonlinear

2.(16 pts.) Find the solution to $3u_t + 2u_x = 0$ in the xt-plane which satisfies the auxiliary condition $u(x,0) = \sin(x)$ for $-\infty < x < \infty$. Sketch some characteristic curves of the pde.

Let $\begin{cases} \xi = 2x + 3t \\ \eta = 3x - 2t \end{cases}$. Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = 2\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta}$

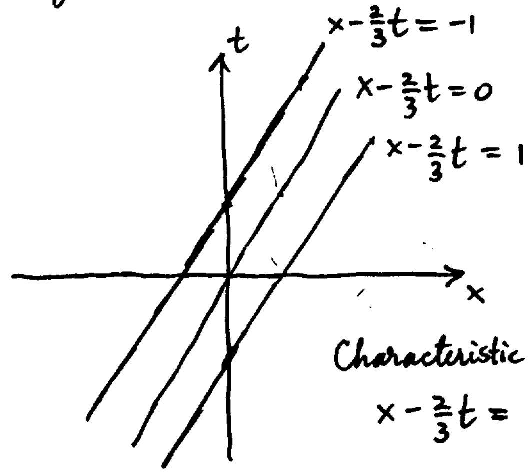
and $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = 3\frac{\partial u}{\partial \xi} - 2\frac{\partial u}{\partial \eta}$

$\therefore\ 0 = 3u_t + 2u_x = 3(3u_\xi - \cancel{2u_\eta}) + 2(2u_\xi + \cancel{3u_\eta}) = 13u_\xi \Rightarrow u = f(\eta)$.

I.e. $u(x,t) = f(3x-2t)$ where f is a C^1 function of a single real variable.

$\sin(x) = u(x,0) = f(3x) \Rightarrow f(\star) = \sin(\star/3)$.

$\therefore\ u(x,t) = \sin\left(\frac{1}{3}(3x-2t)\right) = \boxed{\sin\left(x - \frac{2}{3}t\right)}$



Characteristic Curves:
$x - \frac{2}{3}t = k$

3.(16 pts.) Find the solution to $tu_t - xu_x = 0$ in the xt-plane which satisfies the auxiliary condition $u(x,x) = x^6$ for $-\infty < x < \infty$. Sketch some characteristic curves of the pde.

$(-x,t)\cdot\nabla u = -xu_x + tu_t = 0 \Rightarrow D_{(-x,t)}u = 0$. Thus $u = u(x,t)$ is constant along curves whose tangent remains parallel to $(-x,t)$, i.e. to curves satisfying

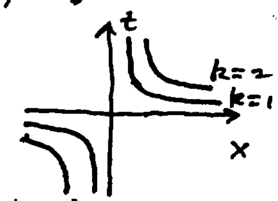
$$\frac{dt}{dx} = \frac{t}{-x} \Rightarrow \frac{1}{t}dt = -\frac{1}{x}dx \Rightarrow \ln|t| = -\ln|x| + c \Rightarrow \ln|xt| = c$$

or $\boxed{xt = e^c = k}$ (the characteristic curves are hyperbolas). Along such a curve

$$u(x,t) = u\left(x, \frac{k}{x}\right) \overset{\text{Set } x=1}{=} u(1,k) = f(k).$$

Thus, the general solution is $u(x,t) = f(xt)$ where f is a C^1 function of a single real variable. $x^6 = u(x,x) = f(x\cdot x) = f(x^2) \Rightarrow f(\star) = \star^3$.

Therefore, the particular solution is $\boxed{u(x,t) = (xt)^3}$.



4.(18 pts.) Classify the following pde as hyperbolic, elliptic, parabolic, or none of these. Find the general solution in the xt-plane, if possible.

$$\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u + 2\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u = 0$$

$$(*) \qquad u_{xx} + 3u_{tt} - 4u_{xt} + 2u_x - 2u_t = 0$$

$B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0$ $\boxed{\text{hyperbolic}}$ Let $\begin{cases} \xi = 3x+t, \\ \eta = x+t. \end{cases}$ The chain rule gives

