Mathematics 325 | Exam I | Name: Dr. Grow
Summer 2002

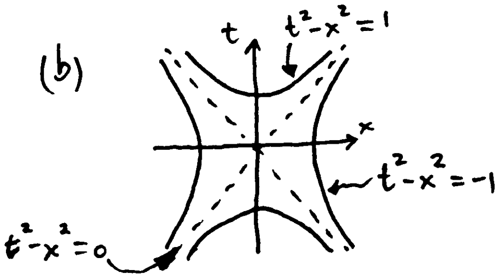
.(25 pts.) Consider

(*) $$tu_x + xu_t = 0.$$

(a) Find the characteristic curves of (*).
(b) Sketch and identify two characteristic curves of (*).
(c) Write the general solution of (*).
(d) Determine the particular solution of (*) that satisfies the auxiliary condition $u(x,0) = e^{-x^2}$ for $-\infty < x < \infty$.
(e) In what region of the xt-plane is the solution in part (d) uniquely determined?

(a) $\frac{dt}{dx} = \frac{x}{t} \Rightarrow t\,dt = x\,dx \Rightarrow \frac{1}{2}t^2 = \frac{1}{2}x^2 + c_1 \Rightarrow \boxed{t^2 - x^2 = c}$

(b)

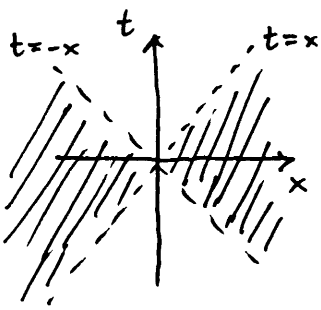


(c) $\boxed{u(x,t) = f(t^2-x^2)}$ where f is any C^1-function of a single real variable.

(d) $e^{-x^2} = u(x,0) = f(-x^2)$ for all real x $\Rightarrow$ $f(w) = e^w$ for all $w \le 0$.

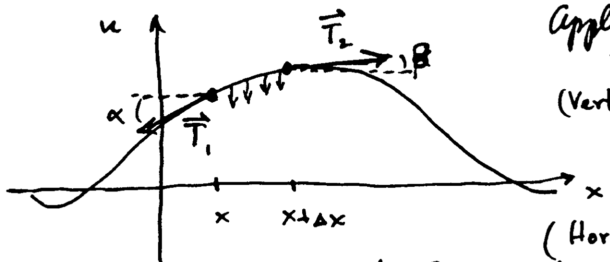
$\therefore \boxed{u(x,t) = e^{t^2-x^2}}$

(e) The solution in part (d) is uniquely determined in the region of the xt-plane given by $\{(x,t) \in \mathbb{R}^2 : t^2 - x^2 \le 0\}$



2. (25 pts.) Carefully derive the equation of motion of a string in a medium which resists motion with a force proportional to the velocity of the motion.

Let $u(x,t)$ denote the vertical displacement of the string at position x and time t. Fix $t>0$ and consider the segment of string between x and $x+\Delta x$.



(Assume motion is upward for sake of argument)

Apply Newton's second law of motion to this segment:

(Vertical) $$\int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi \overset{①}{=} |\vec{T}_2|\sin(\beta) - |\vec{T}_1|\sin(\alpha) - \int_x^{x+\Delta x} r\,u_t(\xi,t)\,d\xi$$

(Horizontal) $$0 \overset{②}{=} |\vec{T}_2|\cos(\beta) - |\vec{T}_1|\cos(\alpha)$$

(Here $\rho=\rho(x)$ denotes the mass density of the string at x, r is a proportionality constant, and $\vec{T}_1$ and $\vec{T}_2$ are the tensions exerted at x and $x+\Delta x$, respectively.) Since $u_x(x,t)=\tan(\alpha)$, it follows that $\sin(\alpha) \overset{③}{=} \dfrac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}$ and $\cos(\alpha) \overset{④}{=} \dfrac{1}{\sqrt{1+u_x^2(x,t)}}$. Similarly $\sin(\beta) \overset{⑤}{=} \dfrac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}$ and $\cos(\beta) \overset{⑥}{=} \dfrac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}}$. Substituting from ④ and ⑥ into ② gives

$$|\vec{T}_2| = \frac{|\vec{T}_1|\cos(\alpha)}{\cos(\beta)} \overset{⑦}{=} \frac{|\vec{T}_1|\sqrt{1+u_x^2(x+\Delta x,t)}}{\sqrt{1+u_x^2(x,t)}}.$$ Substituting ③, ⑤, and ⑦ into ① yields

$$\int_x^{x+\Delta x} \rho(\xi) u_{tt}(\xi,t)\,d\xi \overset{⑧}{=} \frac{|\vec{T}_1|}{\sqrt{1+u_x^2(x,t)}}\Big[u_x(x+\Delta x,t) - u_x(x,t)\Big] - \int_x^{x+\Delta x} r\,u_t(\xi,t)\,d\xi .$$ Dividing ⑧ by Δx and letting $\Delta x \to 0$ produces $$\boxed{\rho(x)u_{tt}(x,t) = \frac{|\vec{T}_1|}{\sqrt{1+u_x^2(x,t)}}\,u_{xx}(x,t) - r\,u_t(x,t)}.$$

For "small" vibrations of the string $u_x^2 \ll 1$ so $\sqrt{1+u_x^2(x,t)} \approx 1 \approx \sqrt{1+u_x^2(x+\Delta x,t)}$ and $|\vec{T}_1| = |\vec{T}_2| = \text{constant} = T_0$. Also, for a homogeneous string $\rho(x) = \text{constant} = \rho$, so the equation of motion becomes

$$\boxed{\rho u_{tt} - T_0 u_{xx} + r u_t = 0}.$$

3. (25 pts.) Consider

(+) $$u_{xx} + 4u_{yy} - 4u_{xy} + u = (x - 2y)^2.$$

(a) Classify the order and type (nonlinear, linear, homogeneous, inhomogeneous, elliptic, hyperbolic, parabolic) of (+).

