Mathematics 325 | Exam I Spring 2002 | Name: Dr. Grow

1. (25 pts.) Consider the partial differential equation

$$(*) \qquad \sqrt{1-x^2}\, u_x + 2xyu_y = 0 .$$

(a) What are the order and type (linear or nonlinear) of (*)?
(b) Sketch three of the characteristic curves of (*).
(c) Find the general solution of (*).
(d) What is the solution to (*) satisfying $u(0,y) = y^2$ for $-\infty < y < \infty$?

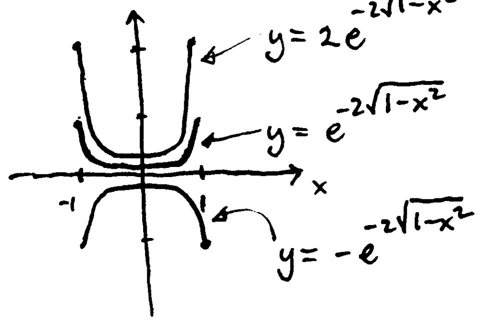
(a) first-order, linear

Let $v = 1-x^2$. Then $dv = -2x\,dx$.

7 (b) $\dfrac{dy}{dx} = \dfrac{2xy}{\sqrt{1-x^2}} \Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{2x\,dx}{\sqrt{1-x^2}} \Rightarrow \ln|y| = -2\sqrt{1-x^2} + c$

$\therefore \boxed{y = A e^{-2\sqrt{1-x^2}}}$ $\quad (A = \pm e^c = \text{constant})$

(Characteristic curves of (*))



6 (c) Along any characteristic curve, u is constant. Thus, along $y = Ae^{-2\sqrt{1-x^2}}$,

$$u(x,y) = u(x, Ae^{-2\sqrt{1-x^2}}) = u(0, Ae^{-2\sqrt{1-0^2}}) = f(A).$$

Therefore the general solution is $\boxed{u(x,y) = f\left(ye^{2\sqrt{1-x^2}}\right)}$ where f is any differentiable function of a single real variable.

6 (d) $y^2 = u(0,y) = f(ye^2) \Rightarrow f(z) = \left(\dfrac{z}{e^2}\right)^2 = e^{-4}z^2$.

$\therefore u(x,y) = f\left(ye^{2\sqrt{1-x^2}}\right) = e^{-4}\left(ye^{2\sqrt{1-x^2}}\right)^2$

$\Rightarrow \boxed{u(x,y) = y^2 e^{4\sqrt{1-x^2}-4}}$ for $-1 < x < 1$ and $-\infty < y < \infty$.

2. (25 pts.) Let D be a closed bounded 3-dimensional region with piecewise smooth orientable boundary surface ∂D, and let $f = f(x,y,z)$ be a continuous function in D. Consider the boundary value problem

$$(*) \qquad \begin{cases} \nabla^2 u = f(x,y,z) & \text{in } D, \\ \partial u / \partial n = 0 & \text{on } \partial D. \end{cases}$$

(a) Is there at most one solution to (*)? Justify your answer.

(b) Use Gauss' divergence theorem to help show that

$$\iiint_D f(x,y,z)\,dV = 0$$

is a necessary condition for (*) to have a solution.

(c) Give a physical interpretation for the result in part (b) in the case of either heat flow or diffusion.

(a) No, for if $u = u(x,y,z)$ is a solution to (*), then so is $u = u(x,y,z) + c$ where c is any real constant.

8 (b) Suppose (*) has a solution, say $u = u(x,y,z)$. Then

$$0 \underset{\text{from B.C. in } (*)}{=} \iint_{\partial D} \frac{\partial u}{\partial n}\,dS = \iint_{\partial D} \nabla u \cdot \vec{n}\,dS \underset{\text{Gauss' Divergence Theorem}}{=} \iiint_D \nabla\cdot(\nabla u)\,dV = \iiint_D \nabla^2 u\,dV \underset{\text{from p.d.e. in } (*)}{=} \iiint_D f(x,y,z)\,dV.$$

7 (c) Suppose that (*) models the (steady-state) temperature $u(x,y,z)$ at each point (x,y,z) in D. Then $f = f(x,y,z)$ indicates a source ($f > 0$) or sink ($f < 0$) of heat energy at the point (x,y,z) in D, while $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0$ indicates that there is no heat energy flowing into or out of the boundary surface of D; i.e. the region D is insulated. Consequently, since the heat energy contained in D must be conserved, the net contribution to heat energy through the sources/sinks in D must be zero. That is, $\iiint_D f(x,y,z)\,dV = 0$.

