1. (25 pts.) Consider the partial differential equation

\[
\frac{1 - x^2}{2} u_x + 2xyu_y = 0 .
\]

\(*)

(a) What are the order and type (linear or nonlinear) of (*)?

(b) Sketch three of the characteristic curves of (*).

(c) Find the general solution of (*).

(d) What is the solution to (*) satisfying \( u(0,y) = y^2 \) for \(-\infty < y < \infty \)?

\[ a) \text{first-order, linear} \quad \text{Let } y = 1 - x^2. \quad \text{Then } dv = 2x dx. \]

\[ b) \frac{dy}{dx} = \frac{2xy}{\sqrt{1-x^2}} \Rightarrow \int \frac{dy}{y} = \int \frac{2x dx}{\sqrt{1-x^2}} \Rightarrow \ln|y| = -2\sqrt{1-x^2} + c \]

\[ y = A e^{-2\sqrt{1-x^2}} \]  

\((\text{Characteristic curves of } \ast))

\[ c) \text{Along any characteristic curve, } u \text{ is constant. Thus, along } y = A e^{-2\sqrt{1-x^2}}, \]

\[ u(x,y) = u(x, Ae^{-2\sqrt{1-x^2}}) = u(0, Ae^{-2\sqrt{1-0^2}}) = f(A). \]

Therefore the general solution is \( u(x,y) = f(y e^{-2\sqrt{1-x^2}}) \) where \( f \) is any differentiable function of a single real variable.

\[ d) \quad y^2 = u(0,y) = f(y e^2) \Rightarrow f(z) = \left(\frac{z}{e^2}\right)^2 = e^{-4} z^2. \]

\[ \therefore u(x,y) = f(y e^{-2\sqrt{1-x^2}}) = e^{4 + \sqrt{1-x^2} - 4} \]

\[ \Rightarrow u(x,y) = y e^{\frac{4 + \sqrt{1-x^2} - 4}{4}} \quad \text{for } -1 < x < 1 \text{ and } -\infty < y < \infty. \]
2. (25 pts.) Let $D$ be a closed bounded 3-dimensional region with piecewise smooth orientable boundary surface $\partial D$, and let $f = f(x,y,z)$ be a continuous function in $D$. Consider the boundary value problem

\[
\begin{cases}
\nabla^2 u = f(x,y,z) & \text{in } D, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D.
\end{cases}
\]

(a) Is there at most one solution to (*)? Justify your answer.
(b) Use Gauss' divergence theorem to help show that

\[
\iiint_D f(x,y,z) \, dV = 0
\]

is a necessary condition for (*) to have a solution.
(c) Give a physical interpretation for the result in part (b) in the case of either heat flow or diffusion.

(a) No, for if $u = u(x,y,z)$ is a solution to (*), then so is $u = u(x,y,z) + c$ where $c$ is any real constant.
(b) Suppose (*) has a solution, say $u = u(x,y,z)$. Then

\[
0 = \iiint_{\partial D} \frac{\partial u}{\partial n} \, dS = \iiint_{\partial D} \nabla u \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \nabla u \, dV = \iiint_D \nabla^2 u \, dV = \iiint_D f(x,y,z) \, dV.
\]

from B.C. in (*)
Gauss' Divergence Theorem
from p.d.e. in (*)

(c) Suppose that (*) models the (steady-state) temperature $u(x,y,z)$ at each point $(x,y,z)$ in $D$. Then $f = f(x,y,z)$ indicates a source ($f > 0$) or sink ($f < 0$) of heat energy at the point $(x,y,z)$ in $D$, while $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0$ indicates that there is no heat energy flowing into or out of the boundary surface of $D$; i.e. the region $D$ is insulated. Consequently, since the heat energy contained in $D$ must be conserved, the net contribution to heat energy through the sources/sinks in $D$ must be zero. That is,

\[
\iiint_D f(x,y,z) \, dV = 0.
\]
3. (25 pts.) Consider the partial differential equation

\[ u_{xx} - 3u_{xt} - 4u_{tt} = 0. \]

(a) Classify (\(*\))'s order and type (linear, nonlinear, parabolic, etc.).
(b) Find the general solution of (\(*\)) in the xt-plane.
(c) Find the solution of (\(*\)) that satisfies

\[ u(x,0) = x^3 \quad \text{and} \quad u_t(x,0) = -3x^2 \]

for \(-\infty < x < \infty\).

\((a) \) \text{Second-order, linear, hyperbolic} \quad (B^2 - 4AC = (3)^2 - 4(1)(-4) > 0).