$\frac{\partial}{\partial x} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$, so $\frac{\partial}{\partial x} - 3\frac{\partial}{\partial t} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 3\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = -2\frac{\partial}{\partial \eta}$

and $\frac{\partial}{\partial x} - \frac{\partial}{\partial t} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 2\frac{\partial}{\partial \xi}$. Therefore (*) is equivalent to $-4\frac{\partial}{\partial \eta}\frac{\partial u}{\partial \xi} + 4\frac{\partial u}{\partial \xi} = 0$

Writing $v = \frac{\partial u}{\partial \xi}$ we see that (*) is equivalent to $-\frac{\partial v}{\partial \eta} + v = 0$. Separation of variables gives the solution $v = c_1(\xi)e^{\eta}$. Thus $u = \int \frac{\partial u}{\partial \xi}d\xi = \int v\,d\xi = \int c_1(\xi)e^{\eta}d\xi = e^{\eta}f(\xi) + g(\eta)$

where $f(\xi)$ is any particular integral of $\int c_1(\xi)d\xi$. Therefore the general solution of (*) in the xt-plane is

$$\boxed{u(x,t) = e^{x+t}f(3x+t) + g(x+t)}$$

where f and g are C^2 functions of a single real variable.

5. (17 pts.) Find the solution to

$$u_{tt} - 4u_{xx} = 0 \quad \text{in } -\infty < x < \infty,\ 0 < t < \infty,$$

satisfying

$$u(x,0) = \frac{1}{1+x^2} \quad \text{and} \quad u_t(x,0) = \frac{-4x}{(1+x^2)^2} \quad \text{for } -\infty < x < \infty.$$

Sketch profiles of the solution at the instants t = 1, 2, and 3.

We use d'Alembert's formula: $u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$

where $c=2$, $\varphi(x) = \frac{1}{1+x^2}$, and $\psi(x) = \frac{-4x}{(1+x^2)^2}$. Thus

$$u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+2t)^2} + \frac{1}{1+(x-2t)^2}\right] + \underbrace{\frac{1}{4}\int_{x-2t}^{x+2t} \frac{-4\xi}{(1+\xi^2)^2}\,d\xi}$$

Let $U = 1+\xi^2$. Then $dU = 2\xi\,d\xi$

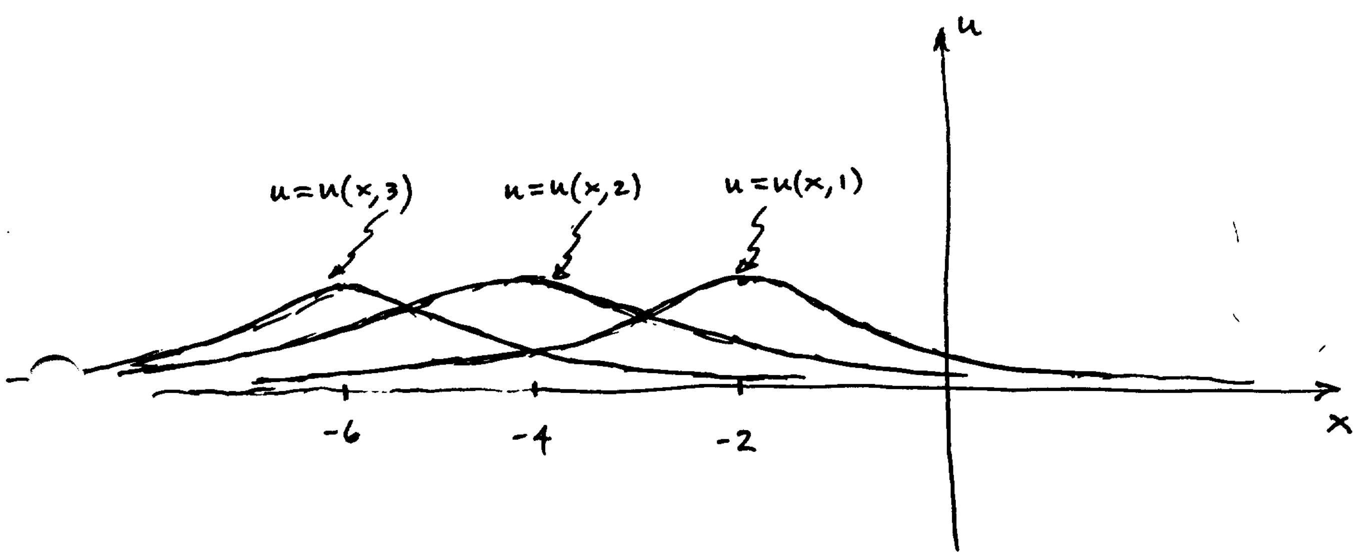
$$\therefore \int \frac{-4\xi\,d\xi}{(1+\xi^2)^2} = \int \frac{-2\,dU}{U^2} = \frac{2}{U} = \frac{2}{1+\xi^2}$$