3. (25 pts.) Consider the partial differential equation

(*) $$u_{xx} - 3u_{xt} - 4u_{tt} = 0.$$

(a) Classify (*)'s order and type (linear, nonlinear, parabolic, etc.).
(b) Find the general solution of (*) in the xt-plane.
(c) Find the solution of (*) that satisfies

$$u(x,0) = x^3 \text{ and } u_t(x,0) = -3x^2$$

for $-\infty < x < \infty$.

(a) $\boxed{\text{second-order, linear, hyperbolic}}$ $(B^2-4AC = (-3)^2-4(1)(-4) > 0)$.

(b) $\left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial t} - 4\frac{\partial^2}{\partial t^2}\right)u = 0 \iff \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0.$

Let $\begin{cases} \xi = 4x+t, \\ \eta = x-t. \end{cases}$ Then $\frac{\partial}{\partial x} = 4\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$

so $\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t} = 5\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = 5\frac{\partial}{\partial \xi}$. Therefore (*) is equivalent to

$\left(\cancel{5}\frac{\partial}{\partial \eta}\right)\left(\cancel{5}\frac{\partial}{\partial \xi}\right)u = 0$, and thus $\frac{\partial u}{\partial \xi} = c_1(\xi)$ and $u = \int c_1(\xi)d\xi + c_2(\eta)$

$= f(\xi) + g(\eta)$. Consequently, the general solution of (*) in the xt-plane is

$\boxed{u(x,t) = f(4x+t) + g(x-t)}$ where f and g are C^2-functions of a single real variable.

(c) ① $x^3 = u(x,0) = f(4x) + g(x)$ for all $-\infty<x<\infty$. $[u_t = f'(4x+t) \quad g'(x-t)]$

② $-3x^2 = u_t(x,0) = f'(4x) - g'(x)$ " " " .

Differentiating ① gives ③: $3x^2 = 4f'(4x) + g'(x)$ for $-\infty<x<\infty$.

Adding ② and ③ gives $0 = 5f'(4x) \Rightarrow f(z) = \text{constant} = c_1$.

Since $f' = 0$, ② becomes $-3x^2 = -g'(x) \Rightarrow x^3 + c_2 = g(x)$.

By ①, $x^3 = c_1 + x^3 + c_2$ for all $-\infty<x<\infty \Rightarrow c_1 + c_2 = 0$. Thus,

$u(x,t) = f(4x+t) + g(x-t) = c_1 + (x-t)^3 + c_2 = \boxed{(x-t)^3}$.

4. (25 pts.) Let $u = u(x,t)$ be a solution to

$$2u_{tt} - 2u_{xx} + u_t = 0$$

in $-\infty < x < \infty$, $0 \le t < \infty$, with the property that

$$\lim_{|x| \to \infty} u_x(x,t) u_t(x,t) = 0$$

for each fixed $t \geq 0$.

(a) Show that $\int_{-\infty}^{\infty} ([u_t(x,t)]^2 + [u_x(x,t)]^2) dx$ is a nonincreasing function of t. (Assume that all relevant improper integrals converge.)

(b) Give a physical interpretation of the result in part (a) in the case of the vibrations of a long string.

(a) Let $E(t) = \int_{-\infty}^{\infty} [u_t^2(x,t) + u_x^2(x,t)] dx$. Then $\frac{dE}{dt} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \{ u_t^2(x,t) + u_x^2(x,t) \} dx$

$= \int_{-\infty}^{\infty} \{ 2u_t(x,t) u_{tt}(x,t) + 2u_x(x,t) u_{tx}(x,t) \} dx$. Substituting $2u_{tt} = 2u_{xx} - u_t$ (from the p.d.e) into the integrand of dE/dt gives

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \{ 2u_t(x,t) u_{xx}(x,t) - u_t^2(x,t) + 2u_x(x,t) u_{tx}(x,t) \} dx$$

$$= \int_{-\infty}^{\infty} - u_t^2(x,t) dx + 2 \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \{ u_t(x,t) u_x(x,t) \} dx$$

$$= \int_{-\infty}^{\infty} - u_t^2(x,t) dx + 2 \lim_{\substack{M \to \infty \\ N \to -\infty}} \left[\cancelto{0}{u_t(x,t) u_x(x,t) \Big|_{x=N}^{x=M}} \right]$$

$$= - \int_{-\infty}^{\infty} u_t^2(x,t) dx \le 0 .$$

Therefore $E = E(t)$ is a nonincreasing function of t.

10 (b) The p.d.e. models the vertical displacement $u(x,t)$ of a long string at position x and time t, taking into account a drag force that is equal in magnitude to the velocity u_t of the string but opposes the motion. Hence the total energy $E(t)$ of the string at time t decreases with time.