\((b) \) \[ \left( \frac{\partial^2}{\partial x^2} - 3 \frac{\partial^2}{\partial x \partial t} - 4 \frac{\partial^2}{\partial t^2} \right) u = 0 \quad \iff \quad \left( \frac{\partial}{\partial x} - 3 \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) u = 0. \]

Let \[ \begin{cases} \tau = 4x + t, & \text{then} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \eta}, \\ \eta = x - t. & \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}. \end{cases} \]

so \[ \frac{\partial}{\partial x} - 3 \frac{\partial}{\partial t} = 5 \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial x} + \frac{\partial}{\partial t} = 5 \frac{\partial}{\partial \eta}, \]

Therefore (\(*\)) is equivalent to \[ \left( \frac{\partial}{\partial \eta} \right)^2 u = 0, \]

and thus \[ \frac{\partial u}{\partial \eta} = c(\eta) \quad \text{and} \quad u = c(\eta) \delta + c_2(\eta), \]

where \( f(\tau) \) and \( g(\eta) \) are \( C^2 \)-functions of a single real variable.

\((c) \) \hspace{0.5cm} 1. \quad x^3 = u(x,0) = f(4x) + g(x) \quad \text{for all} \ -\infty < x < \infty.

\hspace{1.5cm} \left[ u_t = f'(4x+t) \quad g'(x-t) \right]

2. \quad -3x^2 = u_t(x,0) = f'(4x) - g'(x) \quad \text{for} \ -\infty < x < \infty.

Differentiating (1) gives (3): \[ 3x^2 = 4f'(4x) + g'(x) \quad \text{for} \ -\infty < x < \infty, \]

Adding (3) and (2) gives \[ 0 = 5f'(4x) \quad \Rightarrow \quad f(2) = \text{constant} = c_1. \]

Since \( f' = 0 \), (1) becomes \[ -3x^2 = -g'(x) \quad \Rightarrow \quad x^2 + c_2 = g(x). \]

By (1), \[ x^3 = c_1 + x^3 + c_2 \quad \text{for all} \ -\infty < x < \infty \quad \Rightarrow \quad c_1 + c_2 = 0. \]

Thus, \( u(x,t) = f(4x+t) + g(x-t) = c_1 + (x-t)^3 + c_2 = \boxed{(x-t)^3}. \)
4. (25 pts.) Let $u = u(x,t)$ be a solution to

$$2u_{tt} - 2u_{xx} + u_t = 0$$

in $-\infty < x < \infty$, $0 \leq t < \infty$, with the property that

$$\lim_{|x| \to \infty} u(x,t)u_t(x,t) = 0$$

for each fixed $t \geq 0$.

(a) Show that $\int_{-\infty}^{\infty} (|u_t(x,t)|^2 + |u_x(x,t)|^2) \, dx$ is a nonincreasing function of $t$. (Assume that all relevant improper integrals converge.)

(b) Give a physical interpretation of the result in part (a) in the case of the vibrations of a long string.

(a) Let $E(t) = \int_{-\infty}^{\infty} [u_t^2(x,t) + u_x^2(x,t)] \, dx$. Then

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \frac{2}{u} \left\{ u_t^2(x,t) + u_x^2(x,t) \right\} \, dx$$

$$= \int_{-\infty}^{\infty} \{ 2u_t(x,t)u_x(x,t) + 2u_x(x,t)u_{tx}(x,t) \} \, dx.$$ 

Substituting $2u_t = 2u_{xx} - u_x$ (from the p.d.e.) into the integrand of $dE/dt$ gives

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \left\{ 2u_t(x,t)u_x(x,t) - u_x^2(x,t) + 2u_x(x,t)u_{tx}(x,t) \right\} \, dx$$

$$= \int_{-\infty}^{\infty} -u_x^2(x,t) \, dx + 2 \int_{-\infty}^{\infty} \frac{d}{dx} \left\{ u_t(x,t)u_x(x,t) \right\} \, dx$$

$$= \int_{-\infty}^{\infty} -u_x^2(x,t) \, dx + 2 \lim_{M \to \infty} \left[ \frac{u_t(x,t)u_x(x,t)}{x=M} \right]_{x=-\infty}^{x=M}$$

$$= -\int_{-\infty}^{\infty} u_x^2(x,t) \, dx \leq 0.$$ 

Therefore $E = E(t)$ is a nonincreasing function of $t$.

(b) The p.d.e. models the vertical displacement $u(x,t)$ of a long string at position $x$ and time $t$, taking into account a drag force that opposes the motion, which is equal to the velocity $u_x$ of the string. Hence the total energy $E(t)$ of the string at time $t$ decreases with time.