(b) Find, if possible, the general solution of (+) in the xy-plane.

(a) $\boxed{\text{2nd-order, linear, inhomogeneous}}$ $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$ $\boxed{\text{parabolic}}$.

(b) $$\left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} + 4\frac{\partial^2}{\partial y^2}\right)u + u = (x-2y)^2.$$

$$\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)u + u = (x-2y)^2.$$

8 Let $\begin{cases} \xi = 2x + y, \\ \eta = x - 2y. \end{cases}$ Then $\frac{\partial}{\partial x} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}$

So $\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} = 5\frac{\partial}{\partial \eta}$.

12 Hence (+) is equivalent to $25\frac{\partial^2 u}{\partial \eta^2} + u = \eta^2 \Rightarrow \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{25}u = \frac{\eta^2}{25}$. (++)

10 $\therefore$ $u(\xi,\eta) = \underbrace{c_1(\xi)\cos\left(\frac{\eta}{5}\right) + c_2(\xi)\sin\left(\frac{\eta}{5}\right)}_{\text{general solution to } u_{\eta\eta} + \frac{1}{25}u = 0} + u_p$

a particular solution of (++).

We use undetermined coefficients to find u_p. A trial solution is $u_p = A\eta^2 + B\eta + C$

Then $\frac{\partial u_p}{\partial \eta} = 2A\eta + B$ and $\frac{\partial^2 u_p}{\partial \eta^2} = 2A$. Substituting in (++) gives

$$2A + \frac{1}{25}\left(A\eta^2 + B\eta + C\right) = \frac{\eta^2}{25}$$

$\Rightarrow$ $\frac{A}{25} = \frac{1}{25}$, $\frac{B}{25} = 0$, and $2A + \frac{C}{25} = 0$, i.e. $A = 1$, $B = 0$, $C = -50$.

Thus $u_p = \eta^2 - 50$, and hence $u(\xi,\eta) = c_1(\xi)\cos\left(\frac{\eta}{5}\right) + c_2(\xi)\sin\left(\frac{\eta}{5}\right) + \eta^2 - 50$.

Therefore

20 $$\boxed{u(x,y) = f(2x+y)\cos\left(\frac{x-2y}{5}\right) + g(2x+y)\sin\left(\frac{x-2y}{5}\right) + (x-2y)^2 - 50}$$

where f and g are any C^2-functions of a single real variable.

4. (25 pts.) (a) Write (no proof or derivation required) and simplify an expression for the solution to

(*) $$u_{tt} - 4u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to the initial conditions

$$u(x,0) = e^{-x^2} \quad \text{and} \quad u_t(x,0) = 4xe^{-x^2} \quad \text{for } -\infty < x < \infty.$$

(b) Sketch profiles of the solution in part (a), $u = u(x,t)$, at $t = 1$, 2, and 3 in order to illustrate that the solution is a wave traveling to the right along the x-axis. What is its speed?

(c) Derive a general nontrivial relation between ϕ and ψ which will produce a solution to (*) in the xt-plane satisfying

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty$$

and such that u consists solely of a wave traveling to the right along the x-axis.

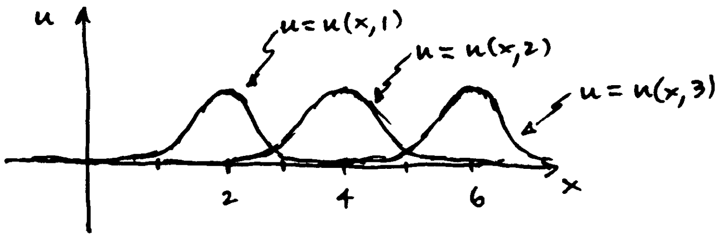
(a) $$u(x,t) = \frac{1}{2}\left[e^{-(x-2t)^2} + e^{-(x+2t)^2}\right] + \frac{1}{4}\int_{x-2t}^{x+2t} 4se^{-s^2}\,ds$$

Let $w = -s^2$. Then $dw = -2s\,ds$ so $\int 4se^{-s^2}ds = \int -2e^{w}dw = -2e^{w} = -2e^{-s^2}$

$$= \frac{1}{2}\left[e^{-(x-2t)^2} + \cancel{e^{-(x+2t)^2}}\right] - \frac{1}{2}\left[\cancel{e^{-(x+2t)^2}} - e^{-(x-2t)^2}\right]$$

$$= \boxed{e^{-(x-2t)^2}}$$

5 (b)



$\boxed{\text{Speed} = 2}$

10 (c) $$u(x,t) = \frac{1}{2}\left[\varphi(x-2t) + \varphi(x+2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \psi(s)\,ds$$

We need $\frac{1}{2}\varphi(x+2t) + \frac{1}{4}\int_0^{x+2t} \psi(s)\,ds = 0$ for all x and t in order for $u = u(x,t)$ to consist solely of a wave traveling to the right along the x-axis.

I.e. $\varphi(z) = -\frac{1}{2}\int_0^z \psi(s)\,ds$ for all real z $\Rightarrow$ $\boxed{\varphi'(z) = -\frac{1}{2}\psi(z) \text{ for all real } z}$.