$$u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+2t)^2} + \frac{1}{1+(x-2t)^2}\right] + \frac{1}{4}\left(\frac{2}{1+\xi^2}\right)\Bigg|_{\xi=x-2t}^{x+2t}$$

$$= \frac{1}{2}\left[\frac{1}{1+(x+2t)^2} + \cancel{\frac{1}{1+(x-2t)^2}}\right] + \frac{1}{2}\left[\frac{1}{1+(x+2t)^2} - \cancel{\frac{1}{1+(x-2t)^2}}\right]$$

$$\boxed{u(x,t) = \frac{1}{1+(x+2t)^2}}$$

The solution is a standing wave that moves to the left along the x-axis with speed 2.



6. (17 pts.) A homogeneous solid material occupying the region

$$D = \{ (x,y,z) : x^2 + y^2 + z^2 \le 100 \}$$

is completely insulated. The initial temperature at each point in D is five times its distance from the center of D.

(a) Write (without proof) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t \ge 0$.

(b) Use Gauss' divergence theorem to help show that the heat energy

$$H(t) = \iiint_D c\rho u(x,y,z,t)\,dxdydz$$

of the material in D at time t is actually a constant function of time. (Here c and ρ denote the (constant) specific heat and density, respectively, of the material in D.)

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

(a) (1) $u_t - \frac{k}{c\rho}\nabla^2 u = 0$ if $x^2+y^2+z^2 < 100$ and $t > 0$,

(2) $u(x,y,z,0) = 5\sqrt{x^2+y^2+z^2}$ if $x^2+y^2+z^2 \le 100$,

(3) $\vec{n}\cdot\nabla u(x,y,z,t) = 0$ if $x^2+y^2+z^2 = 100$ and $t \ge 0$.

(Here $\vec{n} = \dfrac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is the unit outward-pointing normal to ∂D.)

(b) $$\frac{dH}{dt} = \frac{d}{dt}\iiint_D c\rho u\,dxdydz = \iiint_D \frac{\partial}{\partial t}(c\rho u)\,dxdydz = \iiint_D c\rho u_t\,dxdydz \overset{(1)}{=} \iiint_D k\nabla^2 u\,dxdydz$$

$$\overset{\text{Gauss}}{=} \iint_{\partial D} k\vec{n}\cdot\nabla u\,dS \overset{(3)}{=} \iint_{\partial D} 0\,dS = 0.$$ Therefore $H(t) = $ constant for all $t \ge 0$.

(c) For all $t \ge 0$, $\iiint_D c\rho u(x,y,z,0)\,dxdydz = H(0) = H(t) = \iiint_D c\rho u(x,y,z,t)\,dxdydz$. (*)

Let $U = \lim_{t\to\infty} u(x,y,z,t)$ denote the (constant) steady-state temperature that the material reaches after a long time. Taking the limit as $t\to\infty$ of the right member of (*) and using (2) in the left member of (*) gives

$$c\rho\iiint_D 5\sqrt{x^2+y^2+z^2}\,dxdydz = \iiint_D c\rho U\,dxdydz = c\rho U\iiint_D dxdydz = c\rho U\cdot\underbrace{\tfrac{4}{3}\pi(10)^3}_{\text{volume of } D}.$$

Using spherical coordinates to evaluate the integral on the left above gives

$$\iiint_D 5\sqrt{x^2+y^2+z^2}\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{10} 5r\cdot r^2\sin(\varphi)\,dr\,d\varphi\,d\theta = \left(\frac{5}{4}r^4\Big|_0^{10}\right)\left(-\cos\varphi\Big|_0^{\pi}\right)(2\pi).$$

$$\therefore\quad U = \frac{c\rho 5\pi(10)^4}{c\rho\frac{4}{3}\pi(10)^3} = \boxed{\frac{75}{2}